Mathematical Modeling of Complex Microlasers

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A qualitative picture of lasing

resonances: poles in linear(ized) Green’s function

threshold lasing mode

adding gain (increasing pump) pushes up pole(s) to real axis

passive pole

Gain

stable steady-state amplitude > 0

Loss

Mode intensity

pump > threshold: nonlinear gain stabilizes amplitude
Maxwell–Bloch equations: 
simplest accurate spatio-temporal lasing model

- fully time-dependent, multiple unknown fields, nonlinear (Haken, Lamb, 1963): Maxwell + Lorentzian polarization resonance + 2-level atom population inversion

\[ - \nabla \times \nabla \times (E^+) - \varepsilon c \ddot{E}^+ = \frac{1}{\varepsilon_0} \dddot{P}^+ \]

Inversion drives polarization

\[ \dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ + \frac{1}{i\hbar}E^+D \]

Polarization induces inversion

Population inversion:

\[ \dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar}[E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^*] \]
brute-force Maxwell–Bloch FDTD (finite-difference time-domain) simulations very expensive — $E$ and $D$ change on very different timescales — but do-able (barely)

[ Bermel et. al. (PRB 2006) ]
If a steady-state lasing solution exists, we’d rather solve for it directly without time-evolving

[Tureci, Stone, 2006]

\[ \dot{D} = \gamma_{\parallel}(D_0 - D) - \frac{2}{i\hbar}[E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^*] \]

key assumption:  
- “rotating-wave approximation” 
  fast oscillations average out to zero 
  \( \gamma_\perp, \Delta \omega \gg \gamma_{\parallel} \) 
  valid for < 100µm microlasers 
  … all oscillations are fast compared to \( \gamma_{\parallel} \)

… leads to: \( \dot{D} \approx 0 \)

stationary-inversion approximation (SIE)
before:
\[-\nabla \times \nabla \times (E^+) - \varepsilon_c \dot{E}^+ = \frac{1}{\varepsilon_0} \dot{P}^+
\]
\[\dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ + \frac{g^2}{i\hbar} E^+ D\]
\[\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar} [E^+ \cdot (P^+)^* - P^+ \cdot (E^+)^*]\]

after:
Steady-State Ab-Initio Lasing Theory,
“SALT”
[Tureci, Stone, 2006]

\[\nabla \times \nabla \times E_m = \omega_m^2 \varepsilon_m E_m\]

Still nontrivial to solve:
equation is nonlinear in both

**eigenvalue** $\omega_m \leftarrow$ easier

**eigenvector** $E_m \leftarrow$ harder
New numerical solvers:
High-dimensional Newton from threshold modes

\[ \nabla \times \nabla \times \mathbf{E}_m = \omega_m^2 \varepsilon_m \mathbf{E}_m \]

SALT: lasing steady state
= “ordinary” EM eigenproblem

\[ \varepsilon_m = \varepsilon_c(x) + \frac{\gamma_0}{\omega_m - \omega_0 + i\gamma_0} \frac{D_0(x, d)}{1 + \sum_n a_n \frac{\gamma_0}{\omega_n - \omega_0 + i\gamma_0} E_n^2} \]

(Lorentzian gain spectrum, mode amplitudes \( a_n \))
Fully nonlinear inter-modal interactions: "gain-switched lasing modes" in 2d

[ Li Ge et al, Optics Express 24, 41–54 (2016) ]

Mode-switching in Microdisc Laser

![Graph showing mode-switching](image-url)
New analytical formulations (SALT)  
+ new numerical solvers (lasing modes)  

[ many other variations:  
including laser amplification “I-SALT”,  
lasing in diffusive gases “C-SALT”, … ]  

…  

New opportunities for *analytical* results, too.
Laser noise:

random (quantum/thermal) currents
“kick” the laser mode
⇒ Brownian phase drift = finite linewidth
Linewidth formulas: a long history

\[ \Gamma = \frac{\hbar \omega_0 \gamma_c^2}{2P} \cdot \frac{N_2}{N_2 - N_1} \cdot \left| \frac{\int_C dx |E_c|^2}{\int_C dx E_c^2} \right|^2 \cdot \left( \frac{\gamma_\perp}{\gamma_\perp + \frac{\gamma_c}{2}} \right)^2 \cdot (1 + \alpha^2) \]

- Schawlow-Townes ('58) - inverse power 1/P scaling
- Incomplete inversion ('67) - due to partial inversion
- Petermann ('79) - enhancement for lossy cavities
- Bad-cavity ('67) - reduction due to dispersion
- \( \alpha \)-factor ('82) - coupling of intensity/phase fluctuations

... all make approximations invalid for \( \mu \)-scale lasers...

chaotic cavity  photonic crystal  random laser
Starting point:

Maxwell–Bloch

\[
\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2} \ddot{\mathbf{E}} = \frac{4\pi}{c^2} \left[ \ddot{\mathbf{P}}^+ + (\ddot{\mathbf{P}}^+)^* \right]
\]

\[
\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar} \mathbf{E}D
\]

\[
\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar} \mathbf{E} \cdot \left[ (\mathbf{P}^*)^+ - \mathbf{P}^+ \right]
\]

[Arecchi & Bonifacio, 1965]
Starting point:

Langevin Maxwell–Bloch

\[
\nabla \times \nabla \times \mathbf{E} - \frac{\varepsilon_c}{c^2} \ddot{\mathbf{E}} = \frac{4\pi}{c^2} \left[ \dot{\mathbf{P}}^+ + (\mathbf{P}^+)^* \right] - \frac{4\pi}{c} \mathbf{j}
\]

\[
\dot{\mathbf{P}}^+ = -(i\omega_a + \gamma_\perp)\mathbf{P}^+ + \frac{g^2}{i\hbar} \mathbf{E} \mathbf{D}
\]

\[
\dot{D} = \gamma_\parallel (D_0 - D) - \frac{2}{i\hbar} \mathbf{E} \cdot [(\mathbf{P}^*)^+ - \mathbf{P}^+]
\]

[Arechchi & Bonifacio, 1965]

Noise correlations: fluctuation–dissipation theorem at \( T < 0 \)

\[
\langle J_i(\omega, x)J_j^*(\omega, x') \rangle = \frac{\omega}{\pi} \delta_{ij} \delta(x - x') \left[ \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2kT} \right) \right] \text{Im} \varepsilon(x)
\]

[Callen & Welton, 1957]
The **Noisy-SALT linewidth**

[ Pick et al., PRA 91, 063806 (2015) ]

Starting point: Langevin MB. (with **SALT + FDT**)

Maxwell perturbation theory

Dynamical eqs. for lasing mode amplitudes (**oscillator eqs.**)

Formulas for multimode **linewidths** & RO side peaks

ODE linearization + closed-form integration
Oscillator equations

Noise-free SALT: \[ E(x, t) = \sum_{\mu} E_{\mu}(x) a_{\mu 0} e^{-i\omega_{\mu} t} \]

SALT modes

Noisy N-SALT: \[ E(x, t) = \sum_{\mu} E_{\mu}(x) a_{\mu}(t) e^{-i\omega_{\mu} t} \]

Simple limit: Single-mode “class A” lasers
\[ \frac{da_1}{dt} = C_{11} (a_{10}^2 - |a_1|^2) a_1 + f_1 \]

instantaneous restoring force

Most general dynamical equations (class A+B lasers)
\[ \dot{a}_{\mu} = \sum_{\nu} \left[ \int dx c_{\mu \nu}(x) \gamma(x) \int_{-\infty}^{t} dt' e^{-\gamma(x)(t-t')} (a_{\nu 0}^2 - |a_{\nu}(t')|^2) \right] a_{\mu} + f_{\mu} \]

time-delayed, spatially inhomogeneous restoring force

often derived heuristically [ Lax (1967) ]
Solving the oscillator equations

\[
\dot{a}_\mu = \sum_\nu \left[ \int dx \, c_{\mu \nu}(x) \gamma(x) \int_{-\infty}^t dt' e^{-\gamma(x)(t-t')} \left( a_{\nu 0}^2 - |a_\nu(t')|^2 \right) \right] a_\mu + f_\mu
\]

Expand mode amplitudes around steady state:
\[
a_\mu = (a_{\mu 0} + \delta_\mu) \exp(i\varphi_\mu) \text{ [small noise = linearize in } \delta_\mu]\]

- **Miracle #1:** can solve analytically for \( <\varphi_\mu \varphi_\nu> \) correlation function, which gives linewidths.

- **Miracle #2:** \( \gamma(x) \) exactly cancels and gives same answer as instantaneous model! The simple “class A” model is correct for “class B!”
Single-mode linewidth formula

\[ \Gamma = \frac{\hbar \omega_0 \tilde{\gamma}_0^2}{2P} \cdot \tilde{n}_{sp} \cdot \tilde{K} \cdot \tilde{B} \cdot (1 + \tilde{\alpha}^2) \]
Brute-force validation

Brute-force simulations of Langevin–Maxwell–Bloch show excellent agreement with N-SALT linewidth formula

Only N-SALT captures all relevant physics in MB
many other new analytical & computational opportunities...
Lasing of Degenerate Modes

well-studied example: whispering gallery modes

Silica microdisk
[Armani et. al. 2003]

High-symmetry resonant cavities can have degenerate resonances, but almost the cases that have been studied above threshold are ring/disk resonators.

How do you find such modes?

Do SALT or SALT solvers need to be modified for degeneracies?

Lasing stable superposition:

\[ e^{im\phi} = \cos m\phi + i \sin m\phi \]

Intensity in He-Ne laser
[Tamm PRA 1998]
photonic-crystal (and quasicrystal) cavities have discrete rotational symmetries

Group theory: $C_{nv}$ symmetry ($= n$-fold rotation + $n$ mirror planes) can have 2-fold degenerate modes, but how do they lase?
Why not just plug degenerate geometry into SALT?

One of the challenges:

• A nonlinear solver (e.g. Newton) needs an initial guess via the threshold (linear) modes — but now there are two — but what is the “right” (stable) superposition? — are we sure a stable lasing mode exists?
Threshold perturbation theory

At a pump strength $D_T$, suppose we have two degenerate threshold modes $\psi_{1,2}$ (solving linear Maxwell eigenproblem)

... consider pump $D_0 = (1 + d) D_T$ for $0 \leq d << 1$, & solve the nonlinear $d>0$ equations to lowest order in $d$
Perturbative lasing modes

First, find the steady-state $d>0$ modes (possibly unstable)

$$E = \sqrt{d} (a_1 \psi_1 + a_2 \psi_2) + \mathcal{O}(d^{3/2})$$
$$\omega = \omega_0 + \omega_1 d + \mathcal{O}(d^2)$$

… plug into SALT, drop higher-order terms in $d$ …

Straightforward to solve for all allowed $a_{1,2}$ superpositions.
High-symmetry perturbative SALT

- Consider degenerate lasing modes coming from $C_{nv}$ symmetry.

Ring: $C_{\infty v}$  Photonic crystal: $C_{6v}$

Degenerate modes $\psi_{1,2}$ come in cos/sin-like even/odd pairs

Result: $d>0$ SALT solutions are always either standing ($\psi_1$ or $\psi_2$) or circulating ($\psi_1 \pm i \psi_2$)!
Analytical Near-Threshold Stability

[ following Burkhardt, Liertzer, Krimer, & Rotter (2015), who solve linear-stability numerically for any d ]

MB solution = steady state + perturbation

\[ E^+(x, t) = [E(x) + \delta E(x, t)]e^{-i\omega t} \]
\[ P^+(x, t) = [P(x) + \delta P(x, t)]e^{-i\omega t} \]
\[ \tilde{D}(x, t) = D(x) + \delta D(x, t) \]

linearized MB equations, dropping \( O(u^2) \)

\[
\left( C \frac{d^2}{dt^2} + B \frac{d}{dt} + A \right) u(t) = 0
\]

\( ne^{\sigma t} \) eigensolutions: \( (C\sigma^2 + B\sigma + A)v = 0 \)

stability: all eigenvalues \( \sigma \) have \( \text{Re} \ \sigma < 0 \)
Perturbative Stability Analysis

$v e^{\alpha t}$ eigensolutions:  \((C\sigma^2 + B\sigma + A)v = 0\)

expand perturbatively in \(d\):

\[v = v_0 + v_{1/2}\sqrt{d} + v_1 d + \mathcal{O}(d^{3/2})\]
\[\sigma = \sigma_0 + \sigma_{1/2}\sqrt{d} + \sigma_1 d + \mathcal{O}(d^{3/2})\]

solve order-by-order … quite tedious, but analytical!

… many terms simplify depending on symmetry group.
Perturbative Stability Results (in 1d ring example)

- Circulating modes are stable
- Standing modes are unstable

Validated perturbation theory (lines) against brute-force eigenvalues $\sigma$ (dots) for 1d ring.
Result: symmetry + integrals of threshold modes
= stability criteria for circulating/standing modes

stable degenerate solutions are almost always circulating
(from “chiral” group representations)

projection onto circulating mode:

\[ E_+ = \sum_{k=0}^{n-1} \exp \left( -\frac{2\pi imk}{n} \right) R_n^k E_i \]

Correct “initial guess” for above-threshold SALT solver.

[ Interesting point: C4v group (square) is very special,
and can sometimes have stable standing-wave modes ]
Putting it all together: $C_{6v}$ photonic-crystal resonator

Stable lasing intensity:

threshold degenerate modes

[ Omitted details: techniques to correct for numerical symmetry breaking. ]
Symmetry-breaking above threshold

“spiral” intensity pattern of circulating mode generally breaks mirror symmetry above threshold — only $C_n$ symmetry remains!

… what happens to degeneracy above threshold?

“chiral” lasing mode in dielectric square

$C_n$ does not have degeneracy

… except if we also have reciprocity


degenerate passive resonance ≠ mirror flip of lasing mode
New solvers, new formulations = many analytical (& computational) opportunities remaining

- **Lyapunov stability of multi-mode SALT:** well-established numerically & qualitatively plausible, but no rigorous analysis.
- **Exceptional-point lasing**
- **Band-edge** surface-emitting lasers  
  (*continuum* of guided/leaky resonances)
- …
Thanks!

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