Robustness of mixing via bottleneck sequences

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1st August 2017
Take a lazy random walk $X_n$ on a finite connected graph $G = (V, E)$. 

Total variation mixing time: $t_{\text{mix}} = \max_{x \in V} \min_{n \geq 0} \left\{ n : \max_{A \subset V} \left| P_x(X_n \in A) - \pi(A) \right| \leq \frac{1}{4} \right\}$.

There are of course many ways to bound the mixing time. We will look at conductance-based bounds.
The mixing time

- Take a lazy random walk $X_n$ on a finite connected graph $G = (V, E)$.

- Can easily generalise to graphs with conductances, and even to non-reversible chains to some extent.
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An example: the dumbbell graph

Starting from the left-hand side, it takes us time $\approx n^2$ to reach the right-hand side. The invariant measure of the right-hand side is $\frac{1}{2}$, so it seems clear (and it is easy to prove) that the mixing time is at least $cn^2$.
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- Starting from the left-hand side, it takes us time $\propto n^2$ to reach the right-hand side.

- The invariant measure of the right-hand side is $1/2$, so it seems clear (and it is easy to prove) that the mixing time is at least $cn^2$. 
The **conductance** of a set $A \subseteq V$ measures how easy it is to exit $A$ when starting in $A$. 
Conductance

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Note that if \( A \) is the left-hand side of the dumbbell graph, then \( \Phi(A) \approx 1/n^2 \).
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Note that if $A$ is the left-hand side of the dumbbell graph, then $\Phi(A) \asymp 1/n^2$.

First guess: $t_{\text{mix}} \asymp \max_{A \subset V} 1/\Phi(A)$?
A second example: the path of length $n$

Unfortunately our guess is not correct. Again it takes time $\sim n^2$ to reach the right-hand side of this graph, so $t_{\text{mix}} \geq cn^2$. . . .

$$\max_{A \subset V_1/\Phi(A)} \sim n.$$
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A second example: the path of length $n$

Unfortunately our guess is not correct. Again it takes time $\asymp n^2$ to reach the right-hand side of this graph, so $t_{\text{mix}} \geq cn^2 \ldots$ but $\max_{A \subset V} 1/\Phi(A) \asymp n$. 
The Lovász/Kannan/Fountoulakis/Reed/Morris/Peres bound

Let $\phi(r) = \min\{\Phi(A) : A \text{ connected, } r/2 \leq \pi(A) \leq r\}$. 

This built on work of Lovász and Kannan. A similar bound was given by Morris and Peres using evolving sets, which in particular works for non-reversible chains.
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**Theorem (Fountoulakis, Reed)**

$$t_{\text{MIX}} \leq C \left\lceil \log \frac{1}{\pi_{\text{min}}^{-1}} \right\rceil \sum_{j=1}^{\left\lceil \log \frac{1}{\pi_{\text{min}}^{-1}} \right\rceil} \frac{1}{\phi(2^{-j})^2}.$$
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This gives $t_{\text{mix}} \lesssim n^4$. 

Bottleneck sequences

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- $S_j$ and $S_j^c$ are both connected for each $j = 1, \ldots, k$;
- $\mathbb{P}_\pi(\mathbf{X}_0 \in S_j, \mathbf{X}_1 \in S_{j+1} \setminus S_j) \geq \theta \mathbb{P}_\pi(\mathbf{X}_0 \in S_j, \mathbf{X}_1 \in S_j^c)$ for all $j = 1, \ldots, k$. 
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\[ \text{Diagram of a network with highlighted bottlenecks.} \]
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![Diagram showing a graph with a bottleneck sequence highlighted.](image)
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![Diagram](image-url)
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![Diagram of a network with a green subgraph and a larger network connected to it.](image)
Bottleneck sequences

Bounding the mixing time

Let $S_\theta(G)$ be the set of all $\theta$-bottleneck sequences for the graph $G$. 

Theorem (Addario-Berry, R.)

For any $\theta \in (0, 1)$, 

$$t_{\text{mix}} \leq C \max_{S_1, \ldots, S_k \in S_\theta(G)} \sum_{j=1}^k \frac{1}{\Phi(S_j)}.$$ 

For the dumbbell graph, this gives $t_{\text{mix}} \ll n^2$.

For the path, it also gives $t_{\text{mix}} \ll n^2$. 

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Mixing times  
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- For the path, it also gives $t_{\text{MIX}} \lesssim n^2$. 
Gady Kozma asked whether the mixing time is a geometric property. In particular, is the mixing time robust under rough isometry for bounded degree graphs?
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Two graphs $G, H$ are roughly isometric with constant $r$ if there exists a function $f : G \rightarrow H$ such that

$$\frac{1}{r}d_G(x, y) - r \leq d_H(f(x), f(y)) \leq rd_G(x, y) + r;$$

for all $h \in H$, there exists $x \in G$ with $d(f(x), h) \leq r$. If $G$ and $H$ are roughly isometric (with constant $r$) and have bounded degree, are their mixing times within a constant factor (depending only on $r$, not the graphs)?
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If $G$ and $H$ are roughly isometric (with constant $r$) and have bounded degree, are their mixing times within a constant factor (depending only on $r$, not the graphs)?
The mixing time is NOT robust

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- Nonetheless, we may ask: are there large classes of graphs such that the mixing time is robust under rough isometry?

- We start with trees. (Peres and Sousi already proved that the mixing time is robust under rough isometry on trees, but trees give an illuminating application of our bottleneck sequence tools.)
Bounding the mixing time on trees

Recall our first result: for any $\theta \in (0, 1)$,

$$t_{\text{MIX}} \leq C \max_{(S_1, \ldots, S_k) \in S_\theta(G)} \sum_{j=1}^{k} \frac{1}{\Phi(S_j)}.$$

It is also easy to show (an application of Moon's lemma, or prove directly by induction) that on trees,

$$t_{\text{MIX}} \geq c \max_{(S_1, \ldots, S_k) \in S_1(G)} \sum_{j=1}^{k} \frac{1}{\Phi(S_j)}.$$

(Recall: $\theta$-bottleneck sequences "eat at least $\theta$ proportion of the boundary"). But on trees, if $S$ and $S'$ are both connected, then the boundary of $S$ is exactly one vertex, so the two bounds agree. And they are robust under rough isometry.
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But on trees, if $S$ and $S^c$ are both connected, then the boundary of $S$ is exactly one vertex, so the two bounds agree. And they are robust under rough isometry.
A horrible but elementary argument shows that for any graph $G$ that is roughly isometric (with constant $r$) to a tree $T$,

$$t_{\text{MIX}}(G) \geq ct_{\text{MIX}}(T).$$
Graphs roughly isometric to trees

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What about an upper bound?
An example when the bottleneck sequence bound is not tight: the beanstalk graph
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The bottleneck sequence game

We attempt to improve our upper bound by introducing a two-player game.
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We have two players, Crawler and Dasher. The “board” is the subsets of $V$, and each player has a counter on the board which they take turns to move.
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Both players begin from $\emptyset$. Crawler chooses a “target” vertex $s$. Crawler plays first, and the game ends once Dasher’s position includes $s$. 
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**Crawler:** From $(C, D)$, $C'$ valid if

- $C \subset C'$, $C' \setminus C \subset D^c$, $C'$ connected
- $\mathbb{P}_\pi(X_0 \in C, X_1 \in (D \cup C')^c) \leq \gamma \mathbb{P}_\pi(X_0 \in C, X_1 \in D^c)$
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**Dasher:** From $(C, D)$, $D'$ valid if
- $D \cup C \subset D'$, $(D')^c$ connected
- $\partial D'$ is $\alpha$-near to $C$
- $D'$ is a $\beta$-adjustment of $C$
- If $s \in D'$ then $s$ is $\alpha$-near to $C$ and $D' = V(G)$. 
The bottleneck sequence game bound

Theorem (Addario-Berry, R.)

*For any $\alpha, \beta, \gamma \in (0, 1)$, there exists a strategy for Crawler such that for any valid moves by Dasher,*

$$t_{\text{Mix}}(G) \leq C \sum_{j=1}^{k} \frac{1}{\phi(D_j)}.$$
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Note this bound holds for all graphs, not just tree-like graphs.
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Now play the game on a graph $G$ that is roughly isometric (with constant $r$) to a tree $T$ (both with bounded degree).
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Now play the game on a graph $G$ that is roughly isometric (with constant $r$) to a tree $T$ (both with bounded degree).

We devise a strategy for Dasher such that whatever moves Crawler makes,

$$\sum_{j=1}^{k} \frac{1}{\Phi(D_j)} \leq C'(r) t_{\text{mix}}(T)$$
The mixing time is robust on bounded degree graphs that are roughly isometric to trees

Theorem (Addario-Berry, R.)

If $G$ is roughly isometric (with constant $r$) to a tree $T$, and both have degree at most $\Delta$, then

$$c(r, \Delta)t_{\text{MIX}}(T) \leq t_{\text{MIX}}(G) \leq C(r, \Delta)t_{\text{MIX}}(T).$$