Scaling limits of planar random growth models

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Work in progress with
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Physical motivation

DLA aggregate formed on electrode in copper sulphate solution

Photo by Kevin R Johnson

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Physical motivation

Eden cluster formed by lichen growth

Photo by James Wearn
Physical motivation

Electrical “tattoo” on survivor of lightning strike

From “Lichtenberg Figures Due to a Lightning Strike” by Yves Domart, MD, and Emmanuel Garet, MD
Conformal models for planar random growth

Conformal mapping representation of single particle

Let $D_0$ denote the exterior unit disk in the complex plane $\mathbb{C}$ and $P$ denote a particle of size $d$ attached at the point 1.

We typically take $P$ to be the “slit” $(1, d]$ and use the unique conformal mapping $f_P : D_0 \to D_0 \setminus (1, d]$ that fixes $\infty$ as a mathematical description of the particle.

(Usually talk in terms of logarithmic capacity $c = \log f_P'(\infty)$, instead of size $d$. For slit maps, $e^c = 1 + \frac{d^2}{4(1+d)}$ so $c \asymp d^2/4$.)
Suppose \( P_1, P_2, \ldots \) is a sequence of particles, where \( P_n \) has capacity \( c_n \) and attachment angle \( \theta_n, n = 1, 2, \ldots \).

- Set \( \Phi_0(z) = z \).
- Recursively define \( \Phi_n(z) = \Phi_{n-1} \circ f_{P_n}(z) \), for \( n = 1, 2, \ldots \).

This generates a sequence of conformal maps \( \Phi_n : D_0 \to K_n^c \), where \( K_{n-1} \subset K_n \) are growing compact sets, which we call clusters.

- By varying the sequences \( \{\theta_n\} \) and \( \{c_n\} \), it is possible to describe a wide class of growth models.
Cluster formed by iteratively composing slit mappings
Examples of models within this framework

- **Hastings-Levitov family, HL(\(\alpha\)) [1998]:**
  - \(\theta_n\) are i.i.d. \(U(-\pi, \pi)\) random variables;
  - \(c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}\).

- **Dielectric-breakdown models, DBM(\(\eta\)) [due to Mathiesen-Jensen, 2002]:**
  - \(\theta_n\) distributed \(\propto |\Phi'_{n-1}(e^{i\theta})|^{1-\eta} d\theta\);
  - \(c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-2}\).

- **Quantum Loewner Evolution, QLE(\(\gamma, \eta\)) [due to Miller-Sheffield, 2013]:**
  - \(\theta_n\) “distributed” \(\propto e^{a(\gamma)h \circ \Phi_{n-1}(e^{i\theta})}|\Phi'_{n-1}(e^{i\theta})|^{b(\gamma)-1-\eta} d\theta\);
  - \(c_n = c\) for all \(n\), \(P_n\) a \(SLE_\kappa\) conditionally independent of the GFF \(h\), given \(\theta_n\) (\(a, b\), functions depending on \(\kappa\)).
Aggregate Loewner Evolution, $\text{ALE}(\alpha, \eta)$

- $\theta_n$ distributed $\propto |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta$;
- $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$.
Previous results

- Primary interest has been in asymptotic behaviour of large clusters.
- Almost all previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.
  - Norris and T. (2012) showed scaling limit of HL(0) is a growing disk with a branching structure related to the Brownian web.
  - Silvestri (2015) showed fluctuations form a Gaussian field.
- Results for HL(\(\alpha\)) with \(\alpha \neq 0\) have only been shown for regularized versions of the model.
  - Rohde and Zinsmeister (2005) analysed the dimension of scaling limits for HL(0) and for a regularized version of HL(\(\alpha\)) when \(\alpha > 0\).
  - Sola, T., Viklund (2015) showed scaling limit of regularized HL(\(\alpha\)) is a growing disk for all \(\alpha\) provided regularization is strong enough.
Open problems

- Does ALE($\alpha, \eta$) have phase transitions from disks to non-disks along the line $\alpha + \eta = 1$ (within some compact region)?
  
  Longstanding conjectures:
  - HL($\alpha$) has a phase transition at $\alpha = 1$.
  - DBM($\eta$) has a phase transition at $\eta = 0$.

- Does ALE($\alpha, \eta$) have phase transitions to simple paths when $\eta$ or $\alpha$ are large?
  
  Longstanding conjectures:
  - There exists some $\eta_0$ such that DBM($\eta$) converges to a simple path for $\eta > \eta_0$.
  - There exists some $\alpha_0$ such that HL($\alpha$) converges to a simple path for $\alpha > \alpha_0$.

- What other limit shapes are possible?
Natural to consider particle sizes that are very small compared to the overall size of the cluster and scaling limits where $n \to \infty$ while $c \to 0$.

Models are difficult to analyse mathematically as all models (except HL(0)) exhibit long-range dependencies.

Additional difficulty, when $\alpha \neq 0$, is total capacity of cluster is random and cannot, a priori, be bounded above or below, so unclear at what rate to let $n \to \infty$.

When $\alpha = 0$, $K_n$ has capacity $cn$, so natural to look for scaling limits when $n = \lfloor T/c \rfloor$. 
ALE(0,-1) cluster with 10,000 particles for $d = 0.02$
ALE(0,0) cluster with 10,000 particles for $d = 0.02$
ALE(0,1) cluster with 10,000 particles for $d = 0.02$
ALE(0,1.5) cluster with 10,000 particles for $d = 0.02$
ALE(0,2) cluster with 10,000 particles for $d = 0.02$
ALE(0,4) cluster with 10,000 particles for $d = 0.02$
Regularization for ALE(0,η)

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi'_n(e^{i\theta})|$ is badly behaved and takes the values 0 and $\infty$.
- For some values of $\eta$, 
  $$\int_{\pi}^{\pi} |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta = \infty,$$
  so regularization is necessary to even define the measure.
- A solution is to let $\theta_n$ have distribution 
  $$\propto |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$$
  for $\sigma > 0$ and take the limit $\sigma \to 0$. 
Sequences $\{\theta_n\}$ in ALE(0,4) for varying $\gamma$ where $\sigma = c^\gamma$
Sequences \( \{\theta_n\} \) in ALE(0,4) for varying \( \gamma \) where \( \sigma = c^\gamma \)
Results for ALE(0, $\eta$)

Suppose $n = \lceil T/c \rceil$, and ALE(0, $\eta$) is regularized by $\sigma$.

- **Stick Theorem:**
  There exist $\eta_0$ and $\gamma_0$ such that for all $\eta > \eta_0$ and all $\sigma \ll c^{\gamma_0}$,

  $\Phi_n(z) \to e^{i\theta_1} f_T(e^{-i\theta_1} z)$ in probability as $c \to 0$,

  where $f_t$ is the map corresponding to a slit of capacity $t$ at 1.

- **Ball Theorem:**
  For every $\eta \in \mathbb{R}$, there exists a $\gamma_1$ such that for all $\sigma \gg c^{\gamma_1}$

  $\Phi_n(z) \to e^T z$ in probability as $c \to 0$.

(Refining the values of $\eta_0$ and $\gamma_i$ is work in progress.)
Fluctuations about disk ($\eta \leq 1$)

Set

$$\mathcal{F}_n(z) = c^{-1/2}(\Phi_n(z) - e^{cn} z).$$

Then $\mathcal{F}_n(z) \to \mathcal{W}_t(z)$ where

$$\mathcal{W}_t'(z) = (1 - \eta)z\mathcal{W}_t'(z) + \sqrt{2}\xi_t(z)$$

where $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

(Note that if $\eta > 1$ would need $|z| > e^{(\eta-1)t}$ for this SPDE to make sense – beginnings of a phase transition at $|\eta| = 1$?)
Conclusion

Implication of results

- Have family of random growth process for which we are able to prove that, by varying a single parameter, scaling limits transition from being:
  - Deterministic to random;
  - Absolutely continuous to singular.
- Specifically, have shown:
  - Existence of transition from disks to simple paths in $\text{ALE}(0,\eta)$ for fixed $\eta$ as $\sigma$ varies.
  - Existence of transition from disks to simple paths in $\text{ALE}(0,\eta)$ as $\eta$ varies?
  - Existence of phase transition in fluctuations in $\text{ALE}(0,\eta)$ at $\eta = 1$?
References


