Scaling limits of stochastic processes
associated with resistance forms

Markov Processes, Mixing Times and Cutoff
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1. MOTIVATION
E.G. CRITICAL GALTON-WATSON TREES

Let $T_n$ be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have $n$ vertices, then

$$n^{-1/2} T_n \to \mathcal{T},$$

where $\mathcal{T}$ is (up to a constant) the Brownian continuum random tree (CRT) [Aldous 93], also [Duquesne/Le Gall 02].

Convergence in Gromov-Hausdorff-Prohorov topology implies

$$\left(n^{-1/2} X_{n^{3/2}t}^T \right) \to \left(X_t^\mathcal{T}\right)_{t \geq 0},$$

see [Krebs 95], [C. 08] and [Athreya/Löhr/Winter 14].
SOME INTUITION

Suppose $T$ is a graph tree, and $X^T$ is the discrete time simple random walk on $T$, $\pi(\{x\}) = \operatorname{deg}_T(x)$ its invariant measure. The following two properties are then easy to check:

- **[Scale]** For $x, y, z \in T$,

  \[
  P_z^T(\sigma_x < \sigma_y) = \frac{d_T(b_T(x, y, z), y)}{d_T(x, y)}.
  \]

- **[Speed]** Expected number of visits to $z$ when started at $x$ and killed at $y$,

  \[
  d_T(b_T(x, y, z), y)\pi(\{z\}).
  \]

Analogous properties hold for limiting diffusion.

cf. One-dimensional convergence results of [Stone 63].
2. STOCHASTIC PROCESSES ASSOCIATED WITH RESISTANCE METRICS
RANDOM WALKS ON GRAPHS

Let $G = (V, E)$ be a finite, connected graph, equipped with (strictly positive, symmetric) edge conductances $(c(x, y))_{(x,y) \in E}$. Let $\mu$ be a finite measure on $V$ (of full-support).

Let $X$ be the continuous time Markov chain with generator $\Delta$, as defined by:

$$(\Delta f)(x) := \frac{1}{\mu(\{x\})} \sum_{y : y \sim x} c(x, y)(f(y) - f(x)).$$

NB. Common choices for $\mu$ are:
- $\mu(\{x\}) := \sum_{y : y \sim x} c(x, y)$, the constant speed random walk (CSRW);
- $\mu(\{x\}) := 1$, the variable speed random walk (VSRW).
Define a quadratic form on $G$ by setting

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{x, y : x \sim y} c(x, y) (f(x) - f(y)) (g(x) - g(y)).$$

Note that (regardless of the particular choice of $\mu$,) $\mathcal{E}$ is a Dirichlet form on $L^2(\mu)$, and

$$\mathcal{E}(f, g) = -\sum_{x \in V} (\Delta f)(x) g(x) \mu(\{x\}).$$

Suppose we view $G$ as an electrical network with edges assigned conductances according to $(c(x, y))_{\{x, y\} \in E}$. Then the effective resistance between $x$ and $y$ is given by

$$R(x, y)^{-1} = \inf \{ \mathcal{E}(f, f) : f(x) = 1, f(y) = 0 \}.$$

$R$ is a metric on $V$, e.g. [Tetali 91], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 95].
SUMMARY

RANDOM WALK $X$ WITH GENERATOR $\Delta$

$\uparrow$

DIRICHLET FORM $\mathcal{E}$ on $L^2(\mu)$

$\uparrow$

RESISTANCE METRIC $R$ AND MEASURE $\mu$
RESISTANCE METRIC, e.g. [KIGAMI 01]

Let $F$ be a set. A function $R : F \times F \to \mathbb{R}$ is a resistance metric if, for every finite $V \subseteq F$, one can find a weighted (i.e. equipped with conductances) graph with vertex set $V$ for which $R|_{V \times V}$ is the associated effective resistance.
EXAMPLES

- Effective resistance metric on a graph;
- One-dimensional Euclidean (not true for higher dimensions);
- Any shortest path metric on a tree;
- Resistance metric on a Sierpinski gasket, where for ‘vertices’ of limiting fractal, we set

\[ R(x, y) = (3/5)^n R_n(x, y), \]

then use continuity to extend to whole space.
Theorem (e.g. [Kigami 01]) There is a one-to-one correspondence between resistance metrics and a class of quadratic forms called **resistance forms**.

The relationship between a resistance metric $R$ and resistance form $(\mathcal{E}, \mathcal{F})$ is characterised by

$$R(x, y)^{-1} = \inf \{\mathcal{E}(f, f) : f \in \mathcal{F}, f(x) = 1, f(y) = 0\}.$$  

Moreover, if $(F, R)$ is compact, then $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$ for any finite Borel measure $\mu$ of full support. (Version of the statement also hold for locally compact spaces.)
RESISTANCE FORM DEFINITION, e.g. [KIGAMI 12]

[RF1] \( \mathcal{F} \) is a linear subspace of the collection of functions \( \{ f : F \to \mathbb{R} \} \) containing constants, and \( \mathcal{E} \) is a non-negative symmetric quadratic form on \( \mathcal{F} \) such that \( \mathcal{E}(f, f) = 0 \) if and only if \( f \) is constant on \( F \).

[RF2] Let \( \sim \) be the equivalence relation on \( \mathcal{F} \) defined by saying \( f \sim g \) if and only if \( f - g \) is constant on \( F \). Then \( (\mathcal{F}/\sim, \mathcal{E}) \) is a Hilbert space.

[RF3] If \( x \neq y \), then there exists an \( f \in \mathcal{F} \) such that \( f(x) \neq f(y) \).

[RF4] For any \( x, y \in F \),

\[
\sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{E}(f, f)} : f \in \mathcal{F}, \mathcal{E}(f, f) > 0 \right\} < \infty.
\]

[RF5] If \( \bar{f} := (f \wedge 1) \vee 0 \), then \( f \in \mathcal{F} \) and \( \mathcal{E}(\bar{f}, \bar{f}) \leq \mathcal{E}(f, f) \) for any \( f \in \mathcal{F} \).
SUMMARY

RESISTANCE METRIC $R$ AND MEASURE $\mu$

\[ \uparrow \]

RESISTANCE FORM $(\mathcal{E}, \mathcal{F})$, DIRICHLET FORM on $L^2(\mu)$

\[ \uparrow \]

STRONG MARKOV PROCESS $X$ WITH GENERATOR $\Delta$, where

\[ \mathcal{E}(f, g) = -\int_{\mathcal{F}} (\Delta f) g d\mu. \]
A FIRST EXAMPLE

Let \( F = [0, 1] \), \( R = \) Euclidean, and \( \mu \) be a finite Borel measure of full support on \([0, 1]\). Define

\[
\mathcal{E}(f, g) = \int_0^1 f'(x)g'(x)dx,
\]
where \( \mathcal{F} = \{ f \in C([0, 1]) : f \) is abs. cont. and \( f' \in L^2(dx) \}. \)

Then \((\mathcal{E}, \mathcal{F})\) is the resistance form associated with \(([0, 1], R)\). Moreover, \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \(L^2(\mu)\). Note that

\[
\mathcal{E}(f, g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx), \quad \forall f \in \mathcal{D}(\Delta), g \in \mathcal{F},
\]
where \( \Delta f = \frac{d}{d\mu} \frac{df}{dx} \), and \( \mathcal{D}(\Delta) \) contains those \( f \) such that: \( f' \) exists and \( df' \) is abs. cont. w.r.t. \( \mu \), \( \Delta f \in L^2(\mu) \), and \( f'(0) = f'(1) = 0 \).

If \( \mu(dx) = dx \), then the Markov process naturally associated with \( \Delta \) is reflected Brownian motion on \([0, 1]\).
3. CONVERGENCE OF RESISTANCE METRICS AND STOCHASTIC PROCESSES
MAIN RESULT [C. 16]

Write $\mathbb{F}_c$ for the space of marked compact resistance metric spaces, equipped with finite Borel measures of full support. Suppose that the sequence $(F_n, R_n, \mu_n, \rho_n)_{n \geq 1}$ in $\mathbb{F}_c$ satisfies

$$(F_n, R_n, \mu_n, \rho_n) \rightarrow (F, R, \mu, \rho)$$

in the (marked) Gromov-Hausdorff-Prohorov topology for some $(F, R, \mu, \rho) \in \mathbb{F}_c$.

It is then possible to isometrically embed $(F_n, R_n)_{n \geq 1}$ and $(F, R)$ into a common metric space $(M, d_M)$ in such a way that

$$P_{\rho_n}^n \left( (X^n_t)_{t \geq 0} \in \cdot \right) \rightarrow P_\rho \left( (X_t)_{t \geq 0} \in \cdot \right)$$

weakly as probability measures on $D(\mathbb{R}_+, M)$.

Holds for locally compact spaces if $\limsup_{n \to \infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c)$ diverges as $r \to \infty$. (Can also include ‘spatial embeddings’.)
PROOF IDEA 1: RESOLVENTS

For \((F, R, \mu, \rho) \in \mathbb{F}_c\), let

\[
G_x f(y) = E_y \int_0^{\sigma_x} f(X_s) ds
\]

be the resolvent of \(X\) killed on hitting \(x\). NB. Processes associated with resistance forms hit points.

We have [Kigami 12] that

\[
G_x f(y) = \int_F g_x(y, z) f(z) \mu(dz),
\]

where

\[
g_x(y, z) = \frac{R(x, y) + R(x, z) - R(y, z)}{2}.
\]

Metric measure convergence \(\Rightarrow\) resolvent convergence \(\Rightarrow\) semigroup convergence \(\Rightarrow\) finite dimensional distribution convergence.
PROOF IDEA 2: TIGHTNESS

Using that $X$ has local times $(L_t(x))_{x \in F, t \geq 0}$, and

$$E_y L_{\sigma_A}(z) = g_A(y, z) = \frac{R(y, A) + R(z, A) - R_A(y, z)}{2},$$

can establish via Markov's inequality a general estimate of the form:

$$\sup_{x \in F} P_x \left( \sup_{s \leq t} R(x, X_s) \geq \varepsilon \right) \leq \frac{32N(F, \varepsilon/4)}{\varepsilon} \left( \delta + \frac{t}{\inf_{x \in F} \mu(B_R(x, \delta))} \right),$$

where $N(F, \varepsilon)$ is the minimal size of an $\varepsilon$ cover of $F$.

Metric measure convergence $\Rightarrow$ estimate holds uniformly in $n$ $\Rightarrow$ tightness (application of Aldous' tightness criterion).

Similar estimate also gives non-explosion in locally compact case.
4. APPLICATIONS
For any sequence of graph trees \((T_n)_{n \geq 1}\) such that
\[
(V(T_n), a_n R_n, b_n \mu_n) \to (\mathcal{T}, R, \mu),
\]
it holds that
\[
\left(a_n^{-1} X_{t a_n b_n}\right)_{t \geq 0} \to (X_t)_{t \geq 0}.
\]

- Critical Galton-Watson trees with finite variance conditioned on size, \(a_n = n^{1/2}, b_n = n\).
- Uniform spanning tree in two dimensions, \(a_n = n^{5/4}, b_n = n^2\), e.g. after 5,000 and 50,000 steps (picture: Sunil Chhita).

- Many other interesting models...
CONJECTURE FOR CRITICAL PERCOLATION

Bond percolation on integer lattice $\mathbb{Z}^d$:

At criticality $p = p_c(d)$ in high dimensions, incipient infinite cluster (IIC) conjectured to have same scaling limit as Galton-Watson tree, e.g. [Hara/Slade]. So, expect

$$(\text{IIC}, n^{-2}R_{\text{IIC}}, n^{-4}\mu_{\text{IIC}})$$

to converge, and thus obtain scaling limit for random walks. cf. recent work of [Ben Arous, Fribergh, Cabezas 16] for branching random walk.
RANDOM WALK SCALING ON CRITICAL RANDOM GRAPH

Consider largest connected component $C_1^n$ of $G(n, 1/n)$:

It holds that:

\[
\left( C_1^n, n^{-1/3} R_n, n^{-2/3} \mu_n \right) \to (F, R, \mu),
\]

cf. [Addario-Berry, Broutin, Goldschmidt 12]. Hence, as in [C. 12],

\[
\left( n^{-1/3} X_{tn}^n \right)_{t \geq 0} \to (X_t)_{t \geq 0}.
\]
Suppose that $P(c(x, y) \geq u) = u^{-\alpha}$ for $u \geq 1$ and some $\alpha \in (0, 1)$. For gaskets, can then check that resistance homogenises \[ [C., Hambly, Kumagai 16] \]

\[
(V_n, (3/5)^n R_n, 3^{-n} \mu_n) \to (F, R, \mu),
\]

where:
- (up to a deterministic constant) $R$ is the standard resistance,
- $\mu$ is a Hausdorff measure on fractal.

Hence VSRW converges to Brownian motion (spatial scaling assumes graphs already embedded into limiting fractal):

\[
(X_{t5n}^n)_{t \geq 0} \to (X_t)_{t \geq 0}.
\]
It further holds that

\[ \nu_n := 3^{-n/\alpha} \sum_{x \in V_n} c(x) \delta_x \rightarrow \nu = \sum_i v_i \delta_{x_i}, \]

in distribution, where \( \{(v_i, x_i)\} \) is a Poisson point process with intensity \( cv^{-1-\alpha} dv \mu(dx) \). Hence CSRW (and discrete time random walk) converges:

\[ \left( X^{n, \nu_n}_{t(5/3)n3^{n/\alpha}} \right)_{t \geq 0} \rightarrow (X^\nu_t)_{t \geq 0}, \]

where the limiting process \( X^\nu \) is the **Fontes-Isopi-Newman (FIN)** diffusion on the limiting fractal.

Similarly scaling result for heavy-tailed Bouchaud trap model.