Balanced truncation model reduction: algorithms and applications Part III

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Outline

Part I (Tuesday)

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques

Part II (Wednesday)

- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems

Part III (today)

- Balanced truncation for parametric systems
- Related topics and open problems

Outline

- Model order reduction problem
- Balanced truncation model reduction
- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
  - reduced basis method for parametric Lyapunov equations
  - parametric balanced truncation
- Related topics and open problems

Model reduction problem

Given a large-scale parametric control system

$$u \longrightarrow \boxed{\begin{aligned} E(p)\dot{x}(t,p) &= A(p)x(t,p) + B(p)u(t) \\ y(t,p) &= C(p)x(t,p) + D(p)u(t) \end{aligned}} \longrightarrow y$$

where $E(p), A(p) \in \mathbb{R}^{n\times n},\; B(p) \in \mathbb{R}^{n\times m},\; C(p) \in \mathbb{R}^{q\times n},\; D(p) \in \mathbb{R}^{q\times m}$,
$p \in \mathbb{P} \subset \mathbb{R}^d$, find a reduced-order model

$$u \longrightarrow \boxed{\begin{aligned} \widetilde{E}(p)\dot{\widetilde{x}}(t,p) &= \widetilde{A}(p)\widetilde{x}(t,p) + \widetilde{B}(p)u(t) \\ \widetilde{y}(t,p) &= \widetilde{C}(p)\widetilde{x}(t,p) + \widetilde{D}(p)u(t) \end{aligned}} \longrightarrow \widetilde{y}$$

where $\widetilde{E}(p), \widetilde{A}(p) \in \mathbb{R}^{\ell\times\ell},\; \widetilde{B}(p) \in \mathbb{R}^{\ell\times m},\; \widetilde{C}(p) \in \mathbb{R}^{q\times\ell},\; \widetilde{D}(p) \in \mathbb{R}^{q\times m}$.

Balanced truncation algorithm

1. Solve the parametric Lyapunov equations

$$A(p)X(p)E^T(p) + E(p)X(p)A^T(p) = -B(p)B^T(p),$$
$$A^T(p)Y(p)E(p) + E^T(p)Y(p)A(p) = -C^T(p)C(p)$$

for $X(p) \approx \widetilde{R}(p)\widetilde{R}^T(p)$ and $Y(p) \approx \widetilde{L}(p)\widetilde{L}^T(p)$.

2. Compute the SVD

$$\widetilde{L}^T(p)E(p)\widetilde{R}(p) = [\,U_1(p),\ U_2(p)\,]\begin{bmatrix} \Sigma_1(p) & \\ & \Sigma_2(p) \end{bmatrix}\begin{bmatrix} V_1^T(p) \\ V_2^T(p) \end{bmatrix}.$$

3. Compute $\big(\widetilde{E}(p),\ \widetilde{A}(p),\ \widetilde{B}(p),\ \widetilde{C}(p),\ \widetilde{D}(p)\big)$ with

$$\widetilde{E}(p) = W^T(p)E(p)T(p), \qquad \widetilde{A}(p) = W^T(p)A(p)T(p),$$
$$\widetilde{B}(p) = W^T(p)B(p), \qquad \widetilde{C}(p) = C(p)T(p), \qquad \widetilde{D}(p) = D(p),$$
$$W(p) = \widetilde{L}(p)U_1(p)\Sigma_1^{-1/2}(p), \quad T(p) = \widetilde{R}(p)V_1(p)\Sigma_1^{-1/2}(p).$$

Parametric Lyapunov equations

- Lyapunov equation:

$$-A(p)X(p)E^T(p) - E(p)X(p)A^T(p) = B(p)B^T(p),$$

where $E(p), A(p), X(p) \in \mathbb{R}^{n\times n}$, $B(p) \in \mathbb{R}^{n\times m}$

- Operator equation:

$$\mathcal{L}_p(X(p)) = B(p)B^T(p),$$

where $\mathcal{L}_p \,:\, \mathbb{S}_+ \longrightarrow \mathbb{S}_+$ is a *Lyapunov operator*

- Linear system:

$$\boldsymbol{L}(p)\,\boldsymbol{x}(p) = \boldsymbol{b}(p),$$

where $\boldsymbol{L}(p) = -E(p)\otimes A(p) - A(p)\otimes E(p) \in \mathbb{R}^{n^2\times n^2}$,

$$\boldsymbol{x}(p) = \mathrm{vec}(X(p)),\ \ \boldsymbol{b}(p) = \mathrm{vec}(B(p)B^T(p)) \in \mathbb{R}^{n^2}$$

Reduced basis method: idea

Reduced basis method for $\mathcal{L}_p(X(p)) = B(p)B^T(p)$

- Snapshots collection:
  construct the reduced basis matrix $V_k = [\,Z_1, \,\ldots\,, \,Z_k\,]$, where
  $X(p_j) \approx Z_j Z_j^T$ solves $\mathcal{L}_{p_j}(X(p_j)) = B(p_j)B(p_j)^T$
- Galerkin projection:
  approximate the solution $X(p) \approx V_k \widetilde{X}(p) V_k^T$, where $\widetilde{X}(p)$
  solves $-\widetilde{A}(p)\widetilde{X}(p)\widetilde{E}^T(p) - \widetilde{E}(p)\widetilde{X}(p)\widetilde{A}^T(p) = \widetilde{B}(p)\widetilde{B}^T(p)$
  with $\widetilde{E}(p) = V_k^T E(p) V_k$, $\widetilde{A}(p) = V_k^T A(p) V_k$, $\widetilde{B}(p) = V_k^T B(p)$

Questions

- How to choose the parameters $p_1, \ldots, p_k$?
- How to estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$?
- How to make the computations efficient?

Error estimation

Goal: estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$

Residual $\mathcal{R}_k(p) := B(p)B^T(p) - \mathcal{L}_p\big(V_k \widetilde{X}(p) V_k^T\big) = \mathcal{L}_p\big(\mathcal{E}_k(p)\big)$

- Error estimate

$$\|\mathcal{E}_k(p)\|_F \le \|\mathcal{L}_p^{-1}\|_F \|\mathcal{R}_k(p)\|_F = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)}$$

with $\alpha(p) := \|\mathcal{L}_p^{-1}\|_F^{-1} = \inf\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\min}\big(\mathbf{L}(p)\big)$

- Effectivity of the error estimator

$$1 \le \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)\|\mathcal{E}_k(p)\|_F} = \frac{\|\mathcal{L}_p(\mathcal{E}_k(p)\|_F}{\alpha(p)\|\mathcal{E}_k(p)\|_F} \le \frac{\|\mathcal{L}_p\|_F}{\alpha(p)} = \frac{\gamma(p)}{\alpha(p)}$$

with $\gamma(p) := \|\mathcal{L}_p\|_F = \sup\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\max}\big(\mathbf{L}(p)\big)$

Error estimation

Goal: estimate the error $\mathcal{E}_k(p) = X(p) - V_k \widetilde{X}(p) V_k^T$

Residual $\mathcal{R}_k(p) := B(p)B^T(p) - \mathcal{L}_p\big(V_k \widetilde{X}(p) V_k^T\big) = \mathcal{L}_p\big(\mathcal{E}_k(p)\big)$

- Error estimate

$$\|\mathcal{E}_k(p)\|_F \le \|\mathcal{L}_p^{-1}\|_F \|\mathcal{R}_k(p)\|_F = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha(p)} \le \frac{\|\mathcal{R}_k(p)\|_F}{\alpha_{LB}(p)} =: \Delta_k(p)$$

with $\alpha(p) := \|\mathcal{L}_p^{-1}\|_F^{-1} = \inf\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\min}\big(\mathbf{L}(p)\big) \ge \alpha_{LB}(p)$

- Effectivity of the error estimator

$$1 \le \frac{\Delta_k(p)}{\|\mathcal{E}_k(p)\|_F} = \frac{\|\mathcal{R}_k(p)\|_F}{\alpha_{\mathrm{LB}}(p)\|\mathcal{E}_k(p)\|_F} \le \frac{\gamma(p)}{\alpha_{LB}(p)} \le \frac{\gamma_{UB}(p)}{\alpha_{LB}(p)}$$

with $\gamma(p) := \|\mathcal{L}_p\|_F = \sup\limits_{\|X\|_F=1} \|\mathcal{L}_p(X)\|_F = \sigma_{\max}\big(\mathbf{L}(p)\big) \le \gamma_{UB}(p)$

Construction of the reduced basis

Greedy algorithm

Input: tolerance tol, training set $\mathbb{P}_{\rm train} \subset \mathbb{P}$, initial parameter $p_1 \in \mathbb{P}$

- Solve $\mathcal{L}_{p_1}\big(X(p_1)\big) = B(p_1)B^T(p_1)$ for $X(p_1) \approx Z_1 Z_1^T$, $Z_1 \in \mathbb{R}^{n\times r_1}$
- Set $k = 2$, $\Delta_1^{\max} = 1$ and $V_1 = Z_1$
- while $\Delta_{k-1}^{\max} \geq tol$

  $p_k = \arg \max\limits_{p\in\mathbb{P}_{\rm train}} \Delta_{k-1}(p)$ % $\Delta_{k-1}(p) = \dfrac{\|\mathcal{R}_{k-1}(p)\|_F}{\alpha_{LB}(p)}$

  $\Delta_k^{\max} = \Delta_{k-1}(p_k)$

  solve $\mathcal{L}_{p_k}\big(X(p_k)\big) = B(p_k)B^T(p_k)$ for $X(p_k) \approx Z_k Z_k^T$, $Z_k \in \mathbb{R}^{n\times r_k}$

  $V_k = [\,V_{k-1},\ Z_k\,]$

  $k \leftarrow k+1$

  end

Offline-online decomposition

Assumption: affine parameter dependence

$$E(p) = \sum_{i=1}^{n_E} \theta_i^E(p) E_i, \quad A(p) = \sum_{i=1}^{n_A} \theta_i^A(p) A_i, \quad B(p) = \sum_{i=1}^{n_B} \theta_i^B(p) B_i$$

$$\hookrightarrow \quad \mathcal{L}_p(X) = \sum_{i=1}^{n_E} \sum_{j=1}^{n_A} \theta_i^E(p)\, \theta_j^A(p)\, \mathcal{L}_{ij}(X), \quad \mathcal{L}_{ij}(X) = -A_j X E_i^T - E_i X A_j^T,$$

$$B(p)B^T(p) = \sum_{i=1}^{n_B} \sum_{j=1}^{n_B} \theta_i^B(p)\, \theta_j^B(p)\, B_i B_j^T$$

Offline: compute the reduced basis matrix $V_k = [Z_1, \ldots, Z_k] \in \mathbb{R}^{n \times r}$.

Online: for $p \in \mathbb{P}$, compute $X(p) \approx V_k \widetilde{X}(p) V_k^T$, where $\widetilde{X}(p)$ solves

$$-\widetilde{A}(p)\widetilde{X}(p)\widetilde{E}^T(p) - \widetilde{E}(p)\widetilde{X}(p)\widetilde{A}^T(p) = \widetilde{B}(p)\widetilde{B}^T(p)$$

with

$$\widetilde{E}(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_k^T E_j V_k, \quad \widetilde{A}(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_k^T A_j V_k, \quad \widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) V_k^T B_j.$$

Computation of the residual norm

$$\|\mathcal{R}_k(p)\|_F^2 = \|B(p)B^T(p) - \mathcal{L}_p\big(V_k\widetilde{X}(p)V_k^T\big)\|_F^2$$

$$= \sum_{i,j=1}^{n_B}\sum_{f,g=1}^{n_B}\theta_{ijfg}^{B}(p)\,\mathrm{trace}\big((B_i^TB_f)(B_g^TB_j)\big)$$

$$+\ 4\sum_{i,j=1}^{n_B}\sum_{f=1}^{n_E}\sum_{g=1}^{n_A}\theta_{ijfg}^{AEB}(p)\,\mathrm{trace}\big(B_i^T(E_fV_k)\widetilde{X}(p)(A_gV_k)^TB_j\big)$$

$$+\ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A}\theta_{ijfg}^{AE}(p)\,\mathrm{trace}\big((E_fV_k)^T(E_iV_k)\widetilde{X}(p)(A_jV_k)^T(A_gV_k)\widetilde{X}(p)\big)$$

$$+\ 2\sum_{i,f=1}^{n_E}\sum_{j,g=1}^{n_A}\theta_{ijfg}^{AE}(p)\,\mathrm{trace}\big((E_fV_k)^T(A_jV_k)\widetilde{X}(p)(E_iV_k)^T(A_gV_k)\widetilde{X}(p)\big)$$

with $\theta_{ijfg}^{B}(p) = \theta_i^B(p)\theta_j^B(p)\theta_f^B(p)\theta_g^B(p)$, $\quad\theta_{ijfg}^{AEB}(p) = \theta_i^B(p)\theta_j^B(p)\theta_f^E(p)\theta_g^A(p)$,

$\theta_{ijfg}^{AE}(p) = \theta_i^E(p)\theta_j^A(p)\theta_f^E(p)\theta_g^A(p)$.

Error estimation: min-θ approach

Assumption: $E(p) = E^T(p) > 0$, $A(p) + A^T(p) < 0$ for all $p \in \mathbb{P}$

(e.g., $\theta_i^E(p) > 0$, $E_i = E_i^T \geq 0$, $\bigcap \ker(E_i) = \{0\}$ and

$\theta_i^A(p) > 0$, $A_i + A_i^T \leq 0$, $\bigcap \ker(A_i + A_i^T) = \{0\}$)

Let $\hat{p} \in \mathbb{P}$ and

$$\theta_{\min}^{\hat{p}}(p) = \min_{\substack{i=1,\ldots,n_E \\ j=1,\ldots,n_A}} \frac{\theta_i^E(p)\theta_j^A(p)}{\theta_i^E(\hat{p})\theta_j^A(\hat{p})}, \qquad \theta_{\max}^{\hat{p}}(p) = \max_{\substack{i=1,\ldots,n_E \\ j=1,\ldots,n_A}} \frac{\theta_i^E(p)\theta_j^A(p)}{\theta_i^E(\hat{p})\theta_j^A(\hat{p})}.$$

Then $\alpha(p) \geq \theta_{\min}^{\hat{p}}(p)\, \lambda_{\min}\big(-A(\hat{p}) - A^T(\hat{p})\big)\, \lambda_{\min}\big(E(\hat{p})\big) =: \alpha_{LB}(p)$,

$$\gamma(p) \leq \theta_{\max}^{\hat{p}}(p)\, \lambda_{\max}\big(-A(\hat{p}) - A^T(\hat{p})\big)\, \lambda_{\max}\big(E(\hat{p})\big) =: \gamma_{UB}(p)$$

for all $p \in \mathbb{P}$. [Son/St.'17]

Parametric balanced truncation

Offline phase: compute the reduced basis matrices V_X and V_Y for the controllability and observability Lyapunov equations; compute all parameter-independent matrices.

Online phase: for given $p \in \mathbb{P}$,

- solve the reduced Lyapunov equations

$$-\widetilde{A}_X(p)\widetilde{X}(p)\widetilde{E}_X^T(p) - \widetilde{E}_X(p)\widetilde{X}(p)\widetilde{A}_X^T(p) = \widetilde{B}(p)\widetilde{B}^T(p),$$

$$-\widetilde{A}_Y^T(p)\widetilde{Y}(p)\widetilde{E}_Y(p) - \widetilde{E}_Y^T(p)\widetilde{Y}(p)\widetilde{A}_Y(p) = \widetilde{C}^T(p)\widetilde{C}(p)$$

with $\widetilde{E}_X(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_X^T E_j V_X, \quad \widetilde{A}_X(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_X^T A_j V_X,$

$$\widetilde{E}_Y(p) = \sum_{j=1}^{n_E} \theta_j^E(p) V_Y^T E_j V_Y, \quad \widetilde{A}_Y(p) = \sum_{j=1}^{n_A} \theta_j^A(p) V_Y^T A_j V_Y,$$

$$\widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) V_X^T B_j, \quad \widetilde{C}(p) = \sum_{j=1}^{n_C} \theta_j^C(p) C_j V_Y.$$

Parametric balanced truncation

$\hookrightarrow$ Gramians $X(p) \approx V_X \widetilde{X}(p) V_X^T = V_X Z_X(p) Z_X^T(p) V_X^T$

$$Y(p) \approx V_Y \widetilde{Y}(p) V_Y^T = V_Y Z_Y(p) Z_Y^T(p) V_Y^T$$

- Compute the SVD

$$\begin{aligned} Z_Y^T(p) V_Y^T E(p) V_X Z_X(p) &= \sum_{j=1}^{n_E} \theta_j^E(p) Z_Y^T(p) V_Y^T E_j V_X Z_X(p) \\ &= [U_1(p),\, U_2(p)] \begin{bmatrix} \Sigma_1(p) & 0 \\ 0 & \Sigma_2(p) \end{bmatrix} \begin{bmatrix} V_1^T(p) \\ V_2^T(p) \end{bmatrix}. \end{aligned}$$

- Compute the reduced model $\big(\widetilde{E}(p), \widetilde{A}(p), \widetilde{B}(p), \widetilde{C}(p), D(p)\big)$ with

$$\widetilde{E}(p) = \sum_{j=1}^{n_E} \theta_j^E(p) W^T(p) V_Y^T E_j V_X T(p), \quad \widetilde{B}(p) = \sum_{j=1}^{n_B} \theta_j^B(p) W^T(p) V_Y^T B_j,$$

$$\widetilde{A}(p) = \sum_{j=1}^{n_A} \theta_j^A(p) W^T(p) V_Y^T A_j V_X T(p), \quad \widetilde{C}(p) = \sum_{j=1}^{n_C} \theta_j^C(p) C_j V_X T(p),$$

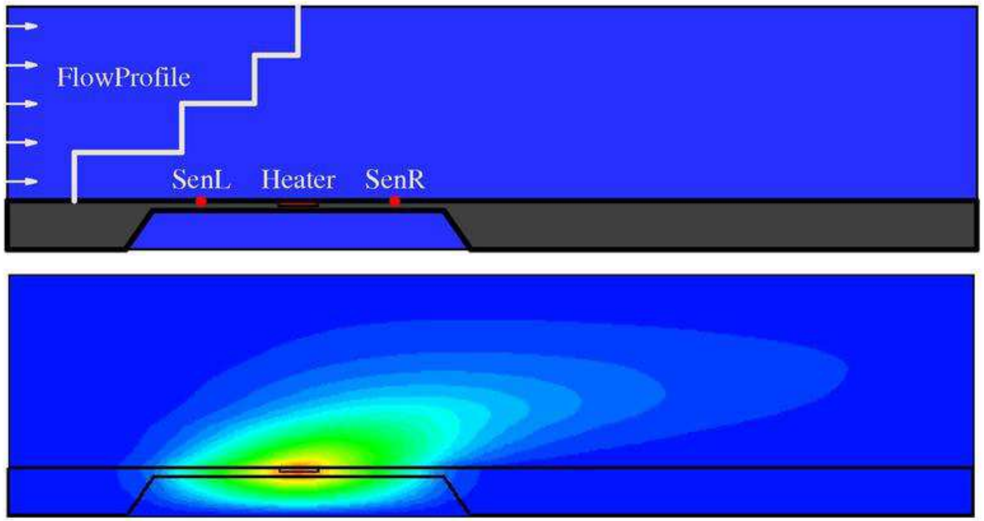
$$T(p) = Z_X(p) V_1(p) \Sigma_1(p)^{-1/2}, \quad W(p) = Z_Y(p) U_1(p) \Sigma_1(p)^{-1/2}.$$

Properties

- Preservation of stability
- Computable error bounds
- Approximation does not rely on solution snapshots and is independent of the training input
- Other error estimation techniques can be used (e.g., successive constraints method)
- Reduced basis method for parametric Riccati equations

[Haasdonk/Schmidt'15]

Example: anemometer



Mathematical model:

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot \kappa \nabla T - \rho c v \cdot \nabla T + \dot{q}$$

boundary / initial conditions

FEM model:

$$\begin{aligned} E(p)\,\dot{x} &= A(p)\,x + B\,u \\ y &= C\,x \end{aligned}$$

with $E(p) = E_1 + p_1 E_2, \quad A(p) = A_1 + p_2 A_2 + p_3 A_3 \in \mathbb{R}^{n\times n}, \quad p = \begin{bmatrix} c_f \\ \kappa_f \\ c_f v \end{bmatrix},$

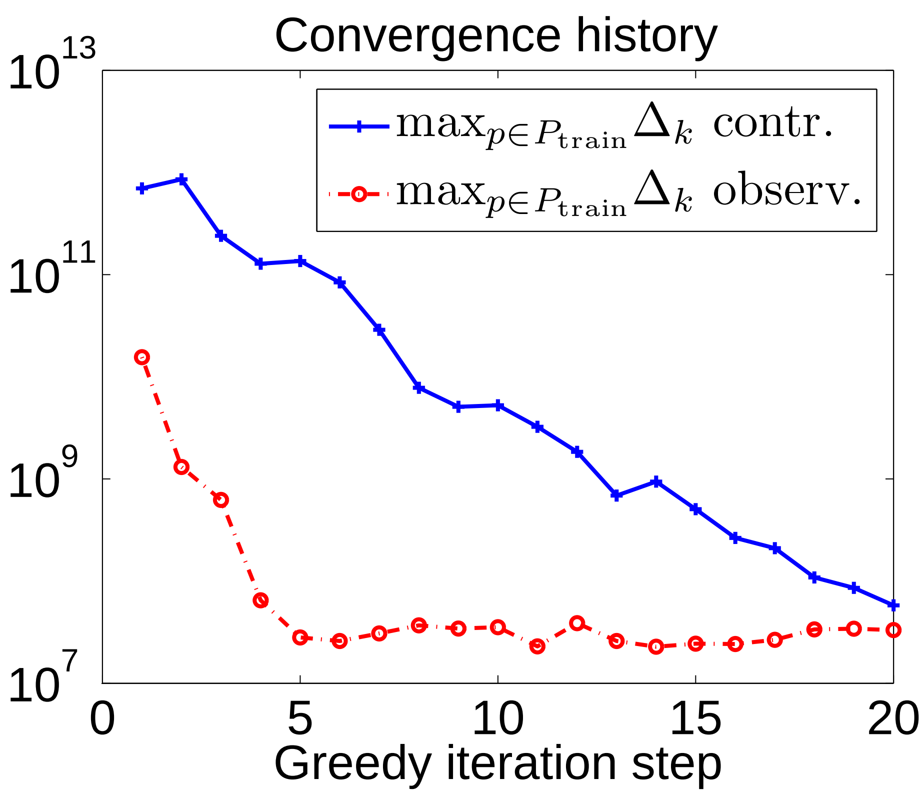
$B, C^T \in \mathbb{R}^n, \quad n = 29008$

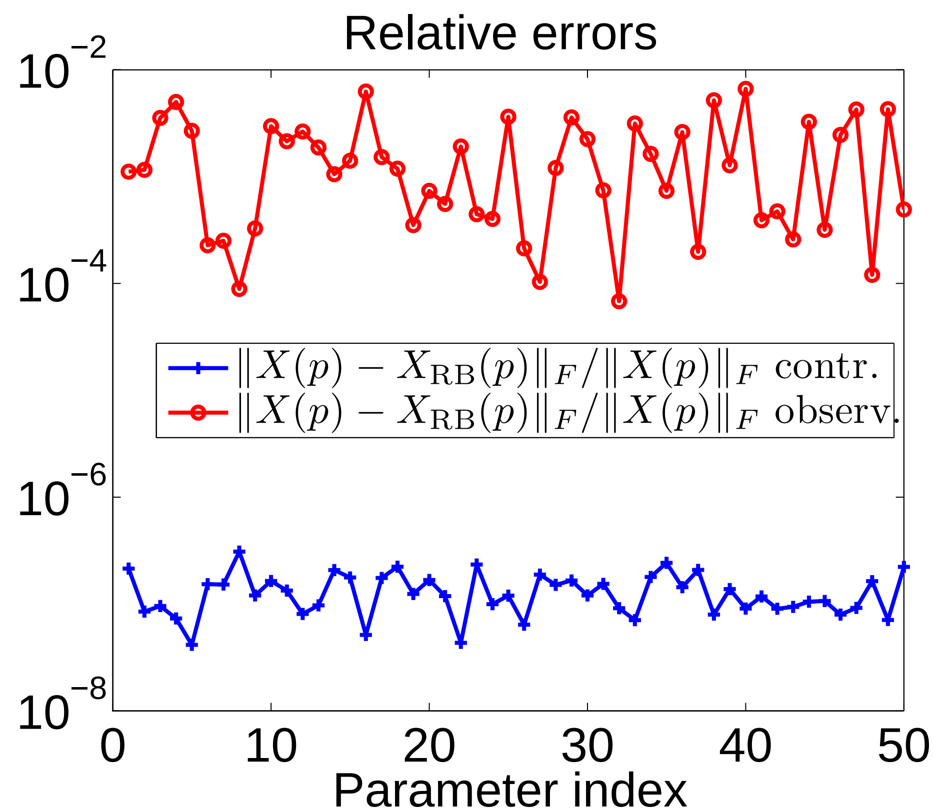
[Moosmann'07, MOR Wiki]

Example: anemometer

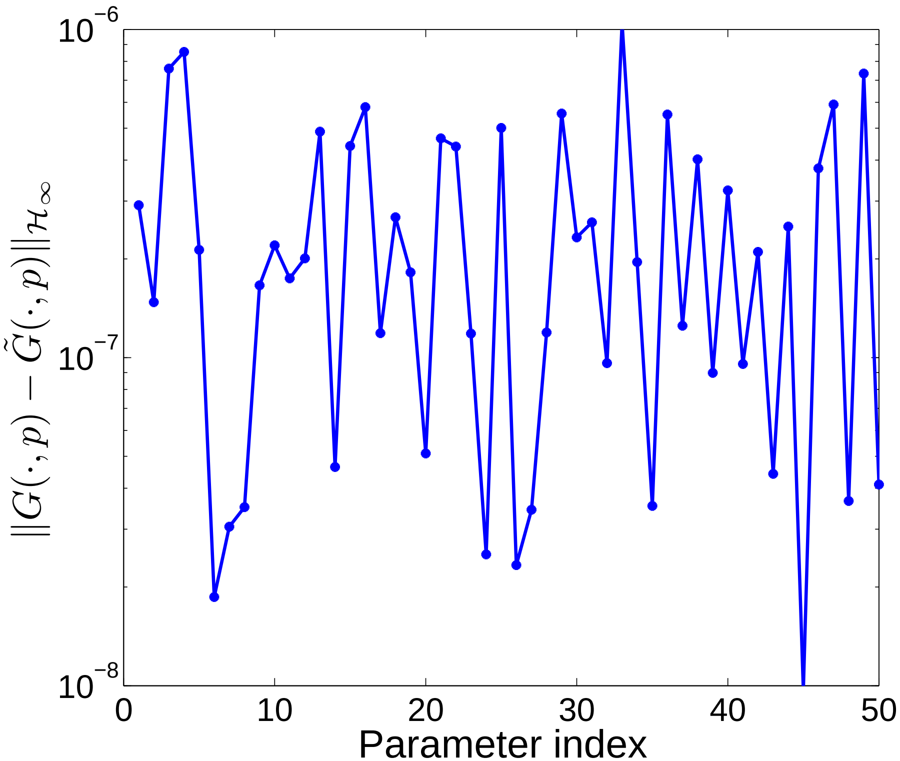
$\mathbb{P}_{\text{train}} = \{10000$ random points$\}$, $\;20$ Greedy iterations

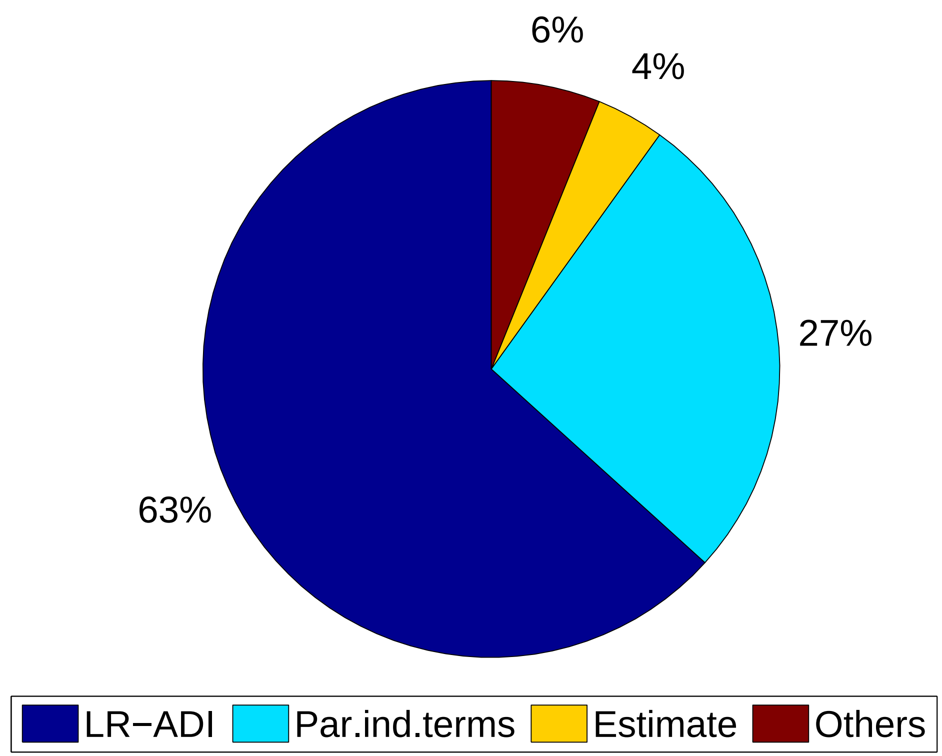
$\mathbb{P}_{\text{test}} = \{50$ random points$\}$





Example: anemometer





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- Balancing-related model reduction techniques
- Balanced truncation for differential-algebraic equations
- Balanced truncation for second-order systems
- Balanced truncation for parametric systems
- Related topics and open problems
  - Balanced truncation for linear time-varying systems
  - Balanced truncation for bilinear systems
  - Balanced truncation for quadratic-bilinear systems
  - Balanced truncation for nonlinear systems
  - Balanced truncation for infinite-dimensional systems

BT for linear time-varying systems

- For linear time-varying systems

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\,u(t), \qquad t \in [0,T], \\ y(t) &= C(t)\,x(t) + D(t)\,u(t), \end{aligned}$$

the Gramians satisfy the Lyapunov differential equations

$$\dot{X}(t) = A(t)\,X(t) + X(t)A^T(t) + B(t)B^T(t),\ X(0) = 0,$$
$$\dot{Y}(t) = A^T(t)Y(t) + Y(t)\,A(t) + C^T(t)C(t),\ \ Y(T) = 0$$

[Shokoohi/Silverman/Van Dooren'83, Sandberg'02]

$\hookrightarrow$ use the BDF or Rosenbrock method combined with the LDL^T-type ADI or Krylov subspace methods [Lang/Saak/St.'16]

- projection matrices are time-dependent
- zero initial and final conditions for the Gramians lead to zero initial and final reduced state

BT for bilinear systems

- For bilinear systems [Benner/Damm'11, Benner/Goyal/Redmann'16]

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + \sum_{k=1}^{m} N_k\,x(t)u_k(t) + B\,u(t), \\ y(t) &= C\,x(t) + D\,u(t), \end{aligned}$$

  the Gramians satisfy the generalized Lyapunov equations

$$\begin{aligned} AX + XA^T + \sum_{k=1}^{m} N_k X N_k^T &= -BB^T, \\ A^TY + YA + \sum_{k=1}^{m} N_k^T X N_k &= -C^TC. \end{aligned}$$

  $\hookrightarrow$ use the ADI or Krylov subspace methods [Benner/Breiten'12]

  $\hookrightarrow$ $(W^TAT,\, W^TN_1T, \ldots, W^TN_mT,\, W^TB,\, CT,\, D)$

  - energy functionals: $E_c(x_0) \geq x_0^TX^{-1}x_0$, $E_o(x_0) \leq x_0^TYx_0$, $x_0 \in \mathcal{B}(0)$
  - computationally expensive $\hookrightarrow$ use truncated Gramians
  - no error bounds

BT for quadratic-bilinear systems

- For quadratic-bilinear systems [Benner/Goyal'17]

$$\begin{aligned}\dot{x}(t) &= A\,x(t) + H\,\big(x(t)\otimes x(t)\big) + \sum_{k=1}^{m} N_k x(t) u_k(t) + B\,u(t)\\ y(t) &= C\,x(t) + D\,u(t)\end{aligned}$$

  the Gramians satisfy the generalized Lyapunov equations

$$AX + XA^T + H\,(X\otimes X)\,H^T + \sum_{k=1}^{m} N_k X N_k^T = -BB^T,$$

$$A^TY + YA + (H^{(2)})^T(X\otimes Y)H^{(2)} + \sum_{k=1}^{m} N_k^T X N_k = -C^TC.$$

  $\hookrightarrow$ use the fix point iteration combined with the ADI method

  $\hookrightarrow$ $(W^TAT,\, W^TH(T\otimes T),\, W^TN_1T, \ldots, W^TN_mT,\, W^TB,\, CT,\, D)$

  - energy functionals: $E_c(x_0) \geq x_0^T X^{-1} x_0,\;\; E_o(x_0) \leq x_0^T Y x_0,\;\; x_0 \in \mathcal{B}(0)$
  - computationally expensive $\hookrightarrow$ use truncated Graminas
  - no error bounds

BT for nonlinear systems

- For nonlinear systems [Scherpen'94, Fujimoto/Scherpen'10]

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \\ y(t) &= h(x(t)), \end{aligned}$$

  the input and output energy functionals $E_u(x_0)$ and $E_y(x_0)$ satisfy the partial differential equations

$$\frac{\partial E_c}{\partial x} f(x) + \frac{1}{4} \frac{\partial E_c}{\partial x} g(x) g^T(x) \frac{\partial^T E_c}{\partial x} = 0, \quad E_c(0) = 0,$$

$$\frac{\partial E_o}{\partial x} f(x) + h(x) h^T(x) = 0, \quad E_o(0) = 0.$$

  - computationally very expensive

BT for infinite-dimensional systems

- For infinite-dimensional systems

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

with $A : \mathcal{D}(A) \subset \mathcal{X} \to \mathcal{X}$, $B : \mathcal{U} \to \mathcal{D}(A^*)'$, $C : \mathcal{X} \to \mathcal{Y}$, $D : \mathcal{U} \to \mathcal{Y}$, where $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, the Gramians satisfy the operator Lyapunov equations

$$2\,\mathrm{Re}\langle Xv, A^*v\rangle_{\mathcal{X}} + \|B'v\|_{\mathcal{U}}^2 = 0 \quad \text{for all } v \in \mathcal{D}(A^*),$$

$$2\,\mathrm{Re}\langle A\,v, Y\,v\rangle_{\mathcal{X}} + \|Cv\|_{\mathcal{Y}}^2 = 0 \quad \text{for all } v \in \mathcal{D}(A).$$

[Glover/Curtain/Partingto'88, Guiver/Opmeer'13, Reis/Selig'14]

$\hookrightarrow$ use the finite-rank ADI iteration [Reis/Opmeer/Wollner'13]

- error bound $\|\boldsymbol{G} - \widetilde{\boldsymbol{G}}\|_{\mathcal{H}_\infty} \le 2 \sum\limits_{j=\ell+1}^{\infty} \xi_j$

Conclusion

- General framework for balanced truncation model reduction
  - input and output energy functionals
  - controllability and observability Gramians
  - (Hankel) singular values
  - balanced realization
- Properties
  - preservation of physical properties
  - computable error bounds
  - independence of the control
- Numerical solution of Lyapunov, Riccati, Lur'e equations