Linear sandwich semigroups

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Permutation groups and transformation semigroups
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Don't mention the cri%$et

Joint work with Igor Dolinka

Sandwiches?

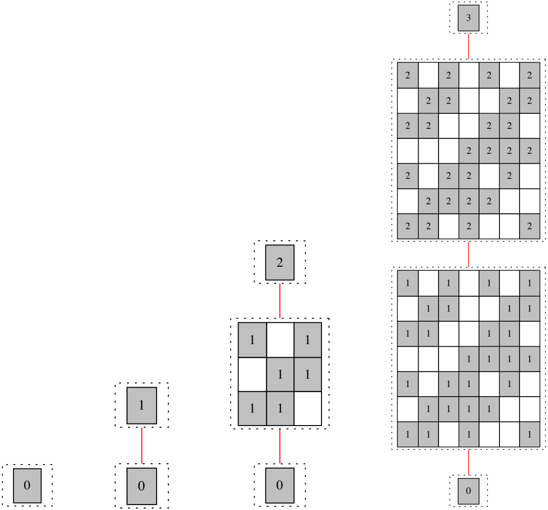
Linear sandwich semigroups (Lyapin, 1960; cf Brown 1955)

- *Let $\mathcal{M}_{mn}$ be the set of all $m \times n$ matrices over a field F.*
- *Fix $A \in \mathcal{M}_{nm}$.*
- *For $X, Y \in \mathcal{M}_{mn}$, define $X \star Y = XAY$.*
- *$\mathcal{M}_{mn}^A = (\mathcal{M}_{mn}, \star)$ is a linear sandwich semigroup.*

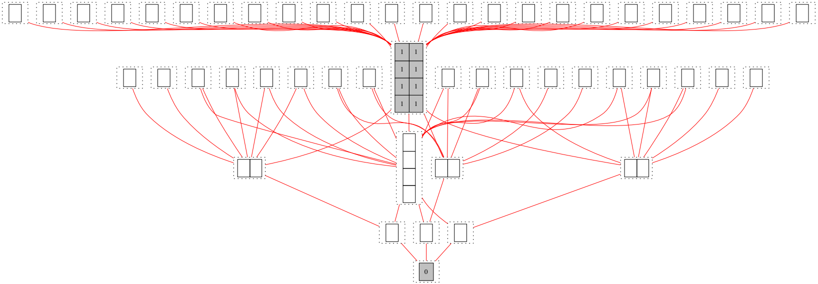
Example

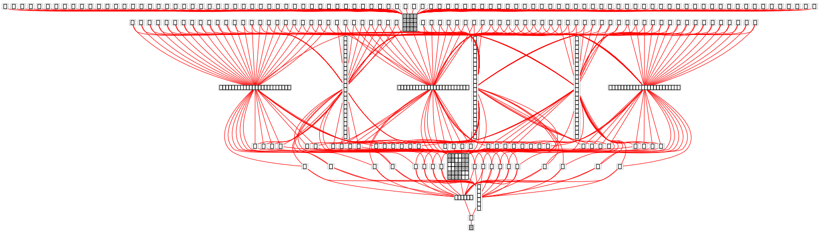
If $m = n$ and $A = I$, then $\mathcal{M}_{mn}^A = \mathcal{M}_n$ is the full linear monoid.

Egg-box diagrams for $\mathcal{M}_n$ ($F = \mathbb{Z}_2$)

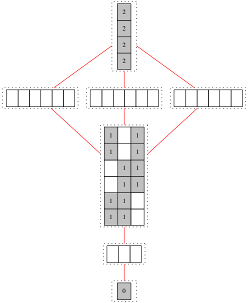


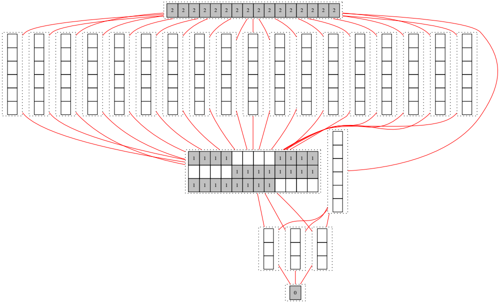
Egg sandwiches

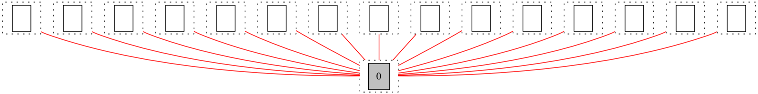




Egg sandwiches







Plan (non-linear)

- *Green's relations*
- *regular elements*
- *ideals*
- *idempotent generation*
- *small (idempotent) generating sets*
- *bigger sandwiches?*

Easy lemmas

Lemma

If $A, B \in \mathcal{M}_{nm}$ and $\operatorname{rank}(A) = \operatorname{rank}(B)$, then $\mathcal{M}_{mn}^A \cong \mathcal{M}_{mn}^B$.

- So we study $\mathcal{M}_{mn}^J$, where

$$J = J_r = \left[\begin{smallmatrix} I_r & O \\ O & O \end{smallmatrix}\right] \in \mathcal{M}_{nm} \qquad (0 \leq r \leq \min(m,n) \text{ is fixed}).$$

- Write elements of $\mathcal{M}_{mn}$ in 2×2 block form:

$$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \qquad \text{where } A \in \mathcal{M}_{rr},\ \ B \in \mathcal{M}_{r,n-r},\ \ \text{etc.}$$

Lemma

$$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \star \left[\begin{smallmatrix} E & F \\ G & H \end{smallmatrix}\right] = \left[\begin{smallmatrix} AE & AF \\ CE & CF \end{smallmatrix}\right].$$

Green's relations

Let $X, Y \in \mathcal{M}_{mn}$. Write

- $X \mathscr{R} Y \Leftrightarrow X\mathcal{M}_n = Y\mathcal{M}_n$,
- $X \mathscr{L} Y \Leftrightarrow \mathcal{M}_m X = \mathcal{M}_m Y$,
- $X \mathscr{J} Y \Leftrightarrow \mathcal{M}_m X \mathcal{M}_n = \mathcal{M}_m Y \mathcal{M}_n$,
- $\mathscr{H} = \mathscr{R} \cap \mathscr{L}$,
- $\mathscr{D} = \mathscr{R} \vee \mathscr{L}$.

For $X \in \mathcal{M}_{mn}$, write $R_X = \{Y \in \mathcal{M}_{mn} : X \mathscr{R} Y\}$, etc.

Proposition

- $R_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Col}(X) = \mathrm{Col}(Y)\}$,
- $L_X = \{Y \in \mathcal{M}_{mn} : \mathrm{Row}(X) = \mathrm{Row}(Y)\}$,
- $J_X = D_X = \{Y \in \mathcal{M}_{mn} : \mathrm{rank}(X) = \mathrm{rank}(Y)\}$

Green's relations

Let $X, Y \in \mathcal{M}_{mn}$. Write

- $X \mathscr{R}^J Y \Leftrightarrow X \star \mathcal{M}_{mn} = Y \star \mathcal{M}_{mn}$, — i.e., $XJ\mathcal{M}_{mn} = YJ\mathcal{M}_{mn}$,
- $X \mathscr{L}^J Y \Leftrightarrow \mathcal{M}_{mn} \star X = \mathcal{M}_{mn} \star Y$,
- $X \mathscr{J}^J Y \Leftrightarrow \mathcal{M}_{mn} \star X \star \mathcal{M}_{mn} = \mathcal{M}_{mn} \star Y \star \mathcal{M}_{mn}$,
- $\mathscr{H}^J = \mathscr{R}^J \cap \mathscr{L}^J$,
- $\mathscr{D}^J = \mathscr{R}^J \vee \mathscr{L}^J$.

These are the usual Green's relations on $\mathcal{M}_{mn}^J$.

For $X \in \mathcal{M}_{mn}$, write $R_X^J = \{Y \in \mathcal{M}_{mn} : X \mathscr{R}^J Y\}$, etc.

Green's relations

Let $X = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{M}_{mn}$. Easy to check:

- $XJ = \left[\begin{smallmatrix} A & O \\ C & O \end{smallmatrix}\right]$, $\quad JX = \left[\begin{smallmatrix} A & B \\ O & O \end{smallmatrix}\right]$, $\quad JXJ = \left[\begin{smallmatrix} A & O \\ O & O \end{smallmatrix}\right]$.

Define sets

- $P_1 = \{X \in \mathcal{M}_{mn} : \mathrm{Col}(XJ) = \mathrm{Col}(X)\}$,
- $P_2 = \{X \in \mathcal{M}_{mn} : \mathrm{Row}(JX) = \mathrm{Row}(X)\}$,
- $P = P_1 \cap P_2 = \{X \in \mathcal{M}_{mn} : \mathrm{rank}(JXJ) = \mathrm{rank}(X)\}$

$$= \mathrm{Reg}(\mathcal{M}_{mn}^J) \leq \mathcal{M}_{mn}^J.$$

Green's relations

Proposition

For $X \in \mathcal{M}_{mn}$,

- $R^J_X = \begin{cases} R_X \cap P_1 & \text{if } X \in P_1 \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus P_1, \end{cases}$

- $L^J_X = \begin{cases} L_X \cap P_2 & \text{if } X \in P_2 \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus P_2, \end{cases}$

- $H^J_X = \begin{cases} H_X & \text{if } X \in P \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus P, \end{cases}$

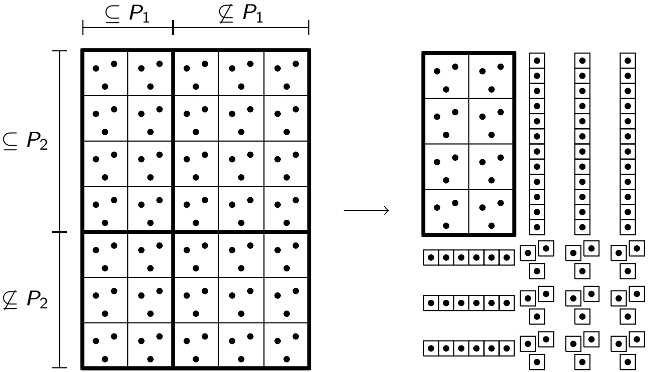
Green's relations

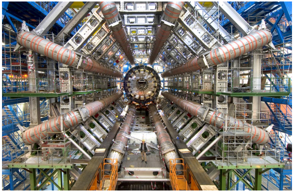
Proposition (continued)

For $X \in \mathcal{M}_{mn}$,

- $D^J_X = \begin{cases} D_X \cap P & \text{if } X \in P \\ L_X \cap P_2 & \text{if } X \in P_2 \setminus P_1 \\ R_X \cap P_1 & \text{if } X \in P_1 \setminus P_2 \\ \{X\} & \text{if } X \in \mathcal{M}_{mn} \setminus (P_1 \cup P_2). \end{cases}$

High energy semigroup theory



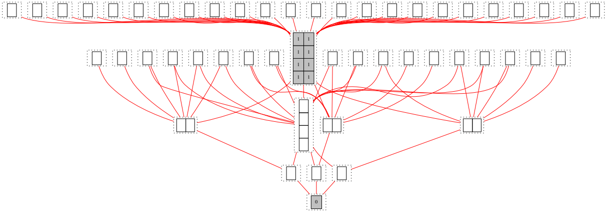


Small generating sets

Theorem

Suppose $r < \min(m,n)$.

- *The $\mathscr{D}^J$-maximal elements generate $\mathcal{M}^J_{mn}$.*
- *Any generating set must contain all $\mathscr{D}^J$-maximal elements.*
- $\operatorname{rank}(\mathcal{M}^J_{mn}) = \sum_{s=r+1}^{\min(m,n)} \begin{bmatrix} m \\ s \end{bmatrix}_q \begin{bmatrix} n \\ s \end{bmatrix}_q q^{\binom{s}{2}}(q-1)^s[s]_q!.$

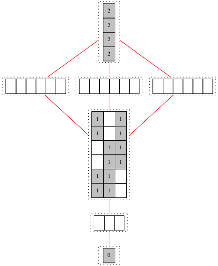


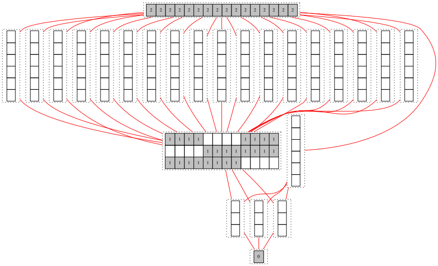
Small generating sets

Theorem

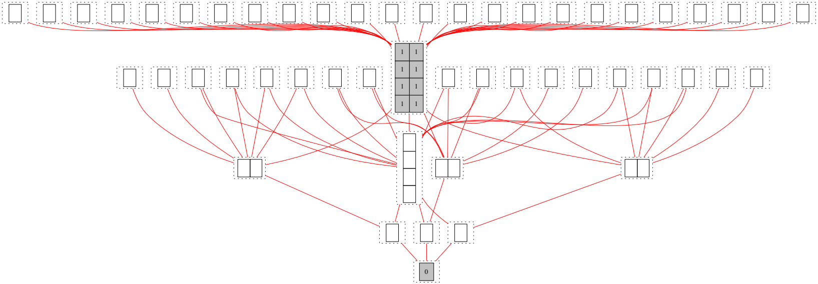
Suppose $r = \min(m, n)$*.*

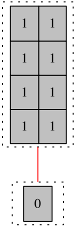
- $\mathcal{M}^J_{mn}$ *has a unique maximal* $\mathscr{D}^J$*-class — a rectangular group.*
- $\operatorname{rank}(\mathcal{M}^J_{mn}) = \begin{bmatrix} \max(m, n) \\ \min(m, n) \end{bmatrix}_q$.



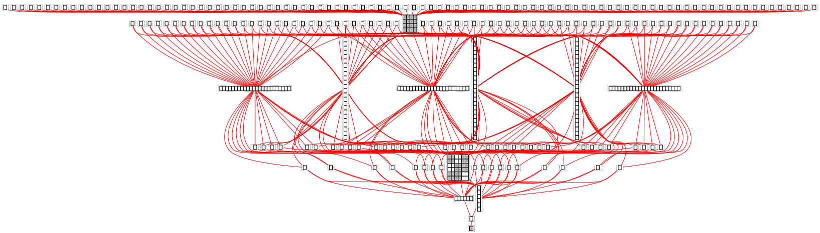


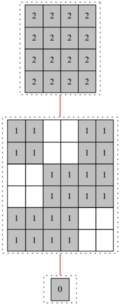
Regular elements — unscrambling the egg — $\mathcal{M}_{32}^{J_1}$



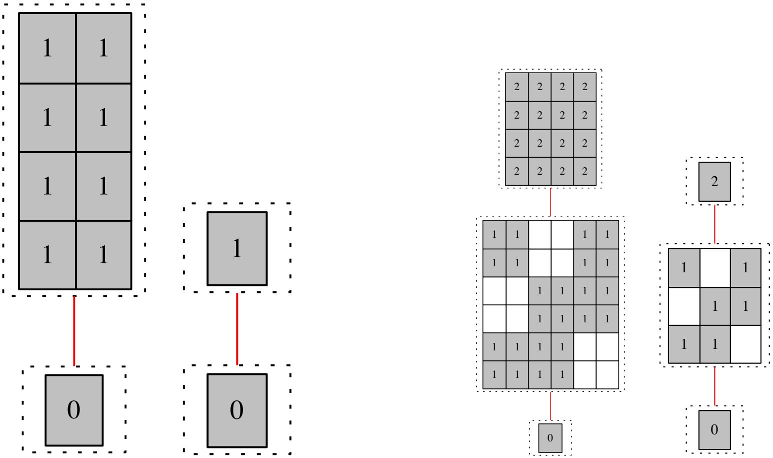


Regular elements — unscrambling the egg — $\mathcal{M}_{33}^{J_2}$



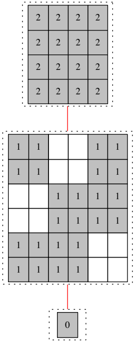


Look familiar?

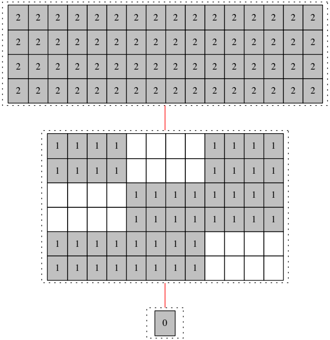


$\mathrm{Reg}(\mathcal{M}_{32}^{J_1})$ $\mathcal{M}_1$ $\mathrm{Reg}(\mathcal{M}_{33}^{J_2})$ $\mathcal{M}_2$

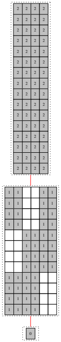
Look familiar?



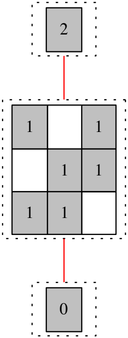
$\mathrm{Reg}(\mathcal{M}_{33}^{J_2})$



$\mathrm{Reg}(\mathcal{M}_{34}^{J_2})$

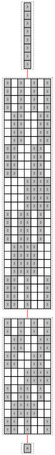


$\mathrm{Reg}(\mathcal{M}_{43}^{J_2})$

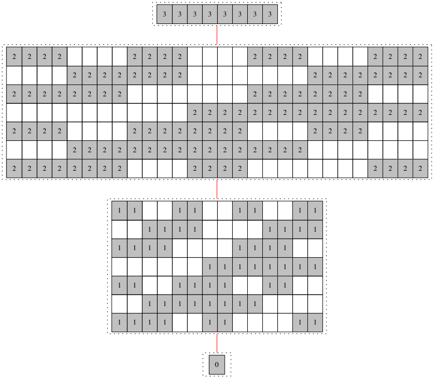


$\mathcal{M}_2$

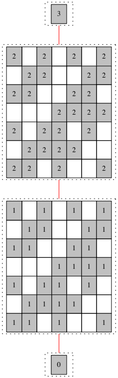
Look familiar?



$\mathrm{Reg}(\mathcal{M}_{43}^{J_3})$



$\mathrm{Reg}(\mathcal{M}_{34}^{J_3})$



$\mathcal{M}_3$

Big bang

Theorem

- $P = \mathrm{Reg}(\mathcal{M}_{mn}^{J_r}) \leq \mathcal{M}_{mn}^{J_r}$ *is an "inflated* $\mathcal{M}_r$*".*
- $\mathcal{M}_r$ *has a chain of* $\mathscr{D}$*-classes:* $D_0 < D_1 < \cdots < D_r$.
- P *has a chain of* $\mathscr{D}^J$*-classes:* $D_0^J < D_1^J < \cdots < D_r^J$.
- $D_s(\mathcal{M}_r)$ *maps onto* $\left[{r \atop s} \right]_q$ $\mathscr{R}$*-classes of* $\mathcal{M}_r$.
- *Each of these expands into* $q^{s(m-r)}$ $\mathscr{R}^J$*-classes in* $D_s^J(P)$.
- *Group* $\mathscr{H}^J$*-classes in* $D_s(\mathcal{M}_r)$ *and* $D_s^J(P)$ *are* $\cong \mathcal{G}_s$.
- $|P| = \sum_{s=0}^r q^{s(m+n-2r)} q^{\binom{s}{2}} (q-1)^s [s]_q! \left[{r \atop s} \right]_q^2$.
- $\mathrm{rank}(P) = q^{r(\max(m,n)-r)} + 1$.

Idempotent generators

Theorem

- $\big|E(\mathcal{M}_r)\big| = \sum_{s=0}^{r} q^{s(r-s)} \left[{r \atop s} \right]_q$.
- $\big|E(\mathcal{M}_{mn}^J)\big| = \sum_{s=0}^{r} q^{s(m+n-r-s)} \left[{r \atop s} \right]_q$.
- $\mathcal{E}_r = \big\langle E(\mathcal{M}_r) \big\rangle = \{I_r\} \cup (\mathcal{M}_r \setminus \mathcal{G}_r)$ — *Erdos.*
- $\operatorname{rank}(\mathcal{E}_r) = \operatorname{idrank}(\mathcal{E}_r) = 1 + (q^r - 1)/(q-1)$ — *Dawlings.*
- $\mathcal{E}_{mn}^J = \big\langle E(\mathcal{M}_{mn}^J) \big\rangle = E(D) \cup (P \setminus D)$, *where* $D = D_r(P)$.
- $\operatorname{rank}(\mathcal{E}_{mn}^J) = \operatorname{idrank}(\mathcal{E}_{mn}^J) = q^{r(\max(m,n)-r)} + (q^r - 1)/(q-1)$.

Ideals

Theorem

- *The ideals of $\mathcal{M}_r$ form a chain:*

$$\{O_r\} = I_0 \subset I_1 \subset \cdots \subset I_r = \mathcal{M}_r.$$

- $\operatorname{rank}(I_s) = \operatorname{idrank}(I_s) = \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_q$ *for* $0 \leq s < r$ *— Gray.*

- *The ideals of* $P = \operatorname{Reg}(\mathcal{M}_{mn}^J)$ *form a chain:*

$$\{O_{mn}\} = I_0^J \subset I_1^J \subset \cdots \subset I_r^J = P.$$

- $\operatorname{rank}(I_s^J) = \operatorname{idrank}(I_s^J) = q^{s(\max(m,n)-r)} \left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]_q$ *for* $0 \leq s < r.$

More sandwiches...

- Let $\mathscr{C} = (\mathrm{Ob}, \mathrm{Hom})$ be a small category.
- $S = \mathrm{Hom}$ has the structure of a partial semigroup.
- $f \circ (g \circ h) = (f \circ g) \circ h$ if the products are defined, etc.
- Partialise notions like Green's relations, regularity, stability.
- If $\theta \in \mathrm{Hom}(Y, X)$ is fixed, $\mathrm{Hom}(X, Y)$ is a semigroup under:

$$f \star g = f \circ \theta \circ g.$$

- $\mathcal{M}_{mn}^A$ arises when $\mathscr{C}$ is a linear category.
- Work is under way for diagram categories and more...

Thank you!

- *Variants of finite full transformation semigroups*
  - Dolinka and East — `http://arxiv.org/abs/1410.5253`
- *Semigroups of rectangular matrices under a sandwich operation*
  - Dolinka and East — `http://arxiv.org/abs/1503.03139`