Nonlinear Schrödinger equation on a periodic graph

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Summary

Introduction: periodic potentials

Periodic graphs - motivations

Linear properties of the periodic graph

Justification of the homogeneous NLS equation

Nonlinear bound states on the periodic graph

Conclusion
Introduction: periodic potentials

Let us consider again the nonlinear Schrödinger (Gross–Pitaevskii) equation

\[ iu_t = -u_{xx} + V(x)u \pm |u|^2u, \]

with a periodic potential, e.g. \( V(x) = V_0 \sin^2(x) \).

Stationary solutions \( u(x, t) = \phi(x)e^{-i\omega t} \) with \( \omega \in \mathbb{R} \) satisfy a stationary Schrödinger equation with a periodic potential

\[ \omega \phi = -\phi_{xx} + V(x)\phi \pm |\phi|^2\phi \]

Spectrum of \( L = -\partial_x^2 + V(x) \) for \( V(x) = V_0 \sin^2(x) \) and \( N = 1 \):

![Graph showing the spectrum of the operator](image)
Floquet–Bloch spectrum

The spectral problem with a bounded $2\pi$-periodic potential $V$,

$$\omega W = -\partial_x^2 W + V(x)W, \quad x \in \mathbb{R},$$

has a purely continuous spectrum, which can be found by using Bloch waves

$$W(x) = e^{i\ell x}f(\ell, x), \quad \ell, x \in \mathbb{R},$$

where $f(\ell, \cdot)$ is a $2\pi$-periodic function for every $\ell \in \mathbb{R}$. Since these functions satisfy the continuation conditions

$$f(\ell, x) = f(\ell, x + 2\pi), \quad f(\ell, x) = f(\ell + 1, x)e^{ix}, \quad \ell, x \in \mathbb{R},$$

we can restrict the definition of $f(\ell, x)$ to $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$ and $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$.

For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

$$-(\partial_x + i\ell)^2 f + V(x)f = \omega(\ell)f, \quad x \in \mathbb{T}_{2\pi}.$$

There exists a Schauder basis $\{f^{(m)}(\ell, \cdot)\}_{m \in \mathbb{N}}$ in $L^2_{\text{per}}(0, 2\pi)$ for an increasing sequence of eigenvalues $\{\omega^{(m)}(\ell)\}_{m \in \mathbb{N}}$. 
Homogenization of the NLS equation

The NLS equation with a bounded periodic potential $V$,

$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

can be reduced to a homogeneous NLS equation

$$i\partial_T A = -\frac{1}{2} \partial_\ell^2 \omega^{(m_0)}(\ell_0) \partial_X^2 A \pm \nu |A|^2 A, \quad \nu = \frac{\|f^{(m_0)}(\ell_0, \cdot)\|^4_{L^4_{\text{per}}}}{\|f^{(m_0)}(\ell_0, \cdot)\|^2_{L^2_{\text{per}}}}$$

**Theorem (Schneider–Uecker, 2006; Dohnal, 2008; Ilan–Weinstein, 2010)**

*Fix* $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, *and assume* $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ *for every* $m \neq m_0$. *Then, for every* $C_0 > 0$ *and* $T_0 > 0$, *there exist* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for all solutions* $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ *of the homogeneous NLS equation with*

$$\sup_{T \in [0, T_0]} \|A(T, \cdot)\|_{H^3} \leq C_0$$

*and for all* $\varepsilon \in (0, \varepsilon_0)$, *there are solutions* $u \in C([0, T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ *of the periodic NLS equation satisfying the bound*

$$\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| u(t, x) - \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_{gr} t)) f^{(m_0)}(\ell_0, x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C \varepsilon^{3/2}.$$
Application of the NLS equation to existence of nonlinear bound states

In the defocusing case, the nonlinear bound states bifurcate if $\partial^2 \omega^{(m_0)}(\ell_0) < 0$. In the focusing case, the nonlinear bound states bifurcate if $\partial^2 \omega^{(m_0)}(\ell_0) > 0$.

For $V(x) = V_0 \sin^2(x)$ and the defocusing case, the bifurcation diagram is
Application of the NLS equation to existence of nonlinear bound states

For $V(x) = V_0 \sin^2(x)$ and the focusing case, the bifurcation diagram is
Let the periodic graph $\Gamma$ consist of the circles of the normalized length $2\pi$ and the horizontal links of the length $L$. Writing the periodic graph as

$$\Gamma = \bigoplus_{n \in \mathbb{Z}} \Gamma_n,$$

with

$$\Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-},$$

we parameterize $\Gamma_{n,0} := [nP, nP + L]$ and $\Gamma_{n,\pm} := [nP + L, (n + 1)P]$, where $P = L + \pi$ is the graph period.

The NLS equation on the periodic graph $\Gamma$,

$$i\partial_t u + \partial_x^2 u + |u|^2 u = 0, \quad t \in \mathbb{R}, \quad x \in \Gamma,$$

subject to the Kirchhoff boundary conditions at the vertices.
Motivations

- Understand differences between analysis of bounded periodic potentials and of singularities related to the periodic graph.

- Study homogenizations of the NLS equation on the periodic graph.

- Construct nonlinear bound states and the ground state on the periodic graph.

D.P. and G. Schneider, arXiv: 1603.05463
Linear spectral problem

The spectral problem with a bounded $2\pi$-periodic potential $V$,

$$\lambda w = -\partial_x^2 w, \quad x \in \Gamma,$$

subject to the Kirchhoff boundary conditions for $n \in \mathbb{Z}$,

$$\begin{cases}
w_{n,0}(nP + L) = w_{n,+}(nP + L) = w_{n,-}(nP + L), \\
w_{n+1,0}((n + 1)P) = w_{n,+}((n + 1)P) = w_{n,-}((n + 1)P),
\end{cases}$$

and

$$\begin{cases}
\partial_x w_{n,0}(nP + L) = \partial_x w_{n,+}(nP + L) + \partial_x w_{n,-}(nP + L), \\
\partial_x w_{n+1,0}((n + 1)P) = \partial_x w_{n,+}((n + 1)P) + \partial_x w_{n,-}((n + 1)P).
\end{cases}$$

Decomposition of the spectrum on $\Gamma$

Lemma

The linear operator $-\partial_x^2 : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint. Its spectrum $\sigma(-\partial_x^2)$ is positive and consists of two parts.

Integrating by parts with Kirchhoff boundary conditions, we have

$$\lambda \|w\|_{L^2(\Gamma)}^2 = \|\partial_x w\|_{L^2(\Gamma)}^2 \geq 0.$$
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Integrating by parts with Kirchhoff boundary conditions, we have

$$\lambda \|w\|_{L^2(\Gamma)}^2 = \|\partial_x w\|_{L^2(\Gamma)}^2 \geq 0.$$ 

The first part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$\begin{cases} 
  w_{n,0}(x) = 0, & x \in [nP, nP + L], \\
  w_{n,+}(x) = -w_{n,-}(x), & x \in [nP + L, (n + 1)P],
\end{cases} \quad n \in \mathbb{Z}.$$ 

Clearly, $\lambda = m^2$, $m \in \mathbb{N}$ is an eigenvalue of infinite multiplicity with the eigenfunction $w_{n,\pm}(x) = \pm \delta_{n,k} \sin[m(x - 2\pi n)]$, $k \in \mathbb{Z}$.

The second part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$w_{n,+}(x) = w_{n,-}(x), \quad x \in [nP + L, (n + 1)P], \quad n \in \mathbb{Z}.$$
Construction of symmetric eigenfunctions

Let us parameterize the spectral parameter $\lambda = \omega^2$. Then, solutions of ODEs are found in terms of the boundary conditions:

$$\begin{cases} 
    w_{n,0}(x) = a_n \cos(\omega(x - nP)) + b_n \sin(\omega(x - nP)), & x \in [nP, nP + L], \\
    w_{n,\pm}(x) = c_n \cos(\omega(x - nP - L)) + d_n \sin(\omega(x - nP - L)), & x \in [nP + L, (n + 1)P],
\end{cases}$$
Construction of symmetric eigenfunctions

Let us parameterize the spectral parameter \( \lambda = \omega^2 \). Then, solutions of ODEs are found in terms of the boundary conditions:

\[
\begin{align*}
  w_{n,0}(x) &= a_n \cos(\omega(x - nP)) + b_n \sin(\omega(x - nP)), & x \in [nP, nP + L], \\
  w_{n,\pm}(x) &= c_n \cos(\omega(x - nP - L)) + d_n \sin(\omega(x - nP - L)), & x \in [nP + L, (n + 1)P],
\end{align*}
\]

Kirchhoff boundary conditions yield

\[
\begin{align*}
  c_n &= a_n \cos(\omega L) + b_n \sin(\omega L), \\
  2d_n &= -a_n \sin(\omega L) + b_n \cos(\omega L),
\end{align*}
\]

and

\[
\begin{align*}
  a_{n+1} &= c_n \cos(\omega \pi) + d_n \sin(\omega \pi), \\
  b_{n+1} &= -2c_n \sin(\omega \pi) + 2d_n \cos(\omega \pi).
\end{align*}
\]

The monodromy matrix

\[
M(\omega) := \begin{bmatrix} \cos(\omega \pi) & \sin(\omega \pi) \\ -2\sin(\omega \pi) & 2\cos(\omega \pi) \end{bmatrix} \begin{bmatrix} \cos(\omega L) & \sin(\omega L) \\ -\frac{1}{2} \sin(\omega L) & \frac{1}{2} \cos(\omega L) \end{bmatrix}
\]

satisfies \( \det(M) = 1 \) and \( \text{tr}(M) = 2 \cos(\omega \pi) \cos(\omega L) - \frac{5}{2} \sin(\omega \pi) \sin(\omega L) \).
The symmetric part of the spectrum

Trace of the monodromy matrix:

$$T(\omega) = 2 \cos(\omega \pi) \cos(\omega L) - \frac{5}{2} \sin(\omega \pi) \sin(\omega L) \in [-2, 2].$$

Note that $T(m) = 2(-1)^m \cos(mL) \in [-2, 2]$ for every $m \in \mathbb{N}$.

The spectrum $\sigma(-\partial_x^2)$ in $L^2(\Gamma)$ consists of eigenvalues $\{m^2\}_{m \in \mathbb{N}}$ of infinite multiplicity and a countable set of spectral bands $\{\sigma_k\}_{k \in \mathbb{N}}$. Moreover, $m^2 \in \bigcup_{k \in \mathbb{N}} \sigma_k$ for every $m \in \mathbb{N}$. 
Floquet–Bloch spectrum

For simplicity, take $L = \pi$ and define the Bloch waves

$$W(x) = e^{i\ell x}f(\ell, x), \quad \ell, x \in \mathbb{R},$$

where $f(\ell, \cdot) = (f_0, f_+, f_-)(\ell, \cdot)$ is a $2\pi$-periodic function for every $\ell \in \mathbb{R}$ satisfying the $\ell$-dependent Kirchhoff boundary conditions

$$\begin{cases} f_0(\ell, \pi) = f_+(\ell, \pi) = f_-(\ell, \pi), \\ f_0(\ell, 0) = f_+(\ell, 2\pi) = f_-(\ell, 2\pi) \end{cases}$$

and

$$\begin{cases} (\partial_x + i\ell)f_0(\ell, \pi) = (\partial_x + i\ell)f_+(\ell, \pi) + (\partial_x + i\ell)f_-(\ell, \pi), \\ (\partial_x + i\ell)f_0(\ell, 0) = (\partial_x + i\ell)f_+(\ell, 2\pi) + (\partial_x + i\ell)f_-(\ell, 2\pi). \end{cases}$$

Note that $e^{i\ell x}$ is defined for $x \in \mathbb{R}$ but is not defined for $x \in \Gamma$.

For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

$$-(\partial_x + i\ell)^2f = \omega(\ell)f, \quad x \in \mathbb{T}_{2\pi}.$$
Numerical approximation of spectral bands: $L = \pi$
Numerical approximation of spectral bands: $L > \pi$
Numerical approximation of spectral bands: semi-rings of different lengths

Figure: The spectral bands $\lambda$ plotted versus the Bloch wave number $\ell$ for the periodic quantum graph $\Gamma$. 
The NLS equation on the periodic graph

Define piecewise functions for solutions of the NLS equation on the periodic graph $\Gamma$:

$$u_0(x) = \bigcup_{n \in \mathbb{Z}} \begin{cases} u_{n,0}(x), & x \in I_{n,0} = [2\pi n, 2\pi n + \pi], \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$u_{\pm}(x) = \bigcup_{n \in \mathbb{Z}} \begin{cases} u_{n,\pm}(x), & x \in I_{n,\pm} = [2\pi n + \pi, 2\pi(n + 1)], \\ 0, & \text{elsewhere}. \end{cases}$$

The NLS equation on the periodic graph $\Gamma$ can be written as the evolutionary problem for $U = (u_0, u_+, u_-)$:

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\},$$

subject to the Kirchhoff boundary conditions at the vertex points.
Homogeneous NLS equation

The asymptotic solution in the form

\[ U(t, x) = \varepsilon A(T, X)f^{(m_0)}(\ell_0, x)e^{i\ell_0 x}e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms}, \]

with \( T = \varepsilon^2 t \) and \( X = \varepsilon(x - c_g t) \) satisfies the homogeneous NLS equation

\[ i\partial_T A + \frac{1}{2}\partial^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0, \cdot)\|_{L^4_{\text{per}}}^4}{\|f^{(m_0)}(\ell_0, \cdot)\|_{L^2_{\text{per}}}^2}. \]

Theorem (Gilg–Schneider-P, 2016)

Fix \( m_0 \in \mathbb{N}, \ell_0 \in \mathbb{T}_1 \), and assume \( \omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0) \) for every \( m \neq m_0 \). Then, for every \( C_0 > 0 \) and \( T_0 > 0 \), there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all solutions \( A \in C(\mathbb{R}, H^3(\mathbb{R})) \) of the homogeneous NLS equation with

\[ \sup_{T \in [0, T_0]} \|A(T, \cdot)\|_{H^3} \leq C_0 \]

and for all \( \varepsilon \in (0, \varepsilon_0) \), there are solutions \( U \in C([0, T_0/\varepsilon^2], L^\infty(\mathbb{R})) \) to the NLS equation on the periodic graph \( \Gamma \) satisfying the bound

\[ \sup_{t \in [0, T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t, x) - \varepsilon A(T, X)f^{(m_0)}(\ell_0, x)e^{i\ell_0 x}e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}. \]
Extension to the Dirac equations

The symmetry constraints $u_{n,+}(t,x) = u_{n,-}(t,x)$ is invariant under the time evolution of the NLS equation on the periodic graph $\Gamma$. Under the constraints, the spectral bands feature Dirac points and no flat bands.
Homogeneous Dirac equations

The asymptotic solution in the form

\[ U(t, x) = \varepsilon A_+ (T, X) f^+ (0, x) e^{-i \omega^+ (0)t} + \varepsilon A_- (T, X) f^- (0, x) e^{-i \omega^- (0)t} + \text{higher-order terms}, \]

with \( T = \varepsilon^2 t \) and \( X = \varepsilon^2 x \) satisfies the homogeneous Dirac equations

\[
\begin{aligned}
&i \partial_T A_+ + i \partial_\ell \omega^+ (0) \partial_x A_+ + \sum_{j_1, j_2, j_3 \in \{+, -\}} \nu_{j_1 j_2 j_3}^+ A_{j_1} A_{j_2} \overline{A_{j_3}} = 0, \\
&i \partial_T A_- + i \partial_\ell \omega^- (0) \partial_x A_- + \sum_{j_1, j_2, j_3 \in \{+, -\}} \nu_{j_1 j_2 j_3}^- A_{j_1} A_{j_2} \overline{A_{j_3}} = 0,
\end{aligned}
\]

Theorem (Gilg–Schneider-P, 2016)

For every \( C_0 > 0 \) and \( T_0 > 0 \), there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all solutions \( A_\pm \in C(\mathbb{R}, H^2 (\mathbb{R})) \) of the Dirac equations with

\[
\sup_{T \in [0, T_0]} \| A_\pm (T, \cdot) \|_{H^2} \leq C_0
\]

and for all \( \varepsilon \in (0, \varepsilon_0) \), there are solutions \( U \in C([0, T_0 / \varepsilon^2], L^\infty (\mathbb{R})) \) of the NLS equation on the periodic graph \( \Gamma \) satisfying the bound

\[
\sup_{t \in [0, T_0 / \varepsilon^2]} \sup_{x \in \mathbb{R}} | U(t, x) - \varepsilon \Psi_{\text{dirac}} (t, x) | \leq C \varepsilon^{3/2}.
\]
Function spaces

The operator $L = -\partial_x^2$ is considered in the space

\[ \mathcal{L}^2 = \{ U = (u_0, u_+, u-) \in (L^2(\mathbb{R}))^3 : \text{supp}(u_{n,j}) = I_{n,j}, \quad n \in \mathbb{Z}, \quad j \in \{0, +, -\} \} \]

with the domain of definition

\[ \mathcal{H}^2 := \{ U \in \mathcal{L}^2 : u_{n,j} \in H^2(I_{n,j}), \quad n \in \mathbb{Z}, \quad j \in \{0, +, -\} \quad \text{Kirchhoff BCs} \}. \]
Function spaces

The operator \( L = -\partial^2_x \) is considered in the space

\[
\mathcal{L}^2 = \{ U = (u_0, u_+, u_-) \in (L^2(\mathbb{R}))^3 : \supp(u_{n,j}) = I_{n,j}, \quad n \in \mathbb{Z}, \quad j \in \{0, +, -\} \}
\]

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\]

▶ The space \( \mathcal{H}^2 \) is closed under pointwise multiplication.

▶ The skew symmetric operator \(-iL\) defines a unitary semi-group \((e^{-iLt})_{t \in \mathbb{R}}\) in \( \mathcal{L}^2 \).

▶ There exists a positive constant \( C_L \) such that

\[
\| e^{-iLt} U \|_{\mathcal{H}^2} \leq C_L \| U \|_{\mathcal{H}^2}
\]

for every \( U \in \mathcal{H}^2 \) and every \( t \in \mathbb{R} \).

▶ There exists a unique local solution \( U \in C([-T_0, T_0], \mathcal{H}^2) \) to the NLS equation on the periodic graph \( \Gamma \).
Bloch transform on the real line

For a function \( f : \mathbb{R} \to \mathbb{C} \), Bloch transform is defined by

\[
\tilde{f}(\ell, x) = (\mathcal{T}f)(\ell, x) = \sum_{j \in \mathbb{Z}} e^{ijx} \hat{f}(\ell + j),
\]

where \( \hat{f}(\xi) = (\mathcal{F}f)(\xi), \xi \in \mathbb{R} \) is the Fourier transform of \( f \). The inverse transform is

\[
f(x) = (\mathcal{T}^{-1}\tilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \tilde{f}(\ell, x) d\ell.
\]

By construction, \( \tilde{f}(\ell, x) \) is extended from \( (\ell, x) \in T_1 \times T_{2\pi} \) to \( (\ell, x) \in \mathbb{R} \times \mathbb{R} \) according to the continuation conditions:

\[
\tilde{f}(\ell, x) = \tilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \tilde{f}(\ell, x) = \tilde{f}(\ell + 1, x)e^{ix}.
\]
Bloch transform on the real line

For a function \( f : \mathbb{R} \to \mathbb{C} \), Bloch transform is defined by

\[
\tilde{f}(\ell, x) = (Tf)(\ell, x) = \sum_{j \in \mathbb{Z}} e^{ijx} \hat{f}(\ell + j),
\]

where \( \hat{f}(\xi) = (\mathcal{F}f)(\xi), \xi \in \mathbb{R} \) is the Fourier transform of \( f \). The inverse transform is

\[
f(x) = (T^{-1}\tilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \tilde{f}(\ell, x) d\ell.
\]

By construction, \( \tilde{f}(\ell, x) \) is extended from \( (\ell, x) \in \mathbb{T}_1 \times \mathbb{T}_{2\pi} \) to \( (\ell, x) \in \mathbb{R} \times \mathbb{R} \) according to the continuation conditions:

\[
\tilde{f}(\ell, x) = \tilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \tilde{f}(\ell, x) = \tilde{f}(\ell + 1, x) e^{ix}.
\]

- \( T \) is an isomorphism between \( H^s(\mathbb{R}) \) and \( L^2(\mathbb{T}_1, H^s(\mathbb{T}_{2\pi})) \).
- Multiplication in \( x \) space corresponds to convolution in Bloch space.
- If \( \chi : \mathbb{R} \to \mathbb{R} \) is \( 2\pi \) periodic, then

\[
T(\chi u)(\ell, x) = \chi(x)(Tu)(\ell, x).
\]

In particular, if \( \chi_j \) are periodic cut-off functions in \( I_j, j \in \{0, +, -\} \), then

\[
T(u_j)(\ell, x) = T(\chi_j u_j)(\ell, x) = \chi_j(x)(Tu_j)(\ell, x).
\]
Function spaces for Bloch transforms

The operator \( \tilde{L}(\ell) = - (\partial_x + i\ell)^2 \) is self-adjoint in the space

\[
L^2_\Gamma := \{ \tilde{U} = (\tilde{u}_0, \tilde{u}_+, \tilde{u}_-) \in (L^2(\mathbb{T}_2\pi))^3 : \text{supp}(\tilde{u}_j) = I_{0,j}, \quad j \in \{0, +, -\} \}
\]

with the domain of definition

\[
H^2_\Gamma := \{ \tilde{U} \in L^2_\Gamma : \tilde{u}_j \in H^2(I_{0,j}), \quad j \in \{0, +, -\}, \quad \text{Kirchhoff BCs} \}.
\]

In Bloch space, we work with functions in \( L^2(\mathbb{T}_1, L^2_\Gamma) \). Local well-posedness applies to smooth functions in \( \tilde{H}^2 = L^2(\mathbb{T}_1, H^2_\Gamma) \).
Function spaces for Bloch transforms

The operator $\tilde{L}(\ell) = -(\partial_x + i\ell)^2$ is self-adjoint in the space

$$L^2_\Gamma := \{ \tilde{U} = (\tilde{u}_0, \tilde{u}_+, \tilde{u}_-) \in (L^2(\mathbb{T}_{2\pi}))^3 : \text{supp}(\tilde{u}_j) = I_{0,j}, \quad j \in \{0, +, -\} \}$$

with the domain of definition

$$H^2_\Gamma := \{ \tilde{U} \in L^2_\Gamma : \tilde{u}_j \in H^2(I_{0,j}), \quad j \in \{0, +, -\}, \quad \text{Kirchhoff BCs} \}.$$

In Bloch space, we work with functions in $L^2(\mathbb{T}_1, \mathbb{T}^2_\Gamma)$. Local well-posedness applies to smooth functions in $\tilde{H}^2 = L^2(\mathbb{T}_1, H^2_\Gamma)$.

**Key Lemma:** The Bloch transform $\mathcal{T}$ is an isomorphism between $H^2$ and $\tilde{H}^2$.

- Extend a piecewise $H^2$ function $u_0$ to $u_{0,\text{ext}} \in H^2(\mathbb{R})$.
- By Bloch transform on the real line, $\mathcal{T}(u_{0,\text{ext}}) \in L^2(\mathbb{T}_1, H^2(\mathbb{T}_{2\pi}))$.
- Compact support persists as $\tilde{u}_0 = \mathcal{T}(u_0) = \mathcal{T}(\chi_0 u_{0,\text{ext}}) = \chi_0 \mathcal{T}(u_{0,\text{ext}})$.
- From the properties of $\mathcal{T}(u_{0,\text{ext}})$, we obtain $\tilde{u}_0 \in L^2(\mathbb{T}_1, H^2(I_{0,0}))$. 
Rest of the proof

- Bloch transform for the NLS equation on the periodic graph $\Gamma$.

- Decomposition of solutions in the Bloch space

$$\tilde{U}(t, \ell, x) = \tilde{V}(t, \ell)f^{(m_0)}(\ell, x) + \tilde{U}^\perp(t, \ell, x)$$

- Approximation of the principal part of the solution

$$\tilde{V}_{\text{app}}(t, \ell) = \tilde{A} \left( \varepsilon^2 t, \frac{\ell - \ell_0}{\varepsilon} \right) e^{-i\omega^{(m_0)}(\ell_0)t} e^{-i\partial_\ell \omega^{(m_0)}(\ell_0)(\ell - \ell_0)t}.$$  

As $\varepsilon \to 0$, $\tilde{A}$ satisfies the homogeneous NLS equation in the Fourier space.

- A near-identity transformation for $\tilde{U}^\perp(t, \ell, x)$ with a suitable chosen approximation $\tilde{U}^\perp_{\text{app}}(t, \ell, x)$.

- Estimates of residual terms in Bloch spaces.

- Estimates of the approximation between the Fourier space and Bloch space.

- Estimates of the error term in time evolution with Gronwall’s inequality.
The asymptotic solution in the form

\[ U(t, x) = \varepsilon A(T, X)f^{(m_0)}(\ell_0, x)e^{i\ell_0 x}e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms}, \]

with \( T = \varepsilon^2 t \) and \( X = \varepsilon(x - c_g t) \) satisfies the homogeneous NLS equation

\[
i\partial_T A + \frac{1}{2} \partial^2_{\ell_0} \omega^{(m_0)}(\ell_0) \partial^2_X A + \nu |A|^2 A = 0, \quad \nu = \left\| f^{(m_0)}(\ell_0, \cdot) \right\|_{L^4_{\text{per}}}^4 \left\| f^{(m_0)}(\ell_0, \cdot) \right\|_{L^2_{\text{per}}}^2.
\]

Theorem (Gilg–Schneider-P, 2016)

Fix \( m_0 \in \mathbb{N}, \ell_0 \in \mathbb{T}_1 \), and assume \( \omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0) \) for every \( m \neq m_0 \). Then, for every \( C_0 > 0 \) and \( T_0 > 0 \), there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for all solutions \( A \in C(\mathbb{R}, H^3(\mathbb{R})) \) of the homogeneous NLS equation with

\[
\sup_{T \in [0, T_0]} \left\| A(T, \cdot) \right\|_{H^3} \leq C_0
\]

and for all \( \varepsilon \in (0, \varepsilon_0) \), there are solutions \( U \in C([0, T_0/\varepsilon^2], L^\infty(\mathbb{R})) \) to the NLS equation on the periodic graph \( \Gamma \) satisfying the bound

\[
\sup_{t \in [0, T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t, x) - \varepsilon A(T, X)f^{(m_0)}(\ell_0, x)e^{i\ell_0 x}e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.
\]
Bifurcations of nonlinear bound states

The stationary NLS equation on the periodic graph $\Gamma$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \quad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}. $$

The effective homogeneous NLS equation on the real line

$$-\frac{1}{2} \partial_{\ell}^2 \omega^{(m_0)}(\ell_0) \partial_X^2 A - \nu |A|^2 A = \Omega A, \quad A(X) : \mathbb{R} \to \mathbb{R}. $$

The stationary reduction is satisfied if $\partial_{\ell} \omega^{(m_0)}(\ell_0) = 0$. 
Nonlinear bound states on the periodic graph

Stable bound states bifurcate from the bottom of the linear spectrum at $\Lambda = 0$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \quad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$ 

Theorem

There are positive constants $\Lambda_0$ and $C_0$ such that for every $\Lambda \in (-\Lambda_0, 0)$, there exist two bound states $\phi \in \mathcal{D}(\Gamma)$ (up to the discrete translational invariance) s.t. either

$$\phi(x - L/2) = \phi(L/2 - x), \quad x \in \Gamma$$

or

$$\phi(x - L - \pi/2) = \phi(L + \pi/2 - x), \quad x \in \Gamma.$$

Moreover, it is true for both bound states that

(i) $\phi$ is symmetric in upper and lower semicircles of $\Gamma$,
(ii) $\phi(x) > 0$ for every $x \in \Gamma$,
(iii) $\phi(x) \to 0$ as $|x| \to \infty$ exponentially fast.
Numerical approximations of the bound states with $L = \pi$

Figure: Profile of the numerically generated bound state on $(x, \phi)$ plane (left) and on $(\phi, \phi')$ plane (right). The red dots show the break points on the periodic graph $\Gamma$. The green dashed line shows the NLS soliton on the infinite line.

Figure: The same but for the other bound state.
Discrete homogenization method

We set $\Lambda = -\epsilon^2$ and consider the limit $\epsilon \to 0$.

For every $(a, b) \in \mathbb{R}^2$ and every $\epsilon \in \mathbb{R}$, there is a unique solution $\psi(x; a, b, \epsilon) \in C^\infty(\mathbb{R})$ of the initial-value problem:

\[
\begin{aligned}
\partial_x^2 \psi - \epsilon^2 \psi + 2|\psi|^2 \psi &= 0, \quad x \in \mathbb{R}, \\
\psi(0) &= a, \\
\partial_x \psi(0) &= b,
\end{aligned}
\]

For each $\Gamma_{n,0}$ and $\Gamma_{n,\pm}$, the solution can be defined in the implicit form:

$\phi_{n,0}(x) = \psi(x - nP; a_n, b_n, \epsilon), \quad \phi_{n,\pm}(x) = \psi(x - nP - L; c_n, d_n, \epsilon)$.

Kirchhoff boundary conditions produces a two-dimensional map:

\[
\begin{aligned}
a_{n+1} &= \psi(\pi; c_n, d_n, \epsilon), \\
b_{n+1} &= 2\partial_x \psi(\pi; c_n, d_n, \epsilon), \\
c_n &= \psi(L; a_n, b_n, \epsilon), \\
2d_n &= \partial_x \psi(L; a_n, b_n, \epsilon),
\end{aligned}
\]

The nonlinear discrete map generalizes the linear transfer matrix method.
Approximate continuous solution

In the limit $\epsilon \to 0$, expand solution $\psi(x; \epsilon \alpha, \epsilon^2 \beta, \epsilon)$ in the power series in $\epsilon$. The two-dimensional map is now available in the perturbative form:

\[
\begin{align*}
\alpha_{n+1} &= \alpha_n + \epsilon (L + \pi/2) \beta_n + \frac{1}{2} \epsilon^2 (L^2 + \pi L + \pi^2) (1 - 2\alpha_n^2) \alpha_n + \mathcal{O}(\epsilon^3), \\
\beta_{n+1} &= \beta_n + \epsilon (L + 2\pi) (1 - 2\alpha_n^2) \alpha_n + \frac{1}{4} \epsilon^2 (2L^2 + 4L\pi + \pi^2) (1 - 6\alpha_n^2) \beta_n + \mathcal{O}(\epsilon^3).
\end{align*}
\]
Approximate continuous solution

In the limit $\epsilon \to 0$, expand solution $\psi(x; \epsilon \alpha, \epsilon^2 \beta, \epsilon)$ in the power series in $\epsilon$. The two-dimensional map is now available in the perturbative form:

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\beta_{n+1} &= \beta_n + \epsilon(L + 2\pi)(1 - 2\alpha_n^2) \alpha_n + \frac{1}{4} \epsilon^2 (2L^2 + 4L \pi + \pi^2)(1 - 6\alpha_n^2) \beta_n + O(\epsilon^3).
\end{align*}
\]

Approximate continuous solution:

\[
\begin{align*}
\alpha_n &= A(X + X_0), \quad \beta_n = B(X + X_0), \quad X = \epsilon n, \quad n \in \mathbb{Z},
\end{align*}
\]

where $X_0$ is arbitrary and $A, B$ satisfy the continuous limit

\[
\begin{align*}
A'(X) &= (L + \pi/2) B(X), \\
B'(X) &= (L + 2\pi)(1 - 2A^2) A(X),
\end{align*}
\]

with the continuous NLS solitons

\[
A(X) = \text{sech}(\nu X), \quad B(X) = -\mu \tanh(\nu X) \text{sech}(\nu X), \quad X \in \mathbb{R},
\]
Justification of the approximate continuous solution

**Key Lemma:** For a given $f \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $f_n = f_{1-n}$ for every $n \in \mathbb{Z}$, consider solutions of the linearized difference equation

$$-\frac{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}{\epsilon^2} + \nu^2 (1 - 6A^2(\epsilon n))\alpha_n = f_n, \quad n \in \mathbb{Z}.$$  

For sufficiently small $\epsilon > 0$, there exists a unique solution $\alpha \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $\alpha_n = \alpha_{1-n}$ for every $n \in \mathbb{Z}$. Moreover there is a positive $\epsilon$-independent constant $C$ such that

$$\epsilon^{-1} \|\sigma_+\alpha - \alpha\|_{\ell^2} \leq C\|f\|_{\ell^2}, \quad \|\alpha\|_{\ell^2} \leq C\|f\|_{\ell^2},$$

where $\sigma_+$ is the shift operator defined by $(\sigma_+\alpha)_n := \alpha_{n+1}$, $n \in \mathbb{Z}$. 

▶ Translational parameter $X_0$ can be chosen to satisfy the reversibility symmetry.

▶ Two reversibility symmetries give two nonlinear bound states.

▶ The symmetry $\phi^+ = \phi^-$ holds by construction.

▶ Positivity and exponential decay are not obtained from this method.
Justification of the approximate continuous solution

Key Lemma: For a given \( f \in \ell^2(\mathbb{Z}) \) satisfying the reversibility symmetry \( f_n = f_{1-n} \) for every \( n \in \mathbb{Z} \), consider solutions of the linearized difference equation

\[
-\frac{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}{\epsilon^2} + \nu^2 (1 - 6A^2(\epsilon n)) \alpha_n = f_n, \quad n \in \mathbb{Z}.
\]

For sufficiently small \( \epsilon > 0 \), there exists a unique solution \( \alpha \in \ell^2(\mathbb{Z}) \) satisfying the reversibility symmetry \( \alpha_n = \alpha_{1-n} \) for every \( n \in \mathbb{Z} \). Moreover there is a positive \( \epsilon \)-independent constant \( C \) such that

\[
\epsilon^{-1} \|\sigma_+ \alpha - \alpha\|_{\ell^2} \leq C \|f\|_{\ell^2}, \quad \|\alpha\|_{\ell^2} \leq C \|f\|_{\ell^2},
\]

where \( \sigma_+ \) is the shift operator defined by \( (\sigma_+ \alpha)_n := \alpha_{n+1}, n \in \mathbb{Z} \).

- Translational parameter \( X_0 \) can be chosen to satisfy the reversibility symmetry.
- Two reversibility symmetries give two nonlinear bound states.
- The symmetry \( \phi_+ = \phi_- \) holds by construction.
- Positivity and exponential decay are not obtained from this method.
Positivity and exponential decay

The perturbative two-dimensional map:

\[
\begin{cases}
\alpha_{n+1} = \alpha_n + \epsilon(L + \pi/2)\beta_n + \frac{1}{2}\epsilon^2(L^2 + \pi L + \pi^2)(1 - 2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\
\beta_{n+1} = \beta_n + \epsilon(L + 2\pi)(1 - 2\alpha_n^2)\alpha_n + \frac{1}{4}\epsilon^2(2L^2 + 4L\pi + \pi^2)(1 - 6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3).
\end{cases}
\]

**Figure:** The plane \((\alpha, \beta)\), where the blue dots denote a sequence \(\{\alpha_n, \beta_n\}_{n \in \mathbb{Z}}\), the green dashed line shows the unstable curve \(\beta = U_\epsilon(\alpha)\), and the red dash-dotted line shows the symmetry curve \(\beta = N_\epsilon(\alpha)\).
Conclusion

For the periodic graph $\Gamma$, we have obtained the following results:

- We developed the Bloch transform on $\Gamma$ and justified homogenization of the NLS equation on $\Gamma$ with the homogeneous NLS or Dirac equations on the line.
- We approximated nonlinear bound states near the lowest spectral band by using NLS solitons.
- We used discrete maps and dynamical system methods to study linear spectrum of the periodic graph $\Gamma$ and the nonlinear bound states on $\Gamma$.
- Scattering and nonlinear dynamics on the periodic graph $\Gamma$ are still to be analyzed in some future.

Thank you!