A MODERN POINT OF VIEW ON ANOMALIES

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Outline

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Motivation

Practical constraints restrain the type of experiments we can perform to test our understanding of the world.

For instance, we are stuck at energies much smaller than the GUT/Planck scale.

Is there a hope of finding a fundamental theory without direct experimental input?

Motivation

Yes. Consistency provides powerful constraints, which may allow us to select a unique theory, string theory.

General relativity was developed similarly using self-consistency.

But it is not a given, cf. the situation in quantum field theory.

When they are sufficiently constrained, it is possible to construct theories relevant to our world without experimental input.

Motivation

A task of prime importance is therefore to understand the constraints that a quantum theory of gravity should satisfy.

There are many constraints on string theory/quantum gravity:

- geometrical/topological nature of the compactifications
- consistency of the worldsheet theory
- dualities
- unitarity
- absence of global symmetries
- weak gravity conjecture
- ...
- *anomaly cancellation*

Motivation

A symmetry of a theory is anomalous if it is broken in a certain controled way.

Anomalies usually occur when quantizing a classical theory invariant under the relevant symmetry. It may happen that the quantization scheme breaks the symmetry, and that it is impossible to restore it in the quantum theory. But it will be useful to keep the definition a bit broader.

If a symmetry is supposed to be gauged, it cannot be anomalous. $\Rightarrow$ anomalies provide constraints on physical gauge theories.

The cancellation of local gravitational and gauge anomalies in type I string theory through the Green-Schwarz mechanism sparkled widespread interest for string theory.

AIM OF THE TALK

The aim of the talk is to present a modern framework for the understanding and study of anomalies.

It encodes all the anomalies of a d-dimensional quantum field theory $\mathcal{F}$ (local, global, Hamiltonian, etc...) in a single well-defined mathematical object: a $d+1$-dimensional field theory functor $\mathcal{A}$.

This is not only a conceptual advance:

1. The constraints satisfied by $\mathcal{A}$ help compute anomalies.
2. The triviality of $\mathcal{A}$ provides a simple condition for the cancellation of *all* anomalies.

Outline

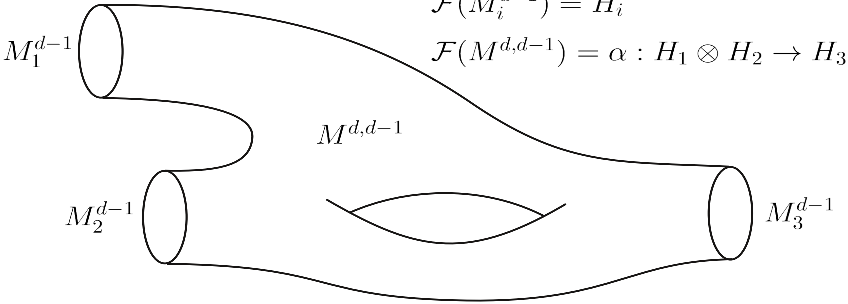
Field theories as functors

A field theory functor is a monoidal functor $\mathcal{F}$ from a bordism category $\mathcal{B}$ into the category $\mathcal{H}$ of Hilbert spaces.

$$\mathcal{F} : \mathcal{B} \to \mathcal{H}$$

$$\mathcal{F}(M_i^{d-1}) = H_i$$

$$\mathcal{F}(M^{d,d-1}) = \alpha : H_1 \otimes H_2 \to H_3$$



Field theories as functors

To include codimension 2 corners and defects: take $\mathcal{F}$ to be a monoidal 2-functor from a bordism 2-category $\mathcal{B}$ into the 2-category $\mathcal{H}_2$ of 2-Hilbert spaces.

2-Hilbert space: category linearly equivalent to $\mathcal{H}^n \sim$ Category whose objects are "vectors of Hilbert spaces".

And so on to include higher codimension objects...

An example: the d-dimensional trivial theory $\mathcal{I}$.

- $\mathcal{I}(M^{d-1}) = \mathbb{C}$ for all $d-1$-manifolds M^d,
- $\mathcal{I}(M^{d,d-1}) = \mathbb{C} \xrightarrow{1} \mathbb{C}$ for all d-bordisms $M^{d,d-1}$.
- $\mathcal{I}(M^{d-2}) = \mathcal{H}$, the 1-dim 2-Hilbert space.

Field theories as functors

A field theory $\mathcal{F}$ is *invertible* if:

1. $\mathcal{F}(M^{d,d-1})$ are invertible homomorphisms for all $M^{d,d-1}$. In particular $\mathcal{F}(M^d) \in \mathbb{C}^*$.
2. $\mathcal{F}(M^{d-1})$ are "invertible Hilbert spaces", i.e. Hermitian lines, which are invertible under the tensor product operation.
3. $\mathcal{F}(M^{d-2})$ are "invertible 2-Hilbert spaces", i.e. categories linearly equivalent to $\mathcal{H}$.
4. ...

Outline

Relative field theories

What is a natural transformation $\eta : \mathcal{I} \to \mathcal{I}$?

- Nat. transformation $\Leftrightarrow \{\eta_{M^{d-1}} : \mathcal{I}(M^{d-1}) \to \mathcal{I}(M^{d-1})\} \Leftrightarrow \{\eta_{M^{d-1}} : \mathbb{C} \to \mathbb{C}\} \Leftrightarrow \{\eta_{M^{d-1}} \in \mathbb{C}\}$
- 2-nat. transformation $\Leftrightarrow \{\eta_{M^{d-2}} : \mathcal{I}(M^{d-2}) \to \mathcal{I}(M^{d-2})\} \Leftrightarrow \{\eta_{M^{d-2}} : \mathcal{H} \to \mathcal{H}\} \Leftrightarrow \{\eta_{M^{d-2}} \in \mathcal{H}\}$

η is nothing but a $d-1$-dimensional field theory.

Relative field theories

$\Rightarrow$ Generalization of the notion of field theory:

A d-dimensional *relative field theory* $\mathcal{F}$ is a natural transformation $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ from the trivial $d+1$-dimensional field theory to a $d+1$-dimensional field theory $\mathcal{A}$. [Freed-Teleman 1212.1692]

$\mathcal{F}(M^d) \in \mathcal{A}(M^d) \leftarrow$ Hilbert space

$\mathcal{F}(M^{d-1}) \in \mathcal{A}(M^{d-1}) \leftarrow$ 2-Hilbert space $\simeq \mathcal{H}^n$

$\mathcal{F}$ is a "d-dimensional field theory valued in the $d+1$-dimensional field theory $\mathcal{A}$".

Anomalous = relative

Claim: anomalous field theories are relative field theories. [Freed 1404.7224]

Given $\mathcal{F} : \mathcal{I} \to \mathcal{A}$:

- $\mathcal{F}$ is the (physical) *anomalous field theory*.
- $\mathcal{A}$ is the *anomaly field theory*

We will see that $\mathcal{A}$ gathers in a convenient package all the information about the anomalies of $\mathcal{F}$.

In the simplest cases, $\mathcal{A}$ is an invertible field theory [Freed 1404.7224], but we will see that more general anomalies require a non-invertible $\mathcal{A}$ [S.M. 1410.7442].

Anomalous = relative

Condition for anomaly cancellation:

$\mathcal{F}$ is non-anomalous if and only if $\mathcal{A}$ is naturally isomorphic to $\mathcal{I}$.

This condition guarantees the cancellation of *all* anomalies (local, global, Hamiltonian, ...).

Examples

$\mathcal{F}$: Chiral (Weyl) fermions in even dimension d, associated to a Dirac operator D.

$\mathcal{A}$: Dai-Freed theory $\mathcal{DF}_D$ based on the corresponding Dirac operator in dimension $d+1$. [Dai, Freed 9405012]

- Partition function: exponentiated eta invariant of the $d+1$-dimensional Dirac operator, $\xi = \exp 2\pi i(\eta + h)/2$.
- State space: determinant line of D.
- $\mathcal{DF}_D$ is invertible.

Examples

$\mathcal{F}$: Prequantum theory associated to a classical Wess-Zumino term
$\mathcal{A}$: Prequantum Chern-Simons theory (invertible)

$\mathcal{F}$: Chiral WZW model: The "partition function" is a vector of conformal blocks.
$\mathcal{A}$: Quantum Chern-Simons theory (non-invertible): The state space of the quantum CS theory is the space of conformal blocks. [Witten, 1989]

$\mathcal{F}$: 6d (2,0) superconformal field theories
$\mathcal{A}$: Product of Dai-Freed theories and discretely gauged Wu Chern-Simons theory (non-invertible), constructed in [S.M. - 1706.01903]

Outline

Local anomaly

The local anomaly of a d-dimensional field theory is a degree $d+2$ characteristic form.

Chiral fermions: I_{d+2} = index density of a Dirac operator $D^{(d+2)}$ in dimension $d+2$ constructed from D.

To recover it, compute $\mathcal{A}(M^d \times S^1)$, with data on $M^d \times S^1$ that extends to $M^d \times D^2$. In all the relevant cases, $\ln \mathcal{A}(M^d \times S^1)$ can be expressed up to locally constant terms as the integral of a characteristic form over $M^d \times D^2$. This form is the local anomaly.

Chiral fermions: Use the Atiyah-Patodi-Singer theorem:

$$\frac{1}{2\pi i} \ln \xi = \text{index}\, D^{(d+2)} + \int_{M^d \times D^2} I_{d+2}$$

Action of symmetries

Suppose that a d-dimensional anomalous field theory $\mathcal{F} : \mathcal{I} \to \mathcal{A}$ has a symmetry group G.

- G acts trivially on $\mathcal{I}$;
- G is a symmetry of $\mathcal{A}$ in the usual QFT sense;
- $\mathcal{F}$ is equivariant, i.e. $\mathcal{F}_{gM^d} \circ \mathcal{I}(g) = \mathcal{A}(g) \circ \mathcal{F}_{M^d}$.

Action of symmetries - partition function

$\mathcal{A}(M^d)$ is a representation of G.

$\mathcal{F}(M^d) \in \mathcal{A}(M^d)$ therefore transforms as a vector in a representation of G.

Chiral fermions: $\mathcal{A}$ is invertible, so $\mathcal{A}(M^d)$ is 1-dimensional, and $\mathcal{F}(M^d)$ transforms by a phase.

Chiral WZW on T^2: The modular S and T matrices define a unitary representation of the diffeomorphism group (constant on the connected component of the identity) in which the conformal blocks transform.

Action of symmetries - state space

G also has an action on the 2-Hilbert space $\mathcal{A}(M^{d-1})$.

Assuming $\mathcal{A}$ invertible, $\mathcal{A}(M^{d-1}) \simeq \mathcal{H}$ and G acts on $\mathcal{H}$ through the tensor product of lines:

$$g \cdot V = L_g \otimes V \ , \quad g \in G \ , \ V \in \mathcal{H} \ .$$

The group law requires isomorphisms

$$L_{g_1 g_2} \otimes L_{g_1^{-1}} \otimes L_{g_2^{-1}} \simeq \mathbb{C} \ .$$

After choosing non-canonical isomorphisms $L_g \simeq \mathbb{C}$, the isomorphisms above yield a $U(1)$-valued group 2-cocycle $\alpha = \{\alpha_{g_1,g_2}\}$.

Action of symmetries - state space

$\mathcal{F}(M^{d-1}) \in \mathcal{A}(M^{d-1})$ can be (non-canonically) identified with a Hilbert space $H \in \mathcal{H}$.

The action of G is $g.H = L_g \otimes H$. After the identification $L_g \simeq \mathbb{C}$ we get an endomorphism ϕ_g of H for each g.

We also have $(g_1 g_2).g_2^{-1}.g_1^{-1}.H = L_{g_1 g_2} \otimes L_{g_1^{-1}} \otimes L_{g_2^{-1}} \otimes H$, so we deduce that

$$\phi_{g_1 g_2} \circ \phi_{g_2^{-1}} \circ \phi_{g_1^{-1}} = \alpha_{g_1, g_2} \mathbb{1}_H$$

ϕ is a *projective* representation of G on H.

We recovered the main characteristic of Hamiltonian anomalies [Faddeev, 1984], [Mickelsson, 1985].

Action of symmetries - state space

There are many arbitrary choices of isomorphisms appearing in the derivation of the Hamiltonian anomaly.

One can show that changes of such choices change the cocycle α by a boundary, so the anomaly is characterized by the group cohomology class of α.

Action of symmetries - state space

The story above generalizes to non-invertible anomalies.

- $\mathcal{F}(M^{d-1})$ is a vector of Hilbert spaces.
- The action of G is through "permutation matrices of lines":

$$g.\begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 & L_g^{12} \\ L_g^{21} & 0 \end{pmatrix} \otimes \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} L_g^{12} \otimes H_2 \\ L_g^{21} \otimes H_1 \end{pmatrix}$$

- Making trivializing choices, we can extract a non-abelian group cocycle, whose non-abelian cohomology class characterizes the anomaly.

These "non-invertible Hamiltonian anomalies" have not been described in the physics literature, but should be relevant to $(2,0)$ superconformal field theories in six dimensions.

Setting the quantum integrand

In certain theories, anomalies cancel, but defining their $\mathbb{C}$-valued partition functions on d-dimensional manifolds that are not boundaries of $d+1$-dimesional manifolds require extra choices, known as the "setting of the quantum integrand". [Witten 9610234], [Freed, Moore 0409135], [Witten at Strings'15]

The situation occurs when the anomaly field theory $\mathcal{A}$ is trivial, but not canonically so.

The freedom of choosing the isomorphism $\mathcal{A} \simeq \mathcal{I}$ precisely translates into the freedom of setting the quantum integrand of the theory $\mathcal{F}$.

Outline

Not the Green-Schwarz mechanism

Suppose you have an anomalous field theory $\mathcal{F}$ with invertible anomaly $\mathcal{A}$. What can you do to get rid of the anomaly?

A solution:

Tensor it with another anomalous field theory $\mathcal{F}'$ (maybe a TFT) with the opposite anomaly, i.e. with anomaly $\mathcal{A}^\dagger$.

Then $\mathcal{F} \otimes \mathcal{F}'$ has anomaly $\mathcal{A} \otimes \mathcal{A}^\dagger \simeq \mathcal{I}$ and all the anomalies vanish.

Problem: Adds extra degrees of freedom, in general incompatible with supersymmetry.

This is definitely NOT how the Green-Schwarz mechanism works.

THE GREEN-SCHWARZ MECHANISM

The actual Green-Schwarz mechanism is much more elegant.

1. For any M^d, construct a vector in the state space $\mathcal{A}$ of the anomaly field theory, as the exponential of a term *local* in the fields of $\mathcal{F}$:

$$\exp 2\pi i \int_{M^d} \mathrm{GS} \in \mathcal{A}(M^d)$$

2. Subtract the Green-Schwarz term GS to the action of $\mathcal{F}$, or equivalently, define

$$\mathcal{F}'(M^d) := \mathcal{F}(M^d) \to \mathcal{F}(M^d) \otimes \exp -2\pi i \int_{M^d} \mathrm{GS}\ .$$

THE GREEN-SCHWARZ MECHANISM

As $\mathcal{F}(M^d) \in \mathcal{A}(M^d)$,

$$\mathcal{F}'(M^d) \in \mathcal{A}(M^d) \otimes \mathcal{A}^\dagger(M^d) \simeq \mathcal{I} \ ,$$

and $\mathcal{F}'$ is non-anomalous.

No degrees of freedom are added $\to$ compatible with supersymmetry, as long as GS is supersymmetrizable.

The Green-Schwarz mechanism in 6d sugra

Cf. [S.M, Moore - 1808.01334], see also [S.M, Moore - 1808.01335] and Greg Moore's talk at Strings'18.

6d supergravities require a version of the Green-Schwarz mechanism to cancel their anomalies.

The existing descriptions of the 6d GS mechanism were valid only in flat space for topologically trivial vectormultiplet gauge bundle.

In our recent work, we describe the Green-Schwarz mechanism for 6d sugra in topologically non-trivial situations.

The Green-Schwarz mechanism in 6d sugra

The anomaly field theory $\mathcal{A}$ of the "bare" sugra (i.e. without GS term) is a product of Dai-Freed theories. It is hard to construct local terms valued in $\mathcal{A}^\dagger(M^6)$.

We construct a field theory $\mathcal{A}_{\text{CT}}$ such that $\mathcal{A} \otimes \mathcal{A}_{\text{CT}}$ is a bordism invariant (of $\Omega_7^{\text{Spin}}(BG)$), i.e. the two theories coincides on boundaries.

We then construct a GS term in $\mathcal{A}_{\text{CT}}(M^6)$ that reduces to the usual expression in topologically trivial situations.

The Green-Schwarz mechanism in 6d sugra

We derive two types of constraint from the requirement of anomaly cancellation, and compare them to F-theory realizations of 6d sugras. (This includes all known sugra constructible in string theory).

1. The construction of $\mathcal{A}_{\mathrm{CT}}$ constrains the sugra data. There are otherwise consistent 6d sugra for which it is impossible to construct an appropriate GS term.

For instance, the gravitational anomaly coefficient a should be a characteristic element of the tensormultiplet charge lattice.

This is true in F-theory (where a is the canonical class in the degree 2 homology lattice of the base), but a low energy derivation was missing until now.

The Green-Schwarz mechanism in 6d sugra

2. The requirement that $\mathcal{A}_{\mathrm{CT}} \simeq \mathcal{A}^{\dagger}$ when $\Omega_7^{\mathrm{Spin}}(BG) \neq 0$ provides extra constraints.

If $\mathcal{A}_{\mathrm{CT}} \neq \mathcal{A}^{\dagger}$ the GS term can be constructed, but it does not cancel all global anomalies.

For 6d sugra with gauge group $\mathbb{Z}_n$, we were able to compare $\mathcal{A}_{\mathrm{CT}}$ and $\mathcal{A}$ and provide constraints on matter representations.

They are again compatible with the known F-theory realizations of 6d sugra.

The Green-Schwarz mechanism in 6d sugra

Main point:

Seeing anomalies as field theories is not only an elegant mathematical framework.

It helps computing anomalies and does provide new constraints that can be checked in string theory constructions.

Outline

Anomaly cancellation checked

1. Standard model [Freed - 0607134]
2. Worldsheet anomalies, too many references to cite.
3. Smooth Type I without D-branes [Green, Schwarz - 1984], [Freed (and Hopkins) - 0011220]
4. Smooth M-theory backgrounds without five-branes [Witten - 9609122], [Freed, Moore - 0409135]
5. Smooth M-theory, type IIA and $E_8 \times E_8$ heterotic backgrounds with five-branes [Freed, Harvey, Minasian, Moore - 9803205], [S.M. - 1310.2250]
6. "Cohomological" type IIB supergravity [Alvarez-Gaume, Witten - 1984], [Witten - 1985], [S.M. - 1110.4639] (assuming zero B-field or RR 3-form field strength)

Anomaly cancellation partially checked

Some setups where people have tried to go beyond local anomaly cancellation:

1. 8d F-theory compactifications [García-Etxebarria, Hayashi, Ohmori, Tachikawa, Yonekura - 1710.04218]
2. 6d F-theory compactifications [Kumar, Morrison, Taylor - 1008.1062], [S.M., Moore - 1808.01334]
3. 4d orientifolds [Gato-Rivera, Schellekens - 0510074]
4. MSSM [Garcia-Etxebarria, Monteroo - 1808.00009]
5. ...

Some open problems

Some open problems:

1. Type II, including orientifolds [Distler, Freed, Moore - 0906.0795]
2. Type IIA D-branes [Freed, Hopkins - 0002027]
3. 6d compactifications, in particular $N = (1, 0)$ F-theory compactifications with strongly coupled sectors.
4. ...