Coproducts for Permutation Groups, Transformation Semigroups, Automata and Related Categories

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Motivation: Gluing Faithful Permutation Groups and Transformation Semigroups

We show the existence and describe the structure of coproducts in the following categories with objects (X, S), with X a set and $S \subseteq X^X$ a set of functions on X closed under composition, writing $x \cdot s$ for $s \in S$ applied to $x \in X$:

permutation groups *PermGrp* (each $s \in S$ is a permutation of X and S is group)
transformation monoids *TM* (identity $id_X \in S$)
transformation semigroups *TS*
partial transformation semigroups *PTS*. (Each s partial function from X to X)

Also for the variants $PermGrp_*$, TM_*, TS_*, PTS_* of these categories with base-points $* \in X$ and base-point preserving maps.

All in all these categories actions are *faithful*: if elements s_1, s_2 of the group (resp., monoid, semigroup) act the same on all states, then they are equal.

Related to these we describe coproducts in various automata categories (deterministic partial; complete deterministic; nondeterministic partial; with and without initial state)

A *morphism* ψ of permutation groups (X, S) to (X', S') is a set map $\psi^{state} : X \to X'$ and homomorphism $\psi^{operators} : S \to S'$, with

$$\psi^{state}(x \cdot s) = \psi^{state}(x) \cdot \psi^{operator}(s) \, \forall x \in X, s \in S$$

$$\psi^{operator}(s_1 s_2) = \psi^{operator}(s_1)\psi^{operator}(s_2) \, \forall s_1, s_2 \in S$$

It follows that inverses map to inverses, and identity of S maps to identity element of S' (since idempotents map to idempotents).

A transformation semigroup morphism is defined the same way. For the transformation monoid category, one must require of morphisms, that the identity of S map to that of S'.

Coproduct of Groups, Monoids, or Semigroups

In groups or monoids, the coproduct is the "free product".

$S * T = \{(a_1, \dots, a_k) : k \geq 0,$ with the $a_i \neq 1$ alternating membership in S and $T\}$.

If $k = 0$ this is the identity element of $S * T$.

Multiply: $(a_1, \dots, a_k)(b_1, \dots, b_n) =$

$$\begin{cases} (a_1, \dots, a_k, b_1, \dots, b_n) & \text{if } a_k \in S, b_1 \in T, \text{ or } a_k \in T, b_1 \in S \\ reduce(a_1, \dots, a_k b_1, \dots, b_n) & \text{if } a_k, b_1 \in S \text{ or } a_k, b_1 \in T \end{cases}$$

where *reduce* means removing any 1's that appear, and combine any new neighbors by multiplication if both are from same S or T, and then iterating reduction to get a canonical form.

Coproduct of two groups in the monoid category is the same as their coproduct in the category of groups.

$$S * T = T * S, \qquad 1 * S = S$$

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Coproduct of Semigroups

$S * T = \{(a_1, \ldots, a_k) : k \geq 1,$ with the a_i alternating membership in S and $T\}$.

Multiply: $(a_1, \ldots, a_k)(b_1, \ldots, b_n) =$

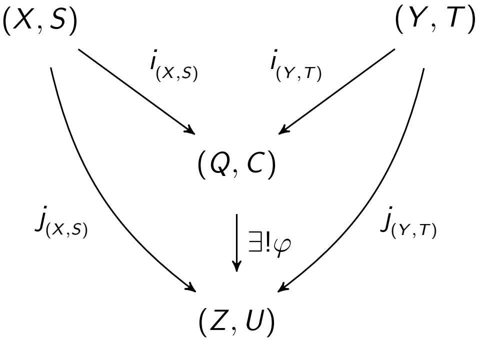
$$\begin{cases} (a_1, \ldots, a_k, b_1, \ldots, b_n) & \text{if } a_k \in S, b_1 \in T, \text{ or } a_k \in T, b_1 \in S \\ (a_1, \ldots, a_k b_1, \ldots, b_n) & \text{if } a_k, b_1 \in S \text{ or } a_k, b_1 \in T \end{cases}$$

For semigroups, coproduct of two nonempty semigroups is always infinite, e.g., 1*1 is infinite.

$S * T = T * S$, $\qquad \emptyset * S = S$,

$S * T$ is not a monoid unless one factor is a monoid and the other is empty. Can make $S * T$ into a monoid by adjoining a new identity element λ (empty sequence).

A **coproduct** $(X, S) \coprod (Y, T)$ of permutation groups (X, S) and (Y, T), if it exists is some (Q, C) with two maps $i_{(X,S)}$ and $i_{(Y,T)}$ to (Q, C) such that when j's are given to some permutation group (Z, U) then these factor uniquely through (Q, C):



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Observe: A coproduct is unique up to isomorphism (if it exists).

Obvious guesses about what the coproduct should be are mostly wrong....

Example: Let $[n] = \{1, \ldots, n\}$. What could $([3], S_3) \coprod ([2], Z_2)$ be? What could $([3], Z_3) \coprod ([2], Z_2)$ be?

Obvious guesses about what the coproduct should be are mostly wrong....

Example: What could $([3], Z_3) \coprod ([2], Z_2)$ be?

Take disjoint union of state sets as new state set?

Group acting should be free product (coproduct) of $Z_3 * Z_2$ or their direct product $Z_3 \times Z_2$?

How to act on states from the *other component* with embedded copies of Z_3 and Z_2?
- Trivially? Undefined?? (Good idea for partial trans. semigroups...)

See black board: What if (Z, U) is given by identifying one of the states of each factor?
NB: Images of Z_3 and Z_2 in U under the j's do not commute!

So their preimages under the unique φ cannot commute either.
So can't act trivially on the other component.
(So coproduct can't have group the direct product).
Also action must be faithful, so if $Z_3 * Z_2$ acts, the state set Q is infinite.
later in talk: Compare this to $([3], S_3) * ([2], Z_2)$ in $PermGrp_*$, TS_*

Coproduct of Permutation Groups

Theorem

In the category of permutation groups PermGrp, given permutation groups (X, S) and (Y, T), their coproduct is

$$((X \sqcup Y) \otimes (S * T), S * T),$$

where $S * T$ is the free product of groups and $(X \sqcup Y) \otimes (S * T)$ denotes $((X \sqcup Y) \times (S * T))/\equiv$ under the equivlaence relation $\equiv$ generated by

$$\begin{aligned} (a, sw) &\sim (a \cdot s, w), \quad \text{if } a \in X, s \in S, \\ (a, tw) &\sim (a \cdot t, w), \quad \text{if } a \in Y, t \in T, \end{aligned} \tag{2}$$

where $i_X : (X, S) \to ((X \sqcup Y) \otimes (S * T), S * T)$ maps $x \mapsto (x, 1)$, $s \mapsto s \in S * T$, and $i_Y : (Y, T) \to ((X \sqcup Y) \otimes (S * T), S * T)$ maps $y \mapsto (y, 1)$, $t \mapsto t \in S * T$.

Outline of Proof for Coproduct of Permutation Groups

Rewrite elements of $(X \sqcup Y) \times (S * T)$ to equivalent elements in canonical form, $a \in X \sqcup Y$, $w \in S * T$: Move letters s and t from w to the left when action of the s or t is defined on a in the factor until impossible .
Action $(a, w) \cdot w' = (a, ww')$, $a \in X \sqcup Y$, $w, w' \in S * T$ is well-defined on equivalence classes.
Action is faithful, so we have a (faithful) permutation group $((X \sqcup Y) \otimes (S * T), S * T)$.
Existence of unique morphism to any (Z, U) making diagram commute: let $\varphi : S * T \to U$ be the unique homomorphism (for the coproduct $S * T$).
For states: Map (a, w) to $j_X(a) \cdot \varphi(w)$ if $a \in X$ or to $j_Y(a) \cdot \varphi(w)$ for $a \in Y$.
This is well-defined on equivalence classes: if we apply an equivalence rule this gives same member of Z. E.g.,
$(x, sw) \mapsto j_X(x) \cdot \varphi(sw) = j_X(x) \cdot \varphi(s)\varphi(w) = (j_X(x) \cdot \varphi(s)) \cdot \varphi(w) = (j_X(x) \cdot j_S(s)) \cdot \varphi(w) = j_X(x \cdot s) \cdot \varphi(w)$, which is where (x, sw) maps.
The diagram commutes as required. Uniqueness of state-map follows easily since it is determined where $(x, 1)$ and $(y, 1)$ must go, and hence where $(x, w) = (x, 1) \cdot w$ and $(y, w) = (y, 1) \cdot w$ go . □

Theorems for Coproducts of Transformation Monoids & Semigroups, and with basepoints and/or parital

- Coproducts for transformation monoids are constructed in exactly the same way.
- Coproducts of transformation semigroups are constructed the same way, semigroup acting is $S * T$, but for states $S * T$ in $(X \sqcup Y) \times (S * T)$ is augmented to $(S * T) \cup \{\lambda\}$, where λ in 2nd coordinate serves the same role 1 did in the permutation group case. (Works for $|X|, |Y| \geq 1$.)
- With basepoints, one also obtains a canonical form for states, adding one more rule $(x_0, w) \sim (y_0, w)$, where x_0 is the basepoint of X, y_0 is the basepoint of Y, and $w \in S * T$. In the semigroup case, we allow $w = \lambda$, the empty word. Coproduct exists for $|X|, |Y| > 1$.
- For partial transformation semigroups, coproduct is very different. States are just the disjoint union. Semigroup acting is just union of S and T which are undefined if they act on the state of the other component. Similarly for partial transformation semigroups with basepoint.

Theorems for Coproducts of Automata

- Theorem (Coproduct for Automata). For complete deterministic reachable automata with initial state, and with distinct inputs 'faithful' (give distinct maps on the state set), the states of the coproduct of automata $\mathcal{A} = (Q_A, X, i_{\mathcal{A}}, \delta_X : Q_A \times X \rightarrow Q_A)$ and $\mathcal{B} = (Q_B, Y, i_{\mathcal{B}}, \delta_Y : Q_B \times B \rightarrow Q_B)$ are the states of the coproduct of the pointed transformation semigroups of the transformation semigroups of its factors, taking the initial states as basepoints.

  That is, the coproduct is the complete deterministic reachable automaton,

  $$\mathcal{A} \sqcup \mathcal{B} = ((Q_A \sqcup Q_B) \otimes (S(\mathcal{A}) * S(\mathcal{B}))^{\lambda}, X \sqcup Y, i, \delta)$$

  with intitial state $i = i_X \otimes \lambda = i_Y \otimes \lambda$, $\delta(a \otimes w, z) = a \otimes wz$ for all $a \in Q_A \sqcup Q_B, w \in (S(\mathcal{A}) * S(\mathcal{B})) \cup \{\lambda\}$, $z \in X \sqcup Y$.
- For partial automata, just put the automata next to each other, identifying their initial states, use disjoint union of input alphabets.
- For nondeterministic partial automata, the states are as for partial transformation semigroups, initial states identified, and input alphabet is joint union of the input alphabets.

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(Slides on Details)

Coproduct of Transformation Semigroups

Let (X, S) and (Y, T) be in TS with $X, Y \neq \emptyset$. Then, the coproduct state Q is given by,

$$((X \sqcup Y) \otimes (S * T)^{\lambda}, S * T) := ((X \sqcup Y) \times (S * T)^{\lambda})/ \equiv, \qquad (3)$$

where $\equiv$ is the symmetric reflexive, transitive closure of $\sim$, where $\sim$ is defined by,

$$\begin{aligned} (a, sw) &\sim (a \cdot s, w), \quad \text{if } a \in X, s \in S, \\ (a, tw) &\sim (a \cdot t, w), \quad \text{if } a \in Y, t \in T. \end{aligned} \qquad (4)$$

- Write $a \otimes w$ for the equivalence class of (a, w).
- Each element of Q can be written in a canonical form $[a, v]$ where $v = \lambda$ or a shortest member of $S * T$ in canonical form, and a is either

$$a \in X \text{ and } v \text{ does not start with a member of } S, \text{ or,}$$
$$a \in Y \text{ and } v \text{ does not start with a member of } T.$$

Coproduct of Transformation Semigroups

- Then $u \in S * T$ acts on Q as determined by

$$(a \otimes w) \cdot u = a \otimes wu. \tag{5}$$

- The action is well-defined. Indeed, if $(x, sw) \sim (x \cdot s, w)$, then, $(x, sw) \cdot \ u = (x, swu) \sim (x \cdot s, wu) = (x \cdot s, w) \cdot u$.
- $S * T$ acts faithfully on Q. To show this, we exhaustively consider different cases where $u \neq u'$ can happen and find a state in Q where they disagree:

1. If $u = sw, w = \lambda$ or starts with t and $u' = s'w'$ where $w' = \lambda$ or starts with t' and $s \neq s'$,
then since (X, S) is faithful, $\exists x \in X, x \cdot s \neq x \cdot s'$. Thus,

$$[x, \lambda] \cdot u = [x \cdot s, w] \neq [x \cdot s', w'] = [x, \lambda] \cdot u'.$$

Coproduct of Transformation Semigroups

2. If $u = sw$ and $u' = sw'$. Then, $u \neq u' \Rightarrow w \neq w'$. Thus,

$$[x, \lambda] \cdot u = [x \cdot s, w] \neq [x \cdot s, w'] = [x, \lambda] \cdot u'.$$

3. If $u = tw$ and $u' = t'w'$, then it is similar to cases 1 and 2.
4. If $u = sw$ and $u' = tw'$ and $\exists x \in X, x \cdot s \neq x$, then

$$[x, \lambda] \cdot u = [x \cdot s, w] \neq [x, tw'] = [x, \lambda] \cdot u'.$$

5. If $u = sw$ and $u' = tw'$ and $\forall x \in X, x \cdot s = x$ but $w \neq tw'$, then

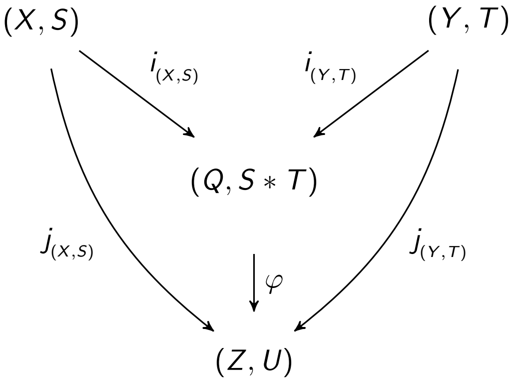
$$[x, \lambda] \cdot u = [x \cdot s, w] = [x, w] \neq [x, tw'] = [x, \lambda] \cdot u'.$$

6. If $u = stw'$ and $u' = tw'$ and $\forall x \in X, x \cdot s = x$, then

$$[y, \lambda] \cdot u = [y, stw'] \neq [y \cdot t, w'] = [y, \lambda] \cdot u'.$$

This establishes faithfulness for all non-trivial cases ($X \neq \emptyset$ and $Y \neq \emptyset$).

Then the coproduct is given by the natural inclusions (X, S) and (Y, T) in $(Q, S * T)$:



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Coproduct of Transformation Semigroups

- $i_{(X,S)}$ and $i_{(Y,T)}$ are defined by $x \mapsto [x, \lambda]$ and $y \mapsto [y, \lambda]$, respectively, on states, and by $s \mapsto s \in S * T$ and $t \mapsto t \in S * T$, respectively, on semigroup elements $s \in S$, $t \in T$.
- $i_{(X,S)}$ and $i_{(Y,T)}$ are injective. Indeed $[x_1, \lambda] = [x_2, \lambda]$ implies $x_1 = x_2$ since both are in canonical form.
- Considering the semigroup component only first, since $S * T$ is the coproduct of semigroups S and T, i.e. their free product, we take $\varphi^{Operator} \colon S * T \to U$ to be the unique semigroup homomorphism making the semigroup part of the diagram commute.
- The state morphism $\varphi^{State} \colon Q \to Z$ is defined by,

$$[a, w] \mapsto \begin{cases} j_{(X,S)}(a), & \text{if } a \in X, w = \lambda, \\ j_{(Y,T)}(a), & \text{if } a \in Y, w = \lambda, \\ j_{(X,S)}(a)\varphi^{Operator}(w), & \text{if } a \in X, w \neq \lambda, \\ j_{(Y,T)}(a)\varphi^{Operator}(w), & \text{if } a \in Y, w \neq \lambda. \end{cases}$$

Well-defined: If $(a, w) \sim (a', w')$ then φ^{State} maps them to same $z \in Z$.

Coproduct of Transformation Semigroups

- φ is a morphism.

$$\begin{aligned}\varphi([a,w]\cdot u) &= \varphi([a,wu]) \\ &= j(a)\varphi(wu) \\ &= j(a)\varphi(w)\varphi(u) \\ &= \varphi([a,w])\cdot\varphi(u).\end{aligned}$$

- The diagram commutes since the semigroup part commutes and $\forall x \in X, x \mapsto [x,\lambda] \mapsto j_{(X,S)}(x)$ and $\forall y \in Y, y \mapsto [y,\lambda] \mapsto j_{(Y,T)}(y)$,
- φ is unique. Indeed, if there is another morphism φ_2 that commutes the diagram, then,

$$\begin{aligned}\varphi_2([a,w]) &= \varphi_2([a,\lambda]\cdot w) \\ &= \varphi_2([a,\lambda])\cdot\varphi_2(w) \\ &= \varphi_2(i(a))\cdot\varphi_2(w) \\ &= j(a)\cdot\varphi(w) = \varphi([a,w]).\end{aligned}$$

- For permutation groups and transformation monoids, the arguments are the same except we interpret λ to denote the identity of M, N and $M * N$, or G, H, $G * H$, respectively.

Theorem

Let (X, M) and (Y, N) be in the category of transformations monoids TM. Then their coproduct is $(X \sqcup Y) \otimes M * N, M * N) := ((X \sqcup Y) \times (M * N))/ \equiv, M * N)$, where $M * N$ is the free product of monoids and $\equiv$ is the symmetric, reflexive, transitive closure of $\sim$ defined by,

$$\begin{aligned} (a, sw) &\sim (a \cdot s, w), \quad \text{if } a \in X, s \in M, \\ (a, tw) &\sim (a \cdot t, w), \quad \text{if } a \in Y, t \in N. \end{aligned} \tag{7}$$

Theorem

In the category of permutation groups PermGrp, given permutation groups (X, G) and (Y, H), their coproduct is $((X \sqcup Y) \otimes G * H, G * H) := ((X \sqcup Y) \times (G * H))/ \equiv, G * H)$, where $G * H$ is the free product of groups and $\equiv$ is defined as above.

Coproduct of Pointed Transformation Semigroups

Let (X, S) and (Y, T) be pointed transformation semigroups with $|X|, |Y| > 1$. Then, the coproduct is given by the natural inclusions (X, S) and (Y, T) in $(Q, S * T)$ where the coproduct state set Q is given by,

$$(X \sqcup Y) \otimes (S * T)^{\lambda} := ((X \sqcup Y) \times (S * T)^{\lambda}) / \equiv \qquad (8)$$

where $\equiv$ is the transitive closure of $\sim$, where $\sim$ is defined by,

$$\begin{aligned} (x_0, w) &\sim (y_0, w), && \text{if } x_0 = *_X, y_0 = *_Y, \\ (a, sw) &\sim (a \cdot s, w), && \text{if } a \in X, s \in S, \\ (a, tw) &\sim (a \cdot t, w), && \text{if } a \in Y, t \in T. \end{aligned} \qquad (9)$$

The base point of Q is the equivalence class $x_0 \otimes \lambda = y_0 \otimes \lambda$. Then, $u \in S * T$ acts on Q as determined by

$$(a \otimes w) \cdot u = a \otimes wu. \qquad (10)$$

Coproduct of Pointed Transformation Semigroups

- Then $u \in S * T$ acts on Q as determined by

$$(a \otimes w) \cdot u = a \otimes wu. \tag{11}$$

- The action is well-defined. Indeed,

$$\begin{aligned}
(x_0, w) \sim (y_0, w) &\Rightarrow (x_0, w) \cdot u = (x_0, wu) \sim (y_0, wu) = (y_0, w) \cdot u, \\
(x, sw) \sim (x \cdot s, w) &\Rightarrow (x, sw) \cdot u = (x, swu) \sim (x \cdot s, wu) = (x \cdot s, w) \cdot u, \\
(y, tw) \sim (y, \cdot t, w) &\Rightarrow (y, tw) \cdot u = (y, twu) \sim (y \cdot t, wu) = (y, \cdot t, w) \cdot u.
\end{aligned}$$

- If factors are reachable, each $a \otimes w \in Q$ is reachable from basepoint $x_0 \otimes \lambda = y_0 \otimes \lambda$: If $a \in X$, then $a \in X$ is reachable from basepoint $x_0 \in X$ by some $s \in S$, so

$$(x_0 \otimes \lambda) \cdot sw = x_0 \otimes sw = (x_0 \cdot s) \otimes w = a \otimes w.$$

Similarly, if $a \in Y$, which reachable from basepoint $y_0 \in Y$ by some $t \in T$.

Coproduct of Pointed Transformation Semigroups

- Each element of Q can be written as in a canonical form $a \otimes w_1 w_2 \cdots w_k$ where $w_1 w_2 \cdots w_k = \lambda$ or a shortest member of $S * T$ in canonical form, and a is either

$$a \in X \setminus \{x_0\} \text{ and } w_1 \in T, \text{ or,}$$
$$a \in Y \setminus \{y_0\} \text{ and } w_1 \in S.$$

- To show this is true, first we define a reduction system by the following rewriting rules:

$$(x, sw) \mapsto (x \cdot s, w),$$
$$(yt, w) \mapsto (y \cdot t, w),$$
$$(x_0, tw) \mapsto (y_0, tw),$$
$$(y_0, sw) \mapsto (x_0, sw),$$

where sw and tw are in the $S * T$ canonical form.

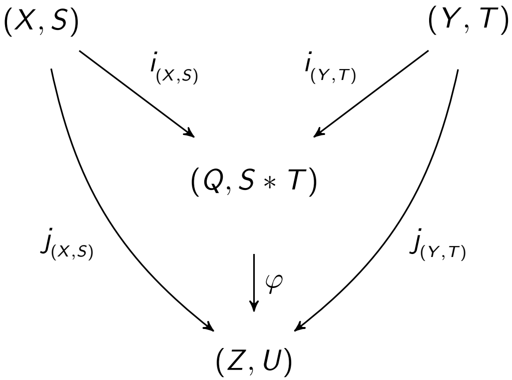
Coproduct of Pointed Transformation Semigroups

- According to the rewriting rules and the fact that sw and tw are in the canonical form of $S * T$, then each (a, w) can be reduced to a unique normal form (a', w') where non of the rewriting rules can be applied anymore. We denote the normal form of (a, w) by $red(a, w)$
- Then it can be seen that $a \otimes u = a' \otimes u'$ implies $red(a, u) = red(a', u')$.
- Suppose $a \otimes u = a' \otimes u'$, then there exists $a_i \in X \sqcup Y$ and $u_i \in S * T$ such that $(a, u) \sim (a_1, u_1) \sim \cdots \sim (a_k, u_k) \sim (a', u')$.
- It is sufficient to show that the neighbouring members $(a_i, u_i) \sim (a_{i+1}, u_{i+1})$ in an equivalence chain are reduced to the same canonical form.
- So it follows that equivalent (a,u)'s have the same canonical form. (And conversely same canonical form implies equivalence.)
- we identify (x_0, λ) and (y_0, λ) as the new base-point $*$.

Coproduct of Pointed Transformation Semigroups

1. Case one: suppose that $(a_i, u_i) = (x_0, w)$ and $(a_{i+1}, u_{i+1}) = (y_0, w)$.
   - If $w = \lambda$ then $red(x_0, \lambda) = red(y_0, \lambda) = *$.
   - Otherwise, if $w = tw'$ then $(x_0, w) = (x_0, tw') \mapsto (y_0, tw') = (y_0, w)$ i.e. $red(p, w) = red(y_0, w)$.
   - If $w = sw'$, then $red(y_0, w) = red(y_0, sw') = red(x_0, sw') = red(x_0, w)$.
2. Case two: suppose $(x, sw) \sim (x \cdot s, w)$ where $x \in X$.
   - If $w = \lambda$, then $(x, sw) = (x, s) \mapsto (x \cdot s, \lambda)$ i.e., $red(x, s) = red(x \cdot s, \lambda)$.
   - If w is nonempty and the canonical form of it in $S * T$ denoted by $Can(w) = tw'$, then $(x, sw) = (x, stw') \mapsto (x \cdot s, tw') = (x \cdot s, w)$, i.e $red(x, sw) = red(x \cdot s, w)$.
   - Otherwise, if $Can(w) = s'w''$, then $(x, sw) = (x, ss'w'') \mapsto (x \cdot ss', w'')$. On the other hand, $(x \cdot s, w) = (x \cdot s, s'w'') \mapsto (x \cdot ss', w'')$ therefore, $red(x \cdot s, w) = (x, sw)$.
3. Case three: suppose $(y, tw) \sim (y \cdot t, w)$ where $y \in Y$, which is similar to case two.

Then the coproduct is given by the natural inclusions (X, S) and (Y, T) in $(Q, S * T)$:



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Coproduct of Pointed Transformation Semigroups

- $i_{(X,S)}$ and $i_{(Y,T)}$ are defined by $x \mapsto [x, \lambda]$ and $y \mapsto [y, \lambda]$, respectively, on states, and by $s \mapsto s \in S * T$ and $t \mapsto t \in S * T$, respectively, on semigroup elements $s \in S$, $t \in T$.
- The state morphism $\varphi^{State} \colon Q \to Z$ is defined by,

$$[a, w] \mapsto \begin{cases} *_Z \cdot \varphi^{Operator}(w) & \text{if } a \in \{x_0, y_0\}, \\ j_{(X,S)}(a) \cdot \varphi^{Operator}(w), & \text{if } a \in X, \\ j_{(Y,T)}(a) \cdot \varphi^{Operator}(w), & \text{if } a \in Y. \end{cases}$$

It is well-defined since it is constant on equivalence classes. Proof: φ^{State} maps $\sim$ related pairs to the same point.

Coproduct of Pointed Transformation Monoids

In the category of pointed transformations monoids TM_* let (X, M) and (Y, N) be two pointed transformation monoids. Then their coproduct is $((X \times (M * N)) \sqcup (Y \times (M * N)))/\equiv, M * N)$, where $M*N$ is the free product of monoids and $\equiv$ is the symmetric, reflexive, transitive closure of $\sim$ defined by,

$$\begin{aligned} (x_0, w) &\sim (y_0, w), && \text{if } x_0 = *_X, y_0 = *_Y, \\ (a, sw) &\sim (a \cdot s, w), && \text{if } a \in X, s \in S, \\ (a, tw) &\sim (a \cdot t, w), && \text{if } a \in Y, t \in T. \end{aligned} \tag{13}$$

Coproduct of Pointed Permutation Groups

In the category of pointed permutation groups $PermGrp_*$, given pointed permutation groups (X, G) and (Y, H) their coproduct is $((X \times (G * H)) \sqcup (Y \times (G * H)))/ \equiv, G * H)$, where $G * H$ is the free product of groups and $\equiv$ is defined as above.

Theorem

Let (X, S) and (Y, T) be pointed transformation semigroups $(|X|, |Y| > 1)$. Then, their coproduct is given by,

$$((X \sqcup Y) \times (S * T)^{\lambda} / \equiv, S * T), \tag{14}$$

where $\equiv$ is the transitive closure of $\sim$, where $\sim$ is defined by,

$$\begin{aligned} (x_0, w) &\sim (y_0, w), && \text{if } x_0 \text{ base point of } X, y_0 \text{ base point of } Y, \\ (a, sw) &\sim (a \cdot s, w), && \text{if } a \in X, s \in S, \\ (a, tw) &\sim (a \cdot t, w), && \text{if } a \in Y, t \in T. \end{aligned} \tag{15}$$

- Caveat: If $|X|, |Y| \leq 1$, then the state set has at most one element, so $S * T$ is infinite and can't be faithful if both S and T are non-empty!

Theorem

In the category of pointed transformations monoids TM_* let (X, M) and (Y, N) be pointed transformation monoids ($X, Y \neq \emptyset$). Then their coproduct is $((X \sqcup Y) \times (M * N))/\equiv, M * N)$, where $M * N$ is the free product of monoids and $\equiv$ is the transitive closure of $\sim$ defined by,

$$
\begin{aligned}
(x_0, w) &\sim (y_0, w), && \text{if } x_0 \text{ base pt of } X, y_0 \text{ base pt of } Y, \\
(a, sw) &\sim (a \cdot s, w), && \text{if } a \in X, s \in S, \\
(a, tw) &\sim (a \cdot t, w), && \text{if } a \in Y, t \in T.
\end{aligned}
\tag{16}
$$

It is reachable if both its factors are.

Theorem

In the category of pointed permutation groups $PermGrp_*$, given pointed permutation groups (X, G) and (Y, H) coproduct is $((X \sqcup Y) \times (G * H))/\equiv, G * H)$, where $G * H$ is the free product of groups and $\equiv$ is defined as above. It is reachable if both (X, G) and (Y, H) are.

Category of Partial Transformation Semigroups (PTS)

- Objects: A **partial transformation semigroup** (X, S) consists of a set X and S is a semigroup consisting of partial transformations, rather than fully defined functions on X.
- A **morphism of partial transformation semigroups** $\varphi\colon (X, S) \to (Y, T)$ consists of two fully-defined functions $\varphi^{State}\colon X \to Y$ and $\varphi^{Operator}\colon S \setminus \{\emptyset\} \to T \setminus \{\emptyset\}$, satisfying two conditions: A relaxed state mapping condition

$$\varphi^{State}(x \cdot s) \subseteq \varphi^{State}(x) \cdot \varphi^{Operator}(s) \text{ , } \forall x \in X, s \in S \qquad (17)$$

and a relaxed homomorphism condition

$$\varphi^{Operator}(ss') \subseteq \varphi^{Operator}(s)\varphi^{Operator}(s') \text{ , } \forall s, s' \in S. \qquad (18)$$

- In fact, morphisms are defined just as for transformation semigroups, except $\varphi(x) \cdot \varphi(s) = \varphi(x \cdot s)$ is only required to hold when $x \cdot s$ is defined.

$\varphi^{State}(x \cdot s) \subseteq \varphi^{State}(x) \cdot \varphi^{Operator}(s)$, $\forall x \in X, s \in S$

- Following Eilenberg, we write $x \cdot s = \emptyset$ when $x \cdot s$ is not defined, and agree to write $\varphi^{State}(\emptyset) = \emptyset$.
- Also following Eilenberg, we identify an element x with the singleton set $\{x\}$. Note that ss' need not be defined anywhere even if s and s' are partial transformations.
- We agree that the completely undefined transformation need not be mapped by the semigroup component of a morphism φ: Writing $\emptyset$ for this nowhere defined transformation, we agree to write $\varphi^{Operator}(\emptyset) = \emptyset$.
- Note $\varphi^{Operator}$ is not assumed to be a semigroup homomorphism, but it will be a homomorphism whenever (X, S) is a (fully-defined) transformation semigroup, and not strictly partial.
- More generally, $\varphi^{Operator}$ will be a semigroup homomorphism as long as S does not contain the empty transformation $\emptyset$.

- The composition $\psi \circ \varphi$ of morphisms

$$(X, S) \overset{(\varphi^{State}, \varphi^{Operator})}{\rightarrow} (Y, T) \overset{(\psi^{State}, \psi^{Opeator})}{\rightarrow} (Z, U) \tag{19}$$

is their componentwise composition as functions

$$(X, S) \overset{(\psi^{State} \circ \varphi^{State}, \psi^{Operator} \circ \varphi^{Operator})}{\longrightarrow} (Z, U). \tag{20}$$

Theorem

With the mentioned definitions of objects and morphisms, partial transformation semigroups comprise a category PTS.

Corollary

The nonempty transformation semigroups TS comprise a full subcategory of PTS.

Theorem (Coproducts of Partial Transformation Semigroups)

Let (X_i, S_i) be partial transformation semigroups for each i in some index set I. (NB: in particular, some or all of the (X_i, S_i) may be fully defined!). Then

$$\coprod (X_i, S_i) = (\bigsqcup X_i, \bigvee S_i) \tag{21}$$

is their coproduct, where $\bigsqcup X_i$ is the disjoint union of sets and $\bigvee S_i$ is the semigroup generated by partial transformations s on $\bigsqcup X_i$ such that s agrees with some $s_i \in S_i$ for some $i \in I$ on X_i, and is undefined on the complement of X_i. That is, the action of elements is

$$x \cdot s = \begin{cases} x \cdot s & \text{if } x \in X_i,\ s \in S_i \\ \text{undefined} & \text{otherwise} \end{cases} \tag{22}$$

and multiplication of semigroup elements is

$$ss' = \begin{cases} ss' \in S_i & s, s \ \in S_i, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{23}$$

For partial automata, coproduct is easy:

The coproduct partial automata (state-transition systems without initial state) in the category of partial automaton is obtained by putting them side by side and taking the disjoint union of their input alphabets as the new input alphabet.

The coproduct of partial automata with state state in the category of paritial automata with start state is obtained by putting them side by side but identifying their start states and taking the disjoint union of their input alphabets as the new input alphabet.

(Remark: The same idea works for nondeterministic partial. The categories of partial nondeterministic automata and labelled directed multigraphs (of a certain kind) are isomorphic categories, and its easy to see what coproducts are in the latter [Karimi & Nehaniv 2014].)

Consequences for Computer Science / Deterministic Automata

For deterministic complete reachable automata (i.e., not partial) coproduct:

The coproduct of deterministic reachable automata $\mathcal{A} = (X, A, \delta : X \times A \to X)$ and $\mathcal{B} = (Y, B, \delta' : Y \times B \to Y)$ has states

$$(X \times (S(\mathcal{A}) * S(\mathcal{B}))^\lambda \sqcup Y \times (S(\mathcal{A}) * S(\mathcal{B}))^\lambda)/\equiv$$

with input alphabet $A \sqcup B$, where $S(\mathcal{A})$ denotes the transition semigroup of $\mathcal{A}$ as in the coproduct for transformation semgroups (using either equivalence relation with basepoint idenitcation if we work in automata with initial state.

Transitions are exactly as in transformation semigroup coproduct action: apply just letters (generators), and reduce to canonical form of states!

This takes us out of the finite realm, since in the deterministic world, the coproduct must account for all possibilities of how transitions can occur and inputs from the two automata are shuffled!

Coproduct for PTS

Proof for Partial Transformation Semigroups.

For all $i \in I$, one has inclusion partial transformation semigroup morphisms $\iota_i \colon (X_i, S_i) \to (\bigsqcup X_i, \bigvee S_i)$ such that if $j_i \colon (X_i, S_i) \to (Q, T)$ are morphisms for some fixed partial transformation semigroup (Q, T), then there is a unique morphism $\varphi \colon (\bigsqcup X_i, \bigvee S_i) \to (Q, T)$ given by defining for $x \in \bigsqcup_{i \in I} X_i$, where $x = x_i$, that $\varphi(x_i) = j_i(x_i)$ and for $s \in \bigvee S_i \setminus \{\emptyset\}$, with s agreeing on X_i with $s_i \in S_i$, that $\varphi(s) = j_i(s_i)$. Any other member of $\bigvee S_i$ must be the empty transformation (which is not in the domain of $\varphi^{Operator}$). Clearly, φ is a morphism. Then we have $j_i = \varphi \circ \iota_i$ holds for all i. Moreover, φ is unique since the equation says $\varphi(x_i) = \varphi(\iota_i(x_i)) = j_i(x_i)$, and, similarly for the nonempty semigroup elements in $\bigvee S_i$, uniquely determining φ.

Proof for PTS_* is similar.