Error Estimation for the Simulation of Elastic Multibody Systems

Jörg Fehr, Dennis Grunert, Ashish Bhatt, Bernard Haasdonk
Advanced System Development Process

Property validation

Model order reduction

Interdisciplinary system design

Virtual test

System simulation

Physical test

Domain specific design
- Mechanics
- Software
- Electronics

Modeling and analysis

System integration

Hybrid test (e.g. HIL)

Institute of Engineering and Computational Mechanics
University of Stuttgart, Germany
Profs. Eberhard / Hanss / Fehr
error by reduction?

- without error quantification simulation cannot be trusted and potentially gives wrong results
- error known
  - simulation results are certified
  - robust simulation process
  - added value for the decision process
Elastic
Multibody Systems

**Elastic multibody systems**

- **rigid body**
- **bearings and coupling elements**
- **elastic body**
- **discretization**
  - finite element, finite difference,
  - ...  

**elastic multibody system**

- **p bodies**
- **f degrees of freedom**
- **q reaction force**

- **reduction of the elastic degrees of freedom**

- **models are getting larger and more detailed**
  - many degrees of freedom
  - FE-models have to be reduced

- **with the floating frame of reference formulation linear model order reduction is possible**

Institute of Engineering and Computational Mechanics
University of Stuttgart, Germany
Profs. Eberhard / Hanss / Fehr
EMBS: The Floating Frame of Reference Approach

floating frame of reference
dividing the motion into
- nonlinear motion of reference frame $K_i$
- linear elastic deformation with respect to $K_i$

\[ r_k(t) = r_i(t) + R_{ik} + u_k(t) \]

equation of motion of the elastic body
nonlinear equation describes the dynamics of the elastic body

\[ M(q) \cdot \ddot{q} + k(q, \dot{q}, t) = g(q, \dot{q}, t) \]

\[
\begin{bmatrix}
    mI \\
    m\ddot{c}(q) \\
    \mathbf{C}_t(q) \\
    \mathbf{C}_r(q) \\
    \mathbf{M}_e
\end{bmatrix}
\cdot
\begin{bmatrix}
    \ddot{q}_t \\
    \ddot{q}_r \\
    \ddot{q}_e
\end{bmatrix}
+
\begin{bmatrix}
    \mathbf{k}_t \\
    \mathbf{k}_r \\
    \mathbf{k}_e
\end{bmatrix}
=
\begin{bmatrix}
    \mathbf{g}_t \\
    \mathbf{g}_r \\
    \mathbf{g}_e
\end{bmatrix}
\]

coupling to reference frame motion
finite element model

\[ M_e \cdot \ddot{q}_e + D_e \cdot \dot{q}_e + K_e \cdot q_e = h_e \]

I/O aspect of forces and moments
- define input or control matrix \( B_e \)
- define output/observation matrix \( C_e \)
- consider EMBS specifica
  - boundary conditions of ref. frame
  - inertia terms coupling forces

node-fixed
- tangent frame, chord frame

fixed to the center of gravity
- Buckens/Tisserand frame

inertia terms introduce coupling forces
- acceleration of reference frame \( K_i \)
  - leads to elastic deformation

\[ y = C_e \cdot q_e \]

model reduction by projection
finite element model
\[ \mathbf{M}_e \dot{\mathbf{q}}_e + \mathbf{D}_e \mathbf{q}_e + \mathbf{K}_e \mathbf{q}_e = \mathbf{h}_e \]

I/O aspect of forces and moments
- define input or control matrix \( \mathbf{B}_e \)
- define output/observation matrix \( \mathbf{C}_e \)
- consider EMBS specifica
  - boundary conditions of ref. frame
  - inertia terms coupling forces

\[ \mathbf{M}_e \dot{\mathbf{q}}_e + \mathbf{D}_e \mathbf{q}_e + \mathbf{K}_e \mathbf{q}_e = \mathbf{B}_e \mathbf{u}_e \]
\[ \mathbf{y} = \mathbf{C}_e \mathbf{q}_e \]

linear model order reduction
- reduced FE equation of motion
  with \( \text{dim}(\mathbf{q}_e) \ll \text{dim}(\mathbf{q}_e) \), \( \mathbf{q}_e \approx \mathbf{V} \cdot \bar{\mathbf{q}}_e \)
  \[ \bar{\mathbf{M}}_e \ddot{\bar{\mathbf{q}}}_e + \bar{\mathbf{D}}_e \dot{\bar{\mathbf{q}}}_e + \bar{\mathbf{K}}_e \bar{\mathbf{q}}_e = \bar{\mathbf{h}}_e \]
  \[ \bar{\mathbf{M}}_e = \mathbf{V}^T \cdot \mathbf{M}_e \cdot \mathbf{V} \ldots \]
  \[ \bar{\mathbf{h}}_e = \mathbf{V}^T \cdot \mathbf{B}_e \cdot \mathbf{u}_e \]

projection matrix \( \mathbf{V} \in \mathbb{R}^{N \times n} \)

reduction algorithms
- modal truncation
- CMS methods
- input-output based methods: Krylov, Balanced Truncation
  - focus on transfer behavior of the system
  - ‘local’ properties

use linear projection space in nonlinear FFR formulation
\[ \bar{\mathbf{M}}(\mathbf{q}) \cdot \ddot{\mathbf{q}} + \bar{\mathbf{k}}(\mathbf{q}, \dot{\mathbf{q}}, t) = \mathbf{g}(\mathbf{q}, \dot{\mathbf{q}}, t) \]
\[
\begin{bmatrix}
\mathbf{mI}
\mathbf{m c(t)}
\mathbf{C}_t\mathbf{q}
\mathbf{C}_r\mathbf{q}
\bar{\mathbf{K}} \cdot \mathbf{q}_e + \bar{\mathbf{D}} \cdot \dot{\mathbf{q}}_e
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{q}}_t \\
\ddot{\mathbf{q}}_r \\
\mathbf{g}_t \\
\mathbf{g}_r \\
\mathbf{g}_e
\end{bmatrix}
\]
H1: single linear FE body expressed as a linear ODE system
\[ \mathbf{M}_e \cdot \ddot{\mathbf{q}}_e + \mathbf{D}_e \cdot \dot{\mathbf{q}}_e + \mathbf{K}_e \cdot \mathbf{q}_e = \mathbf{h}_e \]

H2: single elastic body in the FFR formulation expressed as a nonlinear ODE system
\[
\begin{bmatrix}
\mathbf{M}_r & \mathbf{M}^T_{er} \\
\mathbf{M}_{er} & \mathbf{M}_e
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{q}}_r \\
\ddot{\mathbf{q}}_e
\end{bmatrix}
+ \begin{bmatrix}
\mathbf{k}_r \\
\mathbf{k}_e
\end{bmatrix}
= \begin{bmatrix}
\mathbf{g}_r \\
\mathbf{g}_e
\end{bmatrix}
\]
linear elastic part

H3: Multiple FE bodies linear ODE systems with \( N_i \) DOF

H4: multiple rigid and elastic bodies are nonlinearly coupled with each other in the FFR formulation resulting in an EMBS

H5: EMBS simulates mechanical part of a multiphysics environment
usage of commercial software

- \( \{M_e, D_e, K_e\} \) e.g. from Ansys
- implementation of error estimator in third party code
automated workflow
standard FE programs
  – to describe elasticity

MOR process in Morembs
  – workhorse for \{linear, parametric\} model reduction at ITM [FehrEtAl17]

\[ \begin{bmatrix} M_r & M_{er}^T \\ M_{er} & M_e \end{bmatrix} \cdot \begin{bmatrix} \ddot{q}_r \\ \ddot{q}_e \end{bmatrix} + \begin{bmatrix} k_r \\ k_e \end{bmatrix} = \begin{bmatrix} g_r \\ g_e \end{bmatrix} \]
Illustrative Example

Anti-Roll bar

FEM-Model (Wallrapp anti-roll bar)
anti-roll bar fixed at node 1
force $F=100 \text{N} \sin(2\pi t)$ at node 20 z-direction
full model 120 dof

\[ M_e \cdot \ddot{q}_e + D_e \cdot \dot{q}_e + K_e \cdot q_e = B_e \cdot u_e \]
\[ y = C_e \cdot q_e \]

Laplace Transform
\[ H(s) = C_e (s^2 M_e + s D_e + K_e)^{-1} B_e \]

- error measured in the frequency domain or in a specific system norm [Panzer14]

Krylov-Based/CMS
- find Hermite rational interpolant \( \overline{H} \), s.t. moments match in specified order at specified points
\[ \overline{H}_{ij}(s) = \frac{\sum_{l=0}^{n-2} a_{ij} s^l}{1+\sum_{k=1}^{n} b_{ij,k} s^k} \]
  s.t.
  \[ \overline{H}(s_k) = H(s_k) \]
  \[ \overline{H}'(s_k) = H'(s_k) \]
  \[ \overline{H}''(s_k) = H''(s_k) \]

- \( \mathcal{H}_2 \)-optimal MOR IRKA
  [GugercinAntoulasBeattie08]
  \[ \max_{t>0} |y(t) - \bar{y}(t)| \leq \|H - \overline{H}\|_{\mathcal{H}_2} \]

balanced truncation / Gramian-matrix based reduction
- representation where a specific importance can be identified for each state
- second-order Gramian matrix on position level
  \[ P_p^\omega = \frac{1}{\pi} \int_{\omega_1}^{\omega_2} L^{-1}(\omega) BB^T L^{-H}(\omega) d\omega \]
  with \( L(\omega) = -\omega^2 M_e + i\omega D_e + K_e \)
- solve Eigenproblem
  \[ (\zeta_i I - P_p) \varphi_i = 0 \]
- large generalized Hankel singular values \( \zeta_i \ i = 1 \ldots n \) remain in reduced system
  \[ \|H - \overline{H}\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=n+1}^{N} \zeta_i \]
error in the time domain

error of different methods with the same model size

error in the frequency domain

\[ \varepsilon(f) = \frac{\| H(f) - \bar{H}(f) \|_F}{\| H(f) \|_F} \]
L2-error estimates in state and frequency space are connected by Parseval type equalities.

POD
- a-priori time-domain error bounds for the state-space error [Volkwein13]
- a-priori error bounds
  - worst case behavior bounds
  - ensure **good approximation independent of setting**
  - individual simulation could be much better than worst case
  - largely overestimating the actual error

**Error Estimators / Time Domain**

- certified RB methods
  - a posteriori error control
    - each special input signal, loading case, parameter, etc.
    - reduced model give additional error information
- ingredients
  - norm of the residual
  - efficiently computed by suitable offline/online decomposition
- provable upper bounds
  - rigorosity / reliability
  - not overestimate the true error
  - effectivity / efficiency
efficient a-posteriorri error estimation

- first order state space system
  \[ \dot{x}(t) = A_s \cdot x(t) + B_s \cdot u \]
  \[ y(t) = C_s \cdot x(t) \]

- reduction by two bi-orthonormal projection matrices \( V_s \) and \( W_s \)

- error \( e_s(t) = x(t) - V_s x(t) \)

- residual \( R_s = A_s \cdot V_s \cdot \bar{x} + B_s \cdot u - V_s \cdot \bar{x} \)

- error equation
  \[ e_s = \Phi(t) \cdot e_{s,0} + \int_0^t \Phi(t - \tau) \cdot R_s(\tau) d\tau \]

- fundamental matrix of the system
  \[ \Phi(t) = e^{A_s(t)} \]

- error bound \( \Delta x(t) \)
  \[ ||e_s(t)||_{G_s} \leq \Delta x(t) \]
  \[ = C_1 ||e_{s,0}||_{G_s} + C_1 \int_0^t ||R_s(\tau)||_{G_s} d\tau \]

  with \( C_1 \geq \max \epsilon ||\Phi(t)||_{G_s} \)

- use scaled matrix norm \( || \cdot ||_{G_s} \)

- induced norm with scaled inner product \( <a,b> = b^T \cdot G \cdot a \)

- because \( x \) consists of \( q_i, \phi_i, v_i, \omega_i \)

- apply this error estimator to second order systems
transformation to first order system
\[
\begin{pmatrix}
\dot{q}_e \\
\ddot{q}_e
\end{pmatrix} =
\begin{bmatrix}
0 & I \\
-M_e^{-1} \cdot K_e & -M_e^{-1} \cdot D_e
\end{bmatrix}
\cdot
\begin{pmatrix}
q_e(t) \\
\dot{q}_e(t)
\end{pmatrix}
+ \begin{pmatrix}
0 \\
B_e
\end{pmatrix} \cdot u(t)
\]

\[
y = \begin{bmatrix} C_e & 0 \\ C_s \end{bmatrix} \cdot \begin{pmatrix} q_e(t) \\ \dot{q}_e(t) \end{pmatrix}
\]

\[
e_s(t) = \begin{bmatrix} e_m(t) \\ \dot{e}_m(t) \end{bmatrix} = \begin{bmatrix} q_e(t) - V \cdot \dot{q}_e \\ \dot{q}_e(t) - V \cdot \ddot{q}_e \end{bmatrix}
\]

\[
R_s(t) = \begin{bmatrix} 0 \\ \ddot{R}_m(t) \end{bmatrix} = \begin{bmatrix} 0 \\ M_e^{-1} \cdot R_m(t) \end{bmatrix}
\]

large over prediction of error

\[
\times 10^8
\]
\[ \Delta x(t) = C_1 \| e_{s,0} \|_{G_s} + C_1 \int_0^t \| R_s(\tau) \|_{G_s} d\tau \]

- extreme value of constant
  \[ C_1 \geq \max_t \| e^{A_s(t)} \|_{G_s} \]

simple mass spring damper system

\[
\begin{align*}
\begin{bmatrix}
\Phi_{11}(t) & \Phi_{12}(t) \\
\Phi_{21}(t) & \Phi_{22}(t)
\end{bmatrix} \cdot x_0
\end{align*}
\]

split fundamental matrix

\[ \Phi_{21}(t) \] represent connection initial displacement to velocity

- single error bound for both state variables, \( q_e \) and \( \dot{q}_e \)

\[ \Delta x(t) = C_1 \| e_{s,0} \|_{G_s} + C_1 \int_0^t \| R_s(\tau) \|_{G_s} d\tau \]

\[ \| e^{A_s(t)} \| \] does not decrease monotonically
error estimator delivers impractical results for EMBS
modifed error estimator [FehrEtAl14]
\[
\begin{bmatrix}
e_{m}(t) \\
\dot{e}_{m}(t)
\end{bmatrix} = \begin{bmatrix}
\Phi_{11}(t) & \Phi_{12}(t) \\
\Phi_{21}(t) & \Phi_{22}(t)
\end{bmatrix} \cdot \begin{bmatrix}
e_{m,0} \\
\dot{e}_{m,0}
\end{bmatrix} + \int_{0}^{t} \begin{bmatrix}
\Phi_{11}(t - \tau) & \Phi_{12}(t - \tau) \\
\Phi_{21}(t - \tau) & \Phi_{22}(t - \tau)
\end{bmatrix} \cdot \begin{bmatrix}
0 \\
\overline{R}_{m}(t)
\end{bmatrix} d\tau
\]
relevant \( e_{m}(t) = \Phi_{11}(t) \cdot e_{m,0} + \Phi_{12}(t) \cdot \dot{e}_{m,0} + \int_{0}^{t} \Phi_{12}(t - \tau) \cdot \overline{R}_{m}(t) d\tau \)
term \( \Phi_{21}(t) \), which causes large hump no longer required
three new error estimators \( \Delta_{q}(t) = C_{11} \| e_{m,0} \|_{G_{M}} + C_{12} \| \dot{e}_{m,0} \|_{G_{M}} \)
\[ + C_{12} \int_{0}^{t} \| \overline{R}_{m}(\tau) \|_{G_{M}} d\tau \]
computation time is saved significantly with approximation of fundamental matrix
56.25 s vs. 0.077 s
offline/online decomposition for calculation of residual
H1: single linear FE body expressed as a linear ODE system

\[
\mathbf{M}_e \cdot \ddot{\mathbf{q}}_e + \mathbf{D}_e \cdot \dot{\mathbf{q}}_e + \mathbf{K}_e \cdot \mathbf{q}_e = \mathbf{h}_e
\]

H2: single elastic body in the FFR formulation expressed as a nonlinear ODE system

\[
\begin{bmatrix}
\mathbf{M}_r & \mathbf{M}^T_{er} \\
\mathbf{M}_{er} & \mathbf{M}_e
\end{bmatrix}
\begin{bmatrix}
\ddot{\mathbf{q}}_r \\
\ddot{\mathbf{q}}_e
\end{bmatrix}
+
\begin{bmatrix}
\mathbf{k}_r \\
\mathbf{k}_e
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{g}_r \\
\mathbf{g}_e
\end{bmatrix}
\]

linear elastic part

H3: Multiple FE bodies linear ODE systems with \( N_i \) DOF

H4: multiple rigid and elastic bodies are nonlinearly coupled with each other in the FFR formulation resulting in an EMBS

H5: EMBS simulates mechanical part of a multiphysics environment
very slender beam

finite strain shell

penalty method relate independent rotational degrees of freedom with in-plane components

SHELL181 / SOLID185 / PLANE182

2D-modeling of solids

plane element or axisymmetric element

33 nodes per body, 20 elements

Two Link Flexible Arm

10N \cdot \sin(2 \cdot \pi \cdot t)
- automated workflow
- standard FE programs
  - to describe elasticity

**Workflow for Engineers**

- model reduction
  - preprocessing
  - MOR process in Morembs
    - workhorse for {linear, parametric} model reduction at ITM [FehrEtAl17]

- multibody dynamics
  - simulation
  - in-house EMBS codes
    - combines the benefits of numerical computation (Matlab) and computer algebra (Maple/MuPAD)
    - equation of motion derived in symbolic form
    \[
    \begin{bmatrix}
    \mathbf{M}_r & \mathbf{M}_e^T \\
    \mathbf{M}_{er} & \mathbf{M}_e
    \end{bmatrix}
    \begin{bmatrix}
    \ddot{\mathbf{q}}_r \\
    \ddot{\mathbf{q}}_e
    \end{bmatrix}
    +
    \begin{bmatrix}
    \mathbf{k}_r \\
    \mathbf{k}_e
    \end{bmatrix}
    =
    \begin{bmatrix}
    \mathbf{g}_r \\
    \mathbf{g}_e
    \end{bmatrix}
    \]
radau5Mex integration $t = [0, 2]$s
- implicit Runge-Kutta method of order 5 (Radau IIA) for problems of the form $M y' = f(x, y)$ with possibly singular matrix $M$

sim. time (Intel Xeon E3-1245 3.30 GHz, RAM: 8 GB DDR3-1333)
- full system: $\sim 20$ min
  - PLANE182 model
- red. system: $\sim 37$ s
  - 10 Rational Krylov modes per beam
Sensitivity of Error Estimation

- SHELL 181 results

**Method 2**

- **error bound** $\Delta_q$
- **error bounds** $\sim \Delta_q$

<table>
<thead>
<tr>
<th>time [s]</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>50</td>
<td>150</td>
</tr>
<tr>
<td>1</td>
<td>50</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>100</td>
<td>150</td>
<td>250</td>
</tr>
</tbody>
</table>

- **Large overestimation**

$$e_m(t) = \Phi_{11}(t) \cdot e_{m,0} + \Phi_{12}(t) \cdot \dot{e}_{m,0}$$

$$+ \int_0^t \Phi_{12}(t-\tau) \cdot \frac{M_e^{-1} \cdot R_m(t)}{\tilde{R}_m} d\tau$$

- $R_m(t)$ small $\rightarrow$ multiplication with $M_e^{-1} \rightarrow \tilde{R}_m(t)$ large

- **Condition of mass matrix** $M_e$
  - shells: $10^{14} - 10^{18}$
  - solids: $\sim 100$
  - depends on material, geometry, meshing

- **Scaling** $G_M = M_e^2$ and modal transformation improves results

- **SHELL 181**
  - incorrect modeling approach
  - problem was not well formulated
  - bad input $\rightarrow$ bad output
  - error estimator
    - detects wrong results
- SOLID 185 element
- reduction on dominant eigenspace of second order Gramian matrix $P_p$ (7 modes)
- error bounds are larger than exact error
- conservative estimation!

![Graph](method 3)

- Integration Result
- $\Delta_q$ Error Bound
- $\sim \Delta_q$ Error Bound
error estimator can be used with any MOR technique
- IRKA algorithm
  - local H2-optimality
    - global problem
    - no inclusion of pre-knowledge
  - expansion points distributed over a wide range
error estimator can be used with any MOR technique

CMS-Gram
[HolzwarthEberhard15]

- component mode synthesis
- Gramian based approximation of inner degree of freedoms
error estimator can be used with any MOR technique
rational Krylov (Hermite based reduction)
\( s_k = 0 + i \cdot 0:1:14 \cdot 35/13 \)
\( P_p^\omega \) by a POD-Greedy approach [FehrEtAl12]
\( \omega = [0, 30 \text{ Hz}] \)
nice results
**Sensitivity Analysis with PLANE 182**

[Meral17] PLANE 182
- Consistency with linear elasticity in RBMatlab
- Error bounds for reduction sizes around 10 modes
- POD $\mathbf{P}_p^\omega \omega = [1, 1500 \text{ Hz}]$
  - Standardized settings
  - Deflation tolerance

---

**Krylov approach**
- Distance between shifts

- Krylov and POD reduction methods provide the best results
- Small size of the system is needed to receive small error bounds

---

![Graph showing max. output error bound vs. number of snapshots](image)
Sensitivity Analysis with PLANE 182

modal reduction

balanced reduction

Institute of Engineering and Computational Mechanics
University of Stuttgart, Germany
Profs. Eberhard / Hanss / Fehr
- Influence of input force

- Influence of slender ratio (length/height of beam)

- Output error bounds increase with raising frequency and raising amplitude
- Improved behavior due to reorthogonalization of bases [BuhrEtAl14]

- Speedup of error estimator
  - Small reduced system
  - Error estimation takes as long as simulation

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Integration</th>
<th>$\Delta_y$</th>
<th>$\tilde{\Delta}_y$</th>
<th>$\tilde{C}<em>{11}(t), \tilde{C}</em>{12}(t), C_{11}, C_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>90 (full)</td>
<td>6.16 s</td>
<td>0.070 s</td>
<td>0.078 s</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>5.79 s</td>
<td>0.068 s</td>
<td>0.072 s</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>4.86 s</td>
<td>0.066 s</td>
<td>0.068 s</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>3.96 s</td>
<td>0.063 s</td>
<td>0.065 s</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>3.57 s</td>
<td>0.060 s</td>
<td>0.063 s</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>2.63 s</td>
<td>0.059 s</td>
<td>0.060 s</td>
<td></td>
</tr>
</tbody>
</table>

- Approx. norm of matrix exponential

$$
\| \Phi \|_G = \max_{z \neq 0} \frac{\| \Phi z \|_G}{\| z \|_G} = \max_{z \neq 0} \frac{\| G^{1/2} \Phi z \|_2}{\| G^{1/2} z \|_2} \\
\omega = G^{1/2} z \max_{w \neq 0} \frac{\| G^{1/2} \Phi G^{-1/2} w \|_2}{\| w \|_2} = \| G^{1/2} \Phi G^{-1/2} \|_2 \\
\leq \| G^{1/2} \Phi G^{-1/2} \|_{\text{Fro}} \text{ and} \\
\leq \sqrt{\| G^{1/2} \Phi G^{-1/2} \|_1 \| G^{1/2} \Phi G^{-1/2} \|_\infty}\)
error estimator
\[
\Delta q(t) = C_{11} \|e_{m,0}\|_{G_M} + \\
C_{12} \|\dot{e}_{m,0}\|_{G_M} + C_{12} \int_0^t \|\widetilde{R}_m(\tau)\|_{G_M} d\tau
\]
written as differential equation
\[
\dot{\Delta} q(t) = C_{12} \|\widetilde{R}_m(\tau)\|_{G_M} \\
\Delta q(t_0) = C_{11} \|e_{m,0}\|_{G_M} + \\
C_{12} \|\dot{e}_{m,0}\|_{G_M}
\]
\(\widetilde{R}_m(\tau)\) depends on \(\bar{x}_e\)
add the on \(\bar{x}_e\) depending ODE to Neweul-M²
possible eqm_nonlin_ss.m is given in symbolic form
intrusive approach
calculating error estimator after solver finished with a time step
- hook OUTPUTFcn of Matlab ODESET
minor modification to Neweul-M² core
hook allows solver to stop if error estimator too high
user needs to supply all time steps
- allow optimal preallocation of variables
blue print to other software packages
Summary

- certified MOR adds value
  - a posteriori error bounds in the time domain
  - error estimator from RB community
  - approximation of the residual
- large hump of the fundamental matrix norm $\| \Phi(t) \|$ due to the large submatrix $\Phi_{21}(t)$
- modified error estimator for second order systems
  - does not require this submatrix
- offline/online decomposition for calculation of residual $\| \tilde{R}_m(\tau) \|^3_{G_M}$
- error estimators are sensitive to numerical noise

Outlook

- application to multiphysics system
  - coupled system
- improvement of workflow,
  - automatic implementation
- snapshot based reduction
- search for refined error estimators
Call for Papers

IUTAM Symposium on
Model Order Reduction of Coupled Systems (MORCOS 2018)

Stuttgart, Germany
May 22 – 25, 2018

www.itm.uni-stuttgart.de/iutam2018


