Large scale geometry of automorphism groups

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Third lecture:
Equivariant geometry

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Similarly, linear and affine representations of our groups provides a link to study harmonic-analytic and dynamical features of these.

Cf. earlier work of T. Tsankov on unitary representations of oligomorphic groups.

Linear and affine representations

Let E be a Banach space and G a topological group. A continuous isometric linear representation of G on E is a continuous action

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into the group $\mathrm{Isom}(E)$ of linear isometries of E, equipped with the strong operator topology, that is, the topology of pointwise convergence on E.

By a classical result of Mazur and Ulam, every surjective isometry $A\colon E \to E$ of a Banach space is affine, that is, of the form

$$A(\xi) = T(\xi) + \eta_0$$

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I.e., for $g \in G$ and $\xi \in E$,

$$\alpha(g)\xi = \pi(g)\xi + b(g).$$

In general, given π, for $\alpha(g)\xi = \pi(g)\xi + b(g)$ to define an action, b must satisfy the cocycle equation

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Also, as $b(g) = \alpha(g)0$ and α is an isometric action,

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Definition

The action $\alpha\colon G \curvearrowright E$ is coarsely proper if $b\colon G \to E$ is a coarse embedding.

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Question

When can we turn a coarse embedding into a cocycle?

Amenability

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However, for concrete examples, there is often some more explicit and specific reason for the groups to be amenable bipassing Moore's criterion.

For example, being abelian, solvable, compact, etc.

The Haagerup property

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In the context of countable or locally compact groups, the Haagerup property is often viewed as a strong non-rigidity property.

For general Polish groups, we may also view it as a regularity property, since it allows for an efficient representation of G on the most regular Banach space $\mathcal{H}$.

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1. *G coarsely embeds into a Hilbert space,*
2. *G has the Haagerup property.*

A geometric particuliarity of $\mathcal{H}$ used here is that a Polish group G coarsely embeds into $\mathcal{H}$ if and only if it has a uniformly continuous coarse embedding into $\mathcal{H}$.

This relies on results on the extension of Hölder continuous Hilbert valued functions and was exploited earlier by B. Johnson and L. Randrianarivony.

Observe that, if $\alpha\colon G \curvearrowright \mathcal{H}$ is a non-trivial affine isometric action, the either the linear part

$$\pi\colon G \to U(\mathcal{H})$$

is a non-trivial unitary representation of G or the cocycle

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Composing b with an appropriate linear functional on $\mathcal{H}$, we obtain a non-trivial homomorphism into $\mathbb{C}$.

Corollary

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The above result then shows that provided G is amenable and has non-trivial coarse geometry, this analytical incompatibility with $\mathcal{H}$ must be reflected in a coarse geometric incompatibility with $\mathcal{H}$.

Fixed points on reflexive spaces

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Theorem (Kalton)

Let E be either reflexive or $E = L^1([0,1])$.
Then every bornologous map $\phi\colon c_0 \to E$ is insolvent.

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Paring this with our knowledge of the geometry of $\mathrm{Isom}(\mathbb{QU})$, we can now obtain analytical information from purely geometric data.

Theorem

Every continuous affine isometric action of $\mathrm{Isom}(\mathbb{QU})$ on a reflexive Banach space or on $L^1([0,1])$ has a fixed point.

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Recall that, if $x_0 \in \mathbb{QU}$ is fixed, then the pointwise stabiliser

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After conjugating by a translation, we may assume that $b \equiv 0$ on V.

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More generally,

$$d(y,z) = d(v,w) \quad \Rightarrow \quad \|\phi(y) - \phi(z)\| = \|\phi(v) - \phi(w)\|.$$

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In contradistinction to this, we have

Theorem (Brown–Guentner, Haagerup–Przybyszewska)

Every locally compact Polish group has a coarsely proper continuous affine isometric action on a reflexive space.

Another take on amenability

Definition

A topological group G is said to be approximately compact if there is a countable chain $K_0 \leqslant K_1 \leqslant \ldots \leqslant G$ of compact subgroups whose union $\bigcup_n K_n$ is dense in G.

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E.g., the unitary subgroup $U(M)$ of an approximately finite-dimensional von Neumann algebra M is approximately compact (P. de la Harpe).

In the context of automorphism groups, approximate compactness can be usefully reformulated.

Proposition (A.S. Kechris & C.R.)

Let $\mathcal{K}$ be a Fraïssé class of finite structures with limit $\mathbf{K}$.

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Let $\mathcal{K}$ be a Fraïssé class of finite structures with limit **K**.
Then $\mathrm{Aut}(\mathbf{K})$ is approximately compact if and only if $\mathcal{K}$ has the Hrushovski property, i.e., for every finite substructure $\mathbf{A} \subseteq \mathbf{M}$ and all partial *automorphisms $\phi_1, \ldots, \phi_n$ of* **A**, *there is a larger finite substructure* **B** *with*

$$\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{M}$$

and full *automorphisms $\psi_1, \ldots, \psi_n$ of* **B** *extending $\phi_1, \ldots, \phi_n$ respectively.*

Whereas a locally compact group G is amenable if and only if it admits a Følner sequence, that is, a sequence $F_1, F_2 \ldots \subseteq G$ so that

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A Polish group G is said to be Følner amenable if either

1. *G is approximately compact, or*
2. *there is a continuous homomorphism $\phi \colon H \to G$ from a locally compact second countable amenable group H so that $G = \overline{\phi[H]}$.*

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E.g., the property of being super-reflexive, that is, having a uniformly convex renorming (Enflo).

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Redefining σ to be constant on each left coset gV, we obtain a uniformly continuous coarse embedding

$$\tilde{\sigma}\colon G \to E.$$

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Here is an application in a completely different direction.

Corollary

Let X be a Banach space uniformly embeddable into the unit ball B_E of a super-reflexive Banach space E. Then X contains some ℓ^p, $1 \leqslant p < \infty$.

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Then $\mathrm{Aut}(\mathbf{A})$ admits a coarsely proper continuous affine isometric action on a reflexive Banach space.