BSDEs, martingale problems, pseudo-PDEs and applications.

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Outline

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5. Strong martingale problem.
6. Deterministic problem related to BSDEs driven by a martingale.
7. Special case of the Föllmer-Schweizer decomposition.
Basic Reference

Ismail Laachir and Francesco Russo.

*BSDEs, càdlàg martingale problems and orthogonalization under basis risk.*
Related references.

- A. Barrasso and F. Russo.
  
  
  https://hal.inria.fr/hal-01431559

- A. Barrasso and F. Russo.
  
  *Decoupled Mild solutions for Pseudo Partial Differential Equations versus Martingale driven forward-backward SDEs.*
  
  https://hal.archives-ouvertes.fr/hal-01505974
Available preprints and publications.
http://uma.ensta.fr/~russo/
1 General mathematical context

- Interface between “stochastic processes” and “deterministic world”.
- Benchmark situation: bridge between semilinear PDEs and BSDEs.
PDE:

\[
\begin{cases}
    \partial_s u(s, x) + L_s u(s, x) + f(s, x, u(s, x), \sigma \partial_x u(s, x)) = 0 \\
    u(T, x) = g(x), \ s \in [0, T], \ x \in E = \mathbb{R}^d,
\end{cases}
\]  

(1)

where \( L_s \) is the generator of a diffusion of the type

\[
dX_s = \sigma(s, X_s) dW_s + b(t, X_s) ds, \ X_t = x.
\]  

(2)
BSDE: (2) is coupled with

\[ Y_s = g(X_T) + \int_s^T f(s, X_r, Y_r, Z_r) dr - \int_s^T Z_r dW_r. \]  (3)

The link is the following.
1. If $u$ is a classical solution of (1) then

$$Y_s = u(s, X_s), \ Z_s = \sigma(s, X_s) \nabla u(s, X_s)$$

provide a solution to (3).

2. Viceversa if, given $(t, x) \in [0, T] \times E$ and $X^{t,x}$ is given by (2), $(X^{t,x}, Y^{t,x}, Z^{t,x})$ is a solution to (3), then $u(t, x) := Y^{t,x}_t$ is a viscosity solution to (1).
What about $v(t, x) := Z_{t,x}^t$?

- If $u$ is of class $C^{0,1}$ then $v(t, x) = \sigma(t, x) \nabla u(t, x)$.

- What happens in general? Only partial answers even in the Brownian case.

- This talk and the mentioned references discuss some issues related to this problem when $W$ is replaced by a cadlag martingale.
2 Financial Motivations

2.1 Hedging in a complete market

Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with a filtration $(\mathcal{F}_t)_{t \in [0,T]}$, $\mathcal{F}_0$ being the trivial $\sigma$-algebra.

- $S$ price of a risky asset.
- $B$ price of a riskless asset.
Complete market.

For any random variable $h$, there exists a self-financing strategy $(\nu_t)_{t \in [0,T]}$ perfectly replicating $h$, i.e. a trading strategy that starts from an initial wealth $V_0$ and re-invests the gain/loss from $S$ on the riskless asset $B$.

If we suppose that the riskless asset price is constant, this reduces to

$$V_0 + \int_0^T \nu_u dS_u = h.$$
2.2 Hedging in the presence of basis risk

Basis risk.

Risk arising when a derivative product is based on a non-traded or illiquid underlying, but observable, and the replicating (hedging) portfolio is constituted of traded and liquid additional assets which are correlated with the original one.

Example:

- Basket option hedged with a subset of the composing assets.
- Airline companies hedging kerosene exposure with correlated contacts, as crude oil or heating oil.
Consider a pair of processes \((X, S)\) and a contingent claim of the type \(h := g(X_T, S_T)\).

- \(X\) is a non traded or illiquid, but observable asset.
- \(S\) is a traded asset, correlated to \(X\).
- \(B\) is riskless asset. We suppose \(B\) to be constant.

**Hedging problem**: construct a trading strategy on the assets \((B, S)\) in order to replicate the random variable \(h\).
In this case, the market is **incomplete**: perfect replication with a self-financing strategy is not possible. One should define a risk aversion criterion, for example the following.

6 **Utility-based** criterion.

6 **Quadratic risk criteria**: *local risk minimization* and *mean-variance minimization*.
2.3 Quadratic hedging: local and global risk minimization.

Introduced by Föllmer and Sondermann [1985], for \( S \) being a (local) martingale. In this case, the unique (local) risk-minimizing strategy is determined by the Kunita-Watanabe (K-W) representation of martingales.

Extension to the semimartingale case is more delicate, and was handled by Schweizer [1988, 1991]. Its existence is linked to the existence of the so-called Föllmer-Schweizer (F-S) decomposition, a generalization of the (K-W) representation.

Global risk minimization. Again F-S decomposition.
2.4 Föllmer-Schweizer decomposition

Mean-variance hedging is closely related to the so called Föllmer-Schweizer (F-S) decomposition.

**Definition 1** Let $S = M^S + V^S$, $V^S_0 = 0$ be a special semimartingale. A square integrable random variable $h$ admits an F-S decomposition if

$$h = h_0 + \int_0^T Z_u dS_u + O_T,$$

where $h_0 \in \mathbb{R}$, $Z \in \Theta$ and $O$ is a square integrable martingale, strongly orthogonal to $M^S$. 

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**Definition 2** Let $L$ and $N$ be two $\mathcal{F}_t$-local martingales, with null initial value. $L$ and $N$ are said to be **strongly orthogonal** if $LN$ is a local martingale.

**Example 3** If $L$ and $N$ are locally square integrable, then they are strongly orthogonal if and only if $\langle L, N \rangle = 0$. 
2.5 F-S decomposition via a backward SDE

If \((h_0, Z, O)\) is an F-S decomposition, then the process

\[ Y_t := h_0 + \int_0^t Z_u dS_u + O_t \]

verifies

\[ Y_t := h - \int_t^T Z_u dM_u^S - \int_t^T Z_u dV_u^S - (O_T - O_t), \]

which is a Backward Stochastic Differential Equation, driven by a local martingale, where the final condition \(Y_T = h\) is known.

The resolution of the BSDE is a method to determine the F-S decomposition.
3 Backward Stochastic Differential Equations

3.1 BSDEs driven by a Brownian motion

BSDEs were introduced by Pardoux and Peng [1990]. Pioneering work by Bismut [1973].

Given a pair \((h, \hat{f})\) called terminal condition and driver.

One looks for a pair of (adapted) processes \((Y, Z)\), satisfying
\[ Y_t = h + \int_t^T \hat{f}(\omega, s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad (4) \]

and

\[ \mathbb{E} \int_0^T |Z_t|^2 dt < \infty. \]
3.2 Existence and uniqueness

Pardoux and Peng [1990] showed existence and uniqueness when \( \hat{f} \) is globally Lipschitz with respect to \((y, z)\) and \(h\) being square integrable.

Conditions on the driver \( \hat{f} \) were first relaxed to a monotonicity condition on \(y\), later to a quadratic growth condition and other generalizations, see e.g. Hamadene [1996], Lepeltier and San Martín [1998], Kobylanski [2000], Briand and Hu [2006, 2008].

Applications to finance: El Karoui et al. [1997].

Extension to reflected BSDEs...
3.3 BSDEs and semi-linear parabolic PDEs

Consider the BSDE

\[ Y_{s}^{t,x} = g(X_{T}^{t,x}) + \int_{s}^{T} f(s, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x}) dr - \int_{s}^{T} Z_{r}^{t,x} dW_{r}, \quad (5) \]

where \( \{X_{s}^{t,x}, t \leq s \leq T\} \) is a solution of the SDE

\[ X_{s}^{t,x} = x + \int_{t}^{s} b(r, X_{r}^{t,x}) dr + \int_{t}^{s} \sigma(r, X_{r}^{t,x}) dW_{r}, \quad t \leq s \leq T. \]
Link with the semi-linear parabolic PDE.

\[
\begin{aligned}
\partial_t u(t, x) + L_t u(t, x) + f(t, x, u(t, x), \sigma \partial_x u(t, x)) &= 0 \\
u(T, x) &= g(x), \; t \in [0, T], \; x \in \mathbb{R}.
\end{aligned}
\] (6)
3.4 From semi-linear parabolic PDEs to BSDEs

Theorem 4 (Pardoux and Peng [1992]) Let $u \in C^{1,2}([0, T] \times \mathbb{R}, \mathbb{R})$ be a classical solution of (6) such that

$$|\partial_x u(t, x)| \leq c(1 + |x|^q), \text{ for some } c, q > 0.$$ 

Then, $\forall (t, x), (u(s, X^{t,x}_s), (\sigma \partial_x u)(s, X^{t,x}_s))_{s \in [t, T]}$ is solution of the BSDE (5).

In particular, under the conditions of well-posedness of the BSDE

$$u(t, x) = Y_t^{t,x}.$$
3.5 From BSDEs to semi-linear parabolic PDEs

Theorem 5 (Pardoux and Peng [1992]) Let 
\((Y_{s}^{t,x}, Z_{s}^{t,x})_{s \in [t,T]}\) be the solution of the BSDE (5), then 
\(u(t, x) := Y_{t}^{t,x}\) is a continuous function and it is a viscosity solution of the PDE (6).

This representation theorem can be seen as an extension of Feynman-Kac formula.
3.6 Extensions of BSDEs driven by Brownian Motion

- BSDE driven by a Brownian motion and a compensated random measure.
- BSDE driven by a càdlàg martingale.
3.7 BSDEs driven by a càdlàg Martingale

Given a càdlàg (local) martingale $M^S$ and a bounded variation process $V^S$, one looks for a triplet $(Y, Z, O)$ verifying

$$ Y_t = h + \int_t^T \hat{f}(\omega, s, Y_s-, Z_s) dV_s - \int_t^T Z_s dM_s^S -(O_T-O_t), \quad (7) $$

where $O$ is (local) martingale strongly orthogonal to $M^S$. 
First contribution by Buckdahn [1993].

Other contributions, e.g. El Karoui and Huang [1997]. See also Briand et al. [2002], as side-effect of a convergence scheme.

More recent setting for sufficient conditions for existence and uniqueness for (7) has been given by Carbone et al. [2007].

BSDEs with partial information driven by càdlàg martingales were investigated by Ceci, Cretarola, Russo in Ceci et al. [2014a,b].
4 Contributions of the work

A forward BSDE, where the forward process solves a strong martingale problem. We focus on four tasks.

- Characterize forward-backward SDEs via the solution of a deterministic problem generalizing the classical PDE appearing in the case of Brownian martingales.

- Give applications to the hedging problem in the case of basis risk via the Föllmer-Schweizer decomposition.
Give explicit expressions when the pair of processes $(X, S)$ is an exponential of additive processes.

Extensions to the case when the forward process is given in law: strict and generalized solutions of the deterministic problem.
5 Strong Martingale Problem

5.1 Definition

Definition 6 Let $\mathcal{O}$ be an open set of $\mathbb{R}^2$ and $(A_t)$ be an $\mathcal{F}_t$-adapted b.v. continuous process, such that, a.s.

$$dA_t \ll d\rho_t,$$

for some b.v. function $\rho$, and $A$ a map

$$A : \mathcal{D}(A) \subset \mathcal{C}([0, T] \times \mathcal{O}, \mathbb{C}) \rightarrow \mathcal{L}.$$
We say that \((X, S)\) is a solution of the **strong martingale problem** related to \((\mathcal{D}(\mathcal{A}), \mathcal{A}, \mathcal{A})\), if for any \(g \in \mathcal{D}(\mathcal{A})\), 
\( (g(t, X_t, S_t))_t \) is a semimartingale such that 

\[
t \mapsto g(t, X_t, S_t) - \int_0^t A(g)(u, X_{u^-}, S_{u^-})dA_u
\]

is an \(\mathcal{F}_t\)-local martingale.
Notations 7

6. \( id : (t, x, s) \mapsto s \), \( s^2 : (t, x, s) \mapsto s^2 \).

6. For any \( y \in C([0, T] \times \mathcal{O}) \), \( \tilde{y} := y \times id \).

6. Suppose that \( id \in \mathcal{D}(\mathcal{A}) \). For \( y \in \mathcal{D}(\mathcal{A}) \) such that \( \tilde{y} \in \mathcal{D}(\mathcal{A}) \), we set \( \tilde{\mathcal{A}}(y) := \mathcal{A}(\tilde{y}) - y\mathcal{A}(id) - id\mathcal{A}(y) \).
Proposition 8  Suppose that $id, s^2 \in \mathcal{D}(A)$. Then $S$ is a special semimartingale with decomposition $M^S + V^S$ given below.

1. $V^S_t = \int_0^t A(id)(u, X_{u-}, S_{u-})dA_u$.

2. $\langle M^S \rangle_t = \int_0^t \tilde{A}(id)(u, X_{u-}, S_{u-})dA_u$.
Proof.

Item 2. follows from the following more general result.

**Lemma 9**  If $Y_t = y(t, X_t, S_t)$, $y, y \times id \in \mathcal{D}(A)$, then

$$\langle M^Y, M^S \rangle_t = \int_0^t \tilde{A}(y)(u, X_{u^-}, S_{u^-})dA_u.$$
5.2 Examples

6 Diffusion process: the operator $A$ has the form

$$A(f) = \partial_t f + b_S \partial_s f + b_X \partial_x f$$

$$+ \frac{1}{2} \left\{ |\sigma_S|^2 \partial_{ss} f + |\sigma_X|^2 \partial_{xx} f + 2 \langle \sigma_S, \sigma_X \rangle \partial_{sx} f \right\},$$

6 $S$ is a Markov process, with related Markov semigroup of generator $L$: the operator $A$ has the form

$$A(g)(t, s) = \frac{\partial g}{\partial t}(t, s) + Lg(t, \cdot)(s).$$
5.3 Exponential of additive processes

Definition 10 \((Z^1, Z^2)\) is said to be an additive process if \((Z^1, Z^2)_0 = 0, (Z^1, Z^2)\) is continuous in probability and it has independent increments. The generating function of \((Z^1, Z^2)\) is defined by

\[
\exp(\kappa_t(z_1, z_2)) = \mathbb{E}e^{z_1 Z^1_t + z_2 Z^2_t}, \quad \forall (z_1, z_2) \in D,
\]

where \(D := \{ z = (z_1, z_2) \in \mathbb{C}^2 | \mathbb{E}e^{\Re(z_1) Z^1_T + \Re(z_2) Z^2_T} < \infty \}\).

We denote also, for \((z_1, z_2), (y_1, y_2) \in D/2\)

\[
\rho_t(z_1, z_2, y_1, y_2) := \kappa_t(z_1 + y_1, z_2 + y_2) - \kappa_t(z_1, z_2) - \kappa_t(y_1, y_2),
\]

\[
\rho^S_t := \kappa_t(0, 2) - 2\kappa_t(0, 1), \quad \text{if } (0, 1) \in D/2.
\]
We always suppose the validity of the following.

**Assumption 11 (Basic assumption)** \((0, 2) \in D\). This is equivalent to the existence of the second order moment of \(S = e^{Z^2}\).
5.4 First decomposition

We consider two processes $X = \exp(Z^1)$, $S = \exp(Z^2)$.

**Lemma 12** Let $\lambda : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$ such that, for any $(z_1, z_2) \in D$, $d\lambda(t, z_1, z_2) \ll d\rho_t^S$. Then for any $(z_1, z_2) \in D$,

$$
M^\lambda_t(z_1, z_2) := X_t^{z_1} S_t^{z_2} \lambda(t, z_1, z_2) - \int_0^t X_u^{z_1} S_u^{z_2} \left\{ \frac{d\lambda(u, z_1, z_2)}{d\rho_u^S} + \lambda(u, z_1, z_2) \frac{d\kappa_u(z_1, z_2)}{d\rho_u^S} \right\} \rho_{du},
$$

is a martingale. Moreover, if $(z_1, z_2) \in D/2$ then $M^\lambda(z_1, z_2)$ is a square integrable martingale.
5.5 Strong Martingale Problem for exponential of additive processes

**Theorem 13** Under some technical assumptions, $(X, S)$ is a solution of the strong martingale problem related to $(\mathcal{D}(A), A, \rho^S)$ where, $\mathcal{D}(A)$ is the set of

$$f : (t, x, s) \mapsto \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \lambda(t, z_1, z_2),$$

where $\Pi$ is a finite Borel measure on $\mathbb{C}^2$, $\lambda : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$ Borel verifying a set of conditions,
\[ \mathcal{A}(f)(t, x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \]

\[ \left\{ \frac{d\lambda(t, z_1, z_2)}{d\rho_t^S} + \lambda(t, z_1, z_2) \frac{d\kappa_t(z_1, z_2)}{d\rho_t^S} \right\}. \]
6 Deterministic problem related to BSDEs driven by a martingale

6.1 Forward-backward SDE

We consider a pair of $\mathcal{F}_t$-adapted processes $(X, S)$ fulfilling the martingale problem related to $(\mathcal{D}(A), A, A)$. We are interested in the BSDE

$$Y_t = g(X_T, S_T) + \int_t^T f(r, X_r, S_r, Y_r, Z_r) dA_r - \int_t^T Z_r dM^S_r - (O_T - O_t),$$

where
1. \((Y_t)\) is \(\mathcal{F}_t\)-adapted, \((Z_t)\) is \(\mathcal{F}_t\)-predictable

2. \(\int_0^T |Z_s|^2 d\langle M^S \rangle_s < \infty\) a.s.

3. \(\int_0^t |f(s, X_{s-}, S_{s-}, Y_{s-}, Z_s)| d\|A\|_s < \infty\) a.s.

4. \((O_t)\) is an \(\mathcal{F}_t\)-local martingale such that \(\langle O, M^S \rangle = 0\) and \(O_0 = 0\) a.s.
6.2 Related deterministic analysis

**Goal.** Look for solutions \((Y, Z, O)\) of the BSDE for which there is a function \(y \in \mathcal{D}(A)\) such that \(\tilde{y} = y \times id \in \mathcal{D}(A)\) and a locally bounded Borel function \(z : [0, T] \times O \rightarrow \mathbb{C}\), such that

\[
Y_t = y(t, X_t, S_t), \\
Z_t = z(t, X_{t-}, S_{t-}), \quad \forall t \in [0, T].
\]

- When \(M^S\) is a Brownian motion, \(y\) is a solution of a semilinear PDE.
- General case ?
6.3 Deterministic problem (Pseudo-PDE)

Theorem 14 Suppose the existence of a function $y$, such that $\tilde{y} := y \times \text{id}$ belong to $\mathcal{D}(\mathcal{A})$, and a Borel locally bounded function $z$, solving the system

\[
\begin{align*}
\mathcal{A}(y)(t, x, s) &= -f(t, x, s, y(t, x, s), z(t, x, s)) \\
\tilde{\mathcal{A}}(y)(t, x, s) &= z(t, x, s)\tilde{\mathcal{A}}(\text{id})(t, x, s),
\end{align*}
\]

with the terminal condition $y(T, ., .) = g(., .)$.

Then the triplet $(Y, Z, O)$ defined by

\[
Y_t = y(t, X_t, S_t), \quad Z_t = z(t, X_{t-}, S_{t-})
\]

is a solution to the BSDE (8).
7 Special case of the Föllmer-Schweizer decomposition.

7.1 Weak F-S decomposition

Definition 15 We say that a square integrable $\mathcal{F}_T$-measurable random variable $h$ admits a weak F-S decomposition $(h_0, Z, O)$ with respect to $S$ if it can be written as

$$h = h_0 + \int_0^T Z_s dS_s + O_T, \mathbb{P}-\text{a.s.},$$

(8)
where $h_0$ is an $\mathcal{F}_0$-measurable r.v., $Z$ is a predictable process such that $\int_0^T |Z_s|^2 d\langle M^S \rangle_s < \infty$ a.s.,

$\int_0^T |Z_s| d\|V^S\|_s < \infty$ a.s. and $O$ is a local martingale such that $\langle O, M^S \rangle = 0$ with $O_0 = 0$.  


7.2 Link to BSDEs

Finding a weak F-S decomposition \((h_0, Z, O)\) for some r.v. \(h\) is equivalent to provide a solution \((Y, Z, O)\) of the BSDE

\[
Y_t = h - \int_t^T Z_s dS_s - (O_T - O_t).
\]

The link is given by \(Y_0 = h_0\). Here the driver \(f\) is linear in \(z\), of the form

\[
f(t, x, s, y, z) = -\mathcal{A}(id)(t, x, s)z.
\]

⇒ The weak F-S decomposition can be linked to a deterministic problem (Pseudo-PDE).
7.3 Weak Vs True F-S decomposition

Remark 16 Setting $h_0 = y(0, X_0, S_0)$, the triplet $(h_0, Z, O)$ is a candidate for a true F-S decomposition. Sufficient conditions for this are the following.

1. $h = g(X_T, S_T) \in L^2(\Omega)$.
2. $(z(t, X_{t-}, S_{t-}))_t \in \Theta$ i.e.
   \[ \mathbb{E} \int_0^T |z(t, X_{t-}, S_{t-})|^2 \tilde{A}(id)(t, X_{t-}, S_{t-}) dA_t < \infty. \]
   \[ \mathbb{E} \left( \int_0^T |z(t, X_{t-}, S_{t-})| \| A(id)(t, X_{t-}, S_{t-}) dA \|_t \right)^2 < \infty. \]
3. \( \left( y(t, X_t, S_t) - \int_0^t A(y)(u, X_{u-}, S_{u-}) dA_u \right)_t \) is an $\mathcal{F}_t$-square integrable martingale.
Corollary 17 (Application of the theorem for general BSDEs)

Let \( y \) (resp. \( z \)) : \([0, T] \times \Omega \to \mathbb{C} \). We suppose the following.

1. \( y, \tilde{y} := y \times id \) belong to \( \mathcal{D}(A) \).

2. \( \int_0^T z^2(r, X_{r-}, S_{r-}) \tilde{A}(id)(r, X_{r-}, S_{r-}) dA_r < \infty \) a.s.

3. \( (y, z) \) solves the problem

\[
\begin{align*}
A(y)(t, x, s) &= A(id)(t, x, s)z(t, x, s), \\
\tilde{A}(y)(t, x, s) &= \tilde{A}(id)(t, x, s)z(t, x, s),
\end{align*}
\]

with the terminal condition \( y(T, ., .) = g(., .) \).
Then the triplet \((Y_0, Z, O)\), where

\[
Y_t = y(t, X_t, S_t), \quad Z_t = z(t, X_{t-}, S_{t-}), \quad O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,
\]

is a weak F-S decomposition of \(h\).
7.4 Application 1: exponential of additive processes

\((X, S) = (e^{Z_1}, e^{Z_2})\) is an exponential of additive processes.

**Example 18**  **Goal.** Use the **Pseudo-PDE** to give explicit expressions of a weak F-S of an \(\mathcal{F}_T\)-measurable random variable \(h\) of the form \(h := g(X_T, S_T)\) for a function \(g\) of the form

\[
g(x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2},
\]

where \(\Pi\) is finite Borel complex measure.
Existence and uniqueness.

Proposition 19  
Suppose the validity of the Basic assumption and

\[ \int_0^T \left( \frac{d\kappa_t(0, 1)}{d\rho^S_t} \right)^2 d\rho^S_t < \infty. \]

Then any square integrable variable admits a unique true F-S decomposition.

The proof makes use of a general existence and uniqueness theorem by Monat and Stricker [1995].
Idea.
In agreement with the definition of $\mathcal{D}(\mathcal{A})$, we select $y$ of the form

$$y(t, x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2)x^{z_1} s^{z_2} \lambda(t, z_1, z_2),$$

where $\Pi$ is the same finite complex measure as in the definition of $h$ and $\lambda : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$.

The deterministic equations in the corollary write as
\[
\int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \left\{ \frac{d\lambda(t, z_1, z_2)}{d\rho_t^S} + \lambda(t, z_1, z_2) \frac{d\kappa_t(z_1, z_2)}{d\rho_t^S} \right\} \\
= s \frac{d\kappa_t(0, 1)}{d\rho_t^S} z(t, x, s) \\
\int_{\mathbb{C}^2} d\Pi(z_1, z_2) \lambda(t, z_1, z_2) x^{z_1} s^{z_2+1} \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S} = s^2 z(t, x, s) \\
y(T, \cdot, \cdot) = g.
\]

**Unknown**: $\lambda \Rightarrow$ can be determined through the resolution of an ODE in $t$. 
Theorem 20 (Weak F-S decomposition) Let \( \lambda \) be defined as
\[
\lambda(t, z_1, z_2) = \exp \left( \int_t^T \eta(z_1, z_2, du) \right), \quad \forall (z_1, z_2) \in D/2,
\]
where
\[
\eta(z_1, z_2, t) = \kappa_t(z_1, z_2) - \int_0^t \frac{d\rho_u(z_1, z_2, 0, 1)}{d\rho^S_u} \kappa_{du}(0, 1).
\]

Then, under some technical assumptions, \((Y_0, Z, O)\) is a weak F-S decomposition of \(h\), where
\begin{align*}
Y_t &= \int_{\mathbb{C}^2} \, d\Pi(z_1, z_2) \, X^z_1 \, S^z_2 \, \lambda(t, z_1, z_2), \\
Z_t &= \int_{\mathbb{C}^2} \, d\Pi(z_1, z_2) \, X^z_1 \, S^{z_2-1}_t \, \lambda(t, z_1, z_2) \, \gamma_t(z_1, z_2), \\
O_t &= Y_t - Y_0 - \int_0^t Z_s \, dS_s \quad \text{and} \\
\gamma_t(z_1, z_2) &= \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S}, \quad \forall (z_1, z_2) \in D/2, \, t \in [0, T],
\end{align*}
Proposition 21 (True F-S decomposition)  Under slightly stronger assumptions as in Theorem above, the weak F-S decomposition of

\[ h = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X^{z_1}_T S^{z_2}_T \]

above is a true F-S decomposition. Moreover, if \( h \) is real-valued then the decomposition \( (h_0, Z, O) \) is real-valued and it is therefore the unique F-S decomposition.

Example 22  This statement is a generalization of the results of [Oudjane, Goutte and Russo, 2014] to the case of hedging under basis risk.
7.5 Application 2: diffusion processes

Let \((X, S)\) be a diffusion process with drift \((b_X, b_S)\) and volatility \((\sigma_X, \sigma_S)\).

**Assumption 23** \(b_X, b_S, \sigma_X \text{ and } \sigma_S\) are continuous and globally Lipschitz.

\(g : \mathcal{O} \rightarrow \mathbb{R}\) is continuous.
$\mathcal{A}(y) = \partial_t y + b_S \partial_s y + b_X \partial_x y$ \\
+ $\frac{1}{2} \left\{ |\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} y \right\}$, \\
$\tilde{\mathcal{A}}(y) = |\sigma_S|^2 \partial_s y + \langle \sigma_S, \sigma_X \rangle \partial_x y.$

Example 24 **Goal.** characterize the (weak) F-S decomposition of $h := g(X_T, S_T).$
Theorem 25 (Weak F-S decomposition) We suppose the validity of Assumption \textsuperscript{23}. and that $|\sigma_S|$ is always strictly positive. If $(y, z)$ is a solution of the system

$$
\left\{
\begin{array}{l}
\partial_t y + B \partial_x y + \frac{1}{2} \left( |\sigma_S|^2 \partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2 \langle \sigma_S, \sigma_X \rangle \partial_{sx} y \right) = 0,

y(T, ., .) = g(., .), \text{ where } B = b_X - b_S \frac{\langle \sigma_S, \sigma_X \rangle}{|\sigma_S|^2},

z = \partial_s y + \frac{\langle \sigma_S, \sigma_X \rangle}{|\sigma_S|^2} \partial_x y,
\end{array}
\right.
$$

(10)
such that \( y \in D(A) \), then \((Y_0, Z, O)\) is a weak F-S decomposition of \( g(X_T, S_T) \), where

\[
Y_t = y(t, X_t, S_t), \quad Z_t = z(t, X_t, S_t), \quad O_t = Y_t - Y_0 - \int_0^t Z_s dS_s.
\]

**Remark 26**

1. Under slightly stronger assumption one can give conditions for the existence of a true Föllmer-Schweizer decomposition.

2. Black-Scholes was treated by Hulley and McWalter [2008].
8 Extensions: BSDE vs Pseudo-PDE

Until now we have essentially shown that a solution to a blue Pseudo-PDE provide solutions to BSDEs driven by cadlag martingales.

More problematic is the converse implication. Barrasso and Russo [2017a,b].
Let $E$ be a Polish space. Let $\mathbb{P}^{t,x}$ be a Markov class family of probability measures under which the canonical process $X$ on $D([0, T]; E)$ solves a martingale problem to $\mathcal{D}(\mathcal{A}, \mathcal{A}, \rho)$. Let us denote $M^S := M^{id,t}_s := S_s - x - \int_t^s \mathcal{A}(id)(S_r) d\rho(r)$. We consider $BSDE(f, g(S_T), M)$, i.e.

$$Y_s = g(S_T) + \int_s^T f(r, S_r, Y_r, Z_r) d\rho_r - \int_s^T Z_r dM^S_r$$

$$- (O_T - O_s), s \in [t, T],$$

under $\mathbb{P}^{t,x}$. 
Let us suppose the following.

1. $id \in \mathcal{D}(\mathcal{A})$.
2. $\langle M^S \rangle$ is absolutely continuous with respect to $\rho$.
3. Let us suppose suitable growth condition on $g$ and Lipschitz on $f$. 
"Theorems"

6 Then (11) admits a unique solution \((Y_{t,x}, Z_{t,x}, O_{t,x})\) in some suitable spaces.

6 There is a “unique” couple \((y, z)\) of Borel functions such that \(y(t, x) = Y_{t,x}\), and \(Z_{s,x} = z(s, X_s)\) a.s. under \(\mathbb{P}^{t,x}\).

6 The couple \((y, z)\) is a so called decoupled mild solution of the system.

6 There is a unique decoupled mild solution of Pseudo-PDE\((f, g)\).
Thank you for your attention!
References


Philippe Briand and Ying Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probab. The-


