Homogeneity of the pseudoarc and permutation groups

Sławomir Solecki

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Most of this work is joint with Todor Tsankov.
Outline of Topics

1. The pseudoarc and projective Fraïssé limits
2. Projective “types”
3. Homogeneity for points with minimal types
4. The transfer theorem
5. Questions (and comments on Menger compacta)
Two objects:

1. The pseudoarc $P = \text{a certain compact, connected, second countable space}$
2. The pre-pseudoarc $P = \text{the Cantor set and a certain compact equivalence relation}$ $R$ on it with $P/R = P$ and with a certain relationship to a family of finite structures.
Two objects:

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**Aim:** Develop a model theoretic/combinatorial point of view (projective Fraïssé limit) that can be used to:

1. find canonical "models" for interesting topological spaces, for example, the pseudoarc, Menger compacta, etc;
2. find a unified approach to topological homogeneity results and put these results on firm footing;
3. resolve topological questions about homeomorphism groups.
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The pseudoarc
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This \( P \) is called the \textbf{pseudoarc}. 
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This \( P \) is called the \textbf{pseudoarc}. 

$\mathcal{K}([0, 1]^\omega) =$ compact subsets of $[0, 1]^\omega$ with the Vietoris topology

$\mathcal{K}([0, 1]^\omega)$ is compact

$\mathcal{C} =$ all connected sets in $\mathcal{K}([0, 1]^\omega)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} : P' \text{ homeomorphic to } P\}$$

is a dense $G_\delta$ in $\mathcal{C}$.

This $P$ is called the **pseudoarc**.
**Continuum** = compact and connected
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The pseudoarc is a hereditarily indecomposable continuum
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The pseudoarc is a **hereditarily indecomposable** continuum, that is, if \( C_1, C_2 \subseteq P \) are continua with \( C_1 \cap C_2 \neq \emptyset \), then \( C_1 \subseteq C_2 \) or \( C_2 \subseteq C_1 \).

It was discovered by Knaster in 1922.
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Projective Fraïssé limits
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**Slawomir Solecki (University of Illinois)**

Homogeneity of the pseudoarc

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**Irwin–S:** If \( \mathcal{F} \) is a projective Fraïssé family, then there exists a unique projective limit

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that is projectively universal and projectively homogeneous.
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that is projectively universal and projectively homogeneous.

\( \mathcal{F} \) also has **Projective Extension Property**.
Connection with the pseudoarc
\[ \mathcal{P} = \text{all finite, linear, reflexive graphs with graph relation } R \]
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**Irwin–S.**: \( \mathcal{P} \) is a projective Fraïssé family.
Let $\mathbb{P} = \lim_{\leftarrow} \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^\mathbb{P}$.
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**Irwin–S.:** $\mathbb{P}/R^\mathbb{P}$ is the pseudoarc.
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**Irwin–S.:** $\mathbb{P}/R^\mathbb{P}$ is the pseudoarc.

There is a natural continuous homomorphism

$$\text{Aut}(\mathbb{P}) \to \text{Homeo}(\mathbb{P}/R^\mathbb{P})$$

with dense range.
**Bing:** The pseudoarc is homogeneous
**Bing:** The pseudoarc is homogeneous, that is, for any $x, y \in P$, there exists $f \in \text{Homeo}(P)$ such that $f(x) = y$. 
Projective “types”
$M$ is a **structure** if
$M$ is a **structure** if

- $M$ is a compact, 0-dimensional, second countable space,
- $R^M$ is a closed binary relation on $M$,
- each continuous function $M \to X$, with $X$ finite, factors through an epimorphism $M \to A$ for some $A \in \mathcal{P}$. 

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**Homogeneity of the pseudoarc**

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Define

\[
t_{(M,p)}(f) = \{ f(K) : p \in K \subseteq M, \text{ } K \text{ a structure} \}.
\]
Let $f : M \to X$ be continuous, with $X$ finite. So $f$ is a “projective tuple.”

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Define

$$t_{(M,p)}(f) = \{ f(K) : p \in K \subseteq M, K \text{ a structure} \}.$$ 

$t_{(M,p)}(f)$ is a family of subsets of the finite set $X$. 
$X$ finite set, $x \in X$

c is a **chain at** $x$ if $c$ is a maximal family of subsets of $X$ linearly ordered by inclusion and with $\{x\} \in c$. 
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**minimal** if \( t(M,p)(f) \) is a chain at \( f(p) \);
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t$_{(M,p)}(f)$ is called

minimal if $t_{(M,p)}(f)$ is a chain at $f(p)$;

almost minimal if $t_{(M,p)}(f) = c_1 \cup c_2$, for some chains $c_1$ and $c_2$ at $f(p)$. 
Homogeneity for points with minimal types
Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ continuous, $X$ finite.
Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ continuous, $X$ finite. Then $t_{(\mathbb{P},p)}(f)$ is almost minimal.
\( p \in \mathbb{P} \) has minimal types if \( t(\mathbb{P}, p)(f) \) is minimal for each continuous \( f : \mathbb{P} \to X \) with \( X \) finite.
$p \in \mathbb{P}$ has minimal types if $t_{(\mathbb{P},p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with $X$ finite.

**Theorem (S.–Tsankov, 2015)**

Let $p, q \in \mathbb{P}$. Assume that $R^\mathbb{P}(p) = \{p\}$ and $R^\mathbb{P}(q) = \{q\}$ and $p$ and $q$ have minimal types.
Homogeneity for points with minimal types

\( p \in P \) has minimal types if \( t(P,p)(f) \) is minimal for each continuous \( f : P \to X \) with \( X \) finite.

Theorem (S.–Tsankov, 2015)

Let \( p, q \in P \). Assume that \( R^P(p) = \{ p \} \) and \( R^P(q) = \{ q \} \) and \( p \) and \( q \) have minimal types. Then there exists \( f \in \text{Aut}(P) \) such that \( f(p) = q \).
Proof uses the following strong Projective Extension Property.
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**Lemma**

Given $p \in P$ with minimal types and $R_P(p) = \{p\}$, $A, B \in P$, $a \in A$, $b \in B$; $f : P \to A$, $g : B \to A$ epimorphisms with $f(p) = a$, $g(b) = a$.

Conclusion: there exists an epimorphism $h : P \to B$ such that $h(p) = b$. 

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**Conclusion:** there exists an epimorphism $h: P \to B$ such that $h(p) = b$. 
The transfer theorem
**Aim**: transfer partial homogeneity from $P$ to full homogeneity of $P/R^P$. 
Theorem (S.–Tsankov, 2015)

For each $y \in \mathbb{P}/R^p$, there exists $x \in \mathbb{P}/R^p$ and a homeomorphism $\phi: \mathbb{P}/R^p \to \mathbb{P}/R^p$ such that

(i) $x = p/R^p$ for some $p \in \mathbb{P}$ having minimal types and with $R^p(p) = \{p\}$;

(ii) $\phi(x) = y$. 

An important ingredient of the proof is a notion of weak commutation.
The transfer theorem

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Weak commutation of diagrams
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Given $A \in \mathcal{P}$, define the pre-dual $\hat{A} \in \mathcal{P}$ of $A$ with a bijection

$A \ni a \rightarrow \hat{a}$ an edge in $\hat{A}$.
Weak commutation of diagrams

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Weak commutation for epimorphisms

$f : \mathcal{P} \rightarrow A$, $g : \mathcal{P} \rightarrow B$ and $h : \hat{A} \rightarrow \hat{B}$:
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**Weak commutation** for epimorphisms

$f : \mathbb{P} \rightarrow A$, $g : \mathbb{P} \rightarrow B$ and $h : \hat{A} \rightarrow \hat{B}$:

$$h(\hat{f}(p)) \subseteq \hat{g}(p) \text{ for each } p \in \mathbb{P}.$$
From partial homogeneity of $\mathbb{P}$ and the above transfer theorem we get the following corollary.
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**Corollary (Bing)**

*The pseudoarc is homogeneous.*
Questions
(and comments on Menger compacta)
Define relations $S$ and $T$ on $\mathbb{P}$ as follows

$S(x, y)$ if and only if $x, y \in K$ for some substructure $K \subset \mathbb{P}$;

$T(x, y, z)$ if and only if $x, y \in K$ and $z \not\in K$ for some substructure $K \subset \mathbb{P}$ with $R(p) = \{p\}$.

Theorem (S.–Tsankov, 2015)

Let $F_1, F_2 \subset \mathbb{P}$ be finite sets whose points have minimal types and whose points $p$ are such that $R(p) = \{p\}$.

Let $f : F_1 \to F_2$ be a bijection preserving $S$ and $T$.

Then $f$ extends to an element of $\text{Aut}(\mathbb{P})$. 

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Homogeneity of the pseudoarc

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Is there a maximal homogeneity of $\mathbb{P}$?
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Is every element of $\text{Homeo}(\mathbb{P}/R)$ conjugate to an element of $\text{Aut}(\mathbb{P})$?
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Is every element of $\text{Homeo}(\mathbb{P}/R)$ conjugate to an element of $\text{Aut}(\mathbb{P})$?

Can orbits of the natural action of $\text{Aut}(\mathbb{P})$ on $\mathbb{P}$ be characterized by types or sequences of types?

Can $t_{(M,p)}(f)$ be viewed as actual types?
Menger compacta
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\[ \mathbb{N} \cup \{\infty\} \ni n \rightarrow \mu_n \text{ a compact, second countable space} \]
Menger compacta

\( \mathbb{N} \cup \{\infty\} \ni n \rightarrow \mu_n \) a compact, second countable space

\( \mu_0 = \) Cantor set

\( \mu_\infty = \) Hilbert cube
Menger compacta

\[ \mathbb{N} \cup \{\infty\} \ni n \rightarrow \mu_n \text{ a compact, second countable space} \]

\[ \mu_0 = \text{Cantor set} \]

\[ \mu_\infty = \text{Hilbert cube} \]

\[ \mu_n \text{ is } n\text{-dimensional, universal for } n\text{-dimensional second countable spaces, highly homogeneous} \]
Questions (and comments on Menger compacta)

Joint with Panagiotopoulos
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There exists a projective Fraïssé family \( \mathcal{M}_1 \) such that if \( \mathbb{M}_1 = \lim \leftarrow \mathcal{M}_1 \) is taken with the binary relation \( R_1 \), then
Joint with Panagiotopoulos

There exists a projective Fraïssé family $\mathcal{M}_1$ such that if $\mathcal{M}_1 = \lim \leftarrow \mathcal{M}_1$ is taken with the binary relation $R_1$, then

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$\mathcal{M}_1 / R_1 = \mu_1$

$\mathcal{M}_1$ is highly homogeneous.
In fact, given \( n \in \mathbb{N} \), there exists a projective Fraïssé family \( \mathcal{M}_n \) analogous to \( \mathcal{M}_1 \).

Is it the case that \( \mathbb{M}_n / R_n = \mu_n \)?
In fact, given \( n \in \mathbb{N} \), there exists a projective Fraïssé family \( \mathcal{M}_n \) analogous to \( \mathcal{M}_1 \).

Is it the case that \( \mathbb{M}_n / R_n = \mu_n \)?

For an answer, we need appropriate homology groups for \( \mathbb{M}_n \).