Courant Algebroid Connections: Applications in String Theory

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Generalized geometry and effective actions

- **Main motivation**: understand the geometry behind

$$S[g,B,\phi] = \int_M e^{-2\phi} \{ \mathcal{R}(g) - \frac{1}{2} \langle H', H' \rangle_g + 4 \langle d\phi, d\phi \rangle_g \} \cdot d\,\mathrm{vol}_g \,. \quad (1)$$

  M a manifold, g a **metric** (usually Riemannian), $B \in \Omega^2(M)$ and $\phi \in C^\infty(M)$ a **dilaton field**. Here $H' = H + dB$ for $H \in \Omega^3_{cl}(M)$. Type II supergravity with no fermions and RR fields.
- **Main idea**: generalize the concept of Levi-Civita connection.
- Many people have thought the same:
  Coimbra, Strickland-Constable, Waldram (2011) - supergravity as generalized geometry, M-theory.
  Hohm, Zwiebach (2012) - discussed in the context of DFT.
  Garcia-Fernandez (2013) - modification for heterotic supergravity.
- **Our approach**: Modify and use the geometry to understand some intriguing relations.

Courant algebroids

Definition

Courant algebroid consists of the following data:

- Vector bundle $q : E \to M$;
- Morphism $\rho : E \to TM$ called the **anchor**;
- Fiberwise metric $\langle \cdot, \cdot \rangle_E$ on E;
- $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$.

All objects interplay according to some axioms:

- $[\psi, \cdot]_E$ is a differential operator (**Leibniz rule**):
$$[\psi, f\psi']_E = f[\psi, \psi']_E + \mathcal{L}_{\rho(\psi)}(f) \cdot \psi'. \tag{2}$$
- $[\psi, \cdot]_E$ is a derivation of the bracket (**Jacobi identity**).
- The bracket $[\cdot, \cdot]_E$ and the pairing $g_E = \langle \cdot, \cdot \rangle_E$ are compatible.
- The bracket is not skew-symmetric:
$$\langle [\psi, \psi]_E, \psi' \rangle_E = \frac{1}{2} \mathcal{L}_{\rho(\psi')} \langle \psi, \psi \rangle_E \tag{3}$$

- Algebroid generalization of quadratic Lie algebras (non-degenerate compatible symmetric bilinear form). Reduce to them for $M = \{*\}$.
- Appeared as doubles of Lie bialgebroids (Mackenzie, Xu 1997)
- Every CA is an example of L^∞-algebra (Roytenberg 1999).
- Symplectic NQ-manifolds of degree 2 (Roytenberg 2002).

Example

$E = \mathbb{T}M \equiv (T \oplus T^*)M$ **generalized tangent bundle**, $\rho = \pi_{TM}$, $\langle \cdot, \cdot \rangle_E$ is the canonical pairing of dual vector bundles and

$$[(X,\xi),(Y,\eta)]_E = ([X,Y], \mathcal{L}_X \eta - i_Y d\xi - H(X,Y,\cdot)), \tag{4}$$

where $H \in \Omega^3_{cl}(M)$. H-**twisted Dorfman bracket**.

- Geometry of $\mathbb{T}M$ and its modifications: **generalized geometry**.

Definition

Generalized (Riemannian metric) is a maximal positive subbundle $V_+ \subseteq E$ with respect to $\langle \cdot, \cdot \rangle_E$. Gives a decomposition

$$E = V_+ \oplus V_-, \tag{5}$$

where $V_- = V_+^\perp$. Provides an involution $\tau \in \mathrm{End}(E)$, such that $\tau(V_\pm) = \pm 1 \cdot V_\pm$ and $\mathbf{G}(\psi, \psi') = \langle \psi, \tau(\psi') \rangle_E$ is a positive-definite fiber-wise metric on E.

- On every orthogonal vector bundle $(E, \langle \cdot, \cdot \rangle_E)$, there exists a generalized metric. $\mathrm{O}(E, g_E)$ acts transitively on their space.
- For $E = \mathbb{T}M$, every V_+ is a graph of a bundle map:

$$\Gamma(V_+) = \{(X, (g + B)(x)) \quad X \in \mathfrak{X}(M)\}, \tag{6}$$

where g is a Riemannian metric on M and $B \in \Omega^2(M)$.
- Reduces the structure group of E from $\mathrm{O}(p, q)$ to $\mathrm{O}(p) \times \mathrm{O}(q)$, where $p = \dim(V_+)$ and $q = \dim(V_-)$.

CA connections

Definition

$\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is the **CA connection** if $\nabla_\psi \equiv \nabla(\psi, \cdot)$ satisfies the two axioms

$$\nabla_\psi(f\psi') = f\nabla_\psi(\psi') + \mathcal{L}_{\rho(\psi)}(f) \cdot \psi', \;\; \nabla_{f\psi}(\psi') = f\nabla_\psi(\psi'), \tag{7}$$

and is compatible with $\langle \cdot, \cdot \rangle_E$, that is $\nabla g_E = 0$.

- It contains vector bundle connections compatible with g_E, via the formula $\nabla_\psi = \nabla'_{\rho(\psi)}$. The set of CA connections is non-empty.

Definition (Gualtieri 2007)

Every CA connection allows for a definition of a **torsion** 3-**form** T_∇:

$$T_\nabla(\psi, \psi', \psi'') = \langle \nabla_\psi \psi' - \nabla_{\psi'}\psi - [\psi, \psi']_E, \psi'' \rangle_E + \langle \nabla_{\psi''}\psi, \psi' \rangle_E. \tag{8}$$

It is $C^\infty(M)$-linear and completely skew-symmetric. ∇ is **torsion-free** if $T_\nabla = 0$. This requires full CA connections (not just VB ones).

Definition

Let $V_+ \subseteq E$ be a generalized metric. We say that ∇ is a **Levi-Civita connection** on E with respect to V_+ and write $\nabla \in \mathrm{LC}(E, V_+)$ if

1. $\nabla_\psi(V_+) \subseteq V_+$
2. $T_\nabla = 0$.

One has (Garcia-Fernandez 2016) $LC(E, V_+) \neq \emptyset$.

- There is no closed formula, main reason is that there is quite a lot of them, namely $\mathrm{LC}(E, V_+) \cong \Gamma(LC_0(E, V_+))$, where $\mathrm{LC}_0(E, V_+)$ is a certain vector bundle of rank $\frac{1}{3}p(p^2-1) + \frac{1}{3}q(q^2-1)$.

Definition (Hohm, Zwiebach 2012)

There is a well-defined analogue of the **curvature tensor**:

$$\begin{aligned} R_\nabla(\phi', \phi, \psi, \psi') = {} & \frac{1}{2}\langle([\nabla_\psi, \nabla_{\psi'}] - \nabla_{[\psi,\psi']_E})\phi, \phi'\rangle_E \\ & + \frac{1}{2}\langle([\nabla_\phi, \nabla_{\phi'}] - \nabla_{[\phi,\phi']_E})\psi, \psi'\rangle_E \\ & + \frac{1}{2}\langle\nabla_{\psi_\mu}\psi, \psi'\rangle_E \cdot \langle\nabla_{\psi_E^\lambda}\phi, \phi'\rangle_E. \end{aligned} \tag{9}$$

- Suspicious definition with no clear geometrical meaning. R_∇ is $C^\infty(M)$-linear in all its inputs.
- However, R_∇ has all the usual symmetries including the algebraic Bianchi identity. In particular, there is a unique partial trace:

Definition

The **generalized Ricci tensor** Ric_∇ on E is defined by

$$\mathrm{Ric}_\nabla(\psi, \psi') = R_\nabla(g_E^{-1}(\psi^\mu), \psi, \psi_\mu, \psi'). \tag{10}$$

It is symmetric and $C^\infty(M)$-linear in its inputs. We say that ∇ is **Ricci-compatible** with V_+, if $\mathrm{Ric}_\nabla(V_+, V_-) = 0$.

- As Ric_∇ is well-defined on all sections of E, one may define two scalar curvatures using the trace and metrics g_E and $\mathbf{G}$, respectively:

Definition

We have two canonical functions called the **scalar curvatures of** ∇:

$$\mathcal{R}_\nabla = \mathrm{Ric}_\nabla(\psi_\mu, g_E^{-1}(\psi^\mu)), \quad \mathcal{R}^+_\nabla = \mathrm{Ric}(\psi_\mu, \mathbf{G}^{-1}(\psi^\mu)). \tag{11}$$

Observation

Define a **divergence operator** $\mathrm{div}_{\nabla}(\psi) = \langle \nabla_{\psi_\mu}(\psi), \psi^\mu \rangle$. Suppose $\nabla, \nabla' \in \mathrm{LC}(E, V_+)$ satisfy $\mathrm{div}_{\nabla'} = \mathrm{div}_{\nabla}$. Then

$$\mathcal{R}_{\nabla'} = \mathcal{R}_{\nabla}, \;\; \mathcal{R}^{+}_{\nabla'} = \mathcal{R}^{+}_{\nabla}, \;\; \mathrm{Ric}^{+-}_{\nabla'} = \mathrm{Ric}^{+-}_{\nabla} \, . \tag{12}$$

Theorem (Jurčo & V.)

Let $E = \mathbb{T}M$ with H-twisted Dorfman. Let $V_+ \subset E$ correspond to a pair (g, B). Suppose $\nabla \in \mathrm{LC}(E, V_+)$ satisfies the additional condition

$$\mathrm{div}_{\nabla}(\psi) = \mathrm{div}_{\nabla^{LC}_g}(\rho(\psi)) - \mathcal{L}_{\rho(\psi)}(\phi) \tag{13}$$

for a scalar function $\phi \in C^\infty(M)$. We write $\nabla \in \mathrm{LC}(E, V_+, \phi)$.

Then (g, B, ϕ) satisfies the **equations of motion** *given by action S iff $\mathcal{R}^{+}_{\nabla} = 0$ and ∇ is Ricci compatible with V_+.*

- By the above observation, quantities $\mathcal{R}^{+}_{\nabla}$ and $\mathrm{Ric}^{+-}_{\nabla}$ do not depend on the choice inside $\mathrm{LC}(E, V_+, \phi)$. Also $\mathcal{R}_{\nabla} = 0$.
- All quantities behave as expected under CA isomorphisms, this description is very "covariant" in this sense.

Applications: Kaluza-Klein reduction

- Let $\pi : P \to M$ be a principal G-bundle with compact Lie group G, let $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ its adjoint bundle, and $c = (\cdot, \cdot)_{\mathfrak{g}}$ a corresponding negative-definite Killing form.
- Choose a connection $A \in \Omega^1(P, \mathfrak{g})$ and let $F \in \Omega^2(M, \mathfrak{g}_P)$ be its curvature. Let $H_0 \in \Omega^3(M)$. There is a structure of a **heterotic (almost) Courant algebroid** on $E' = TM \oplus \mathfrak{g}_P \oplus T^*M$:
  1. The pairing uses the canonical one and $(\cdot, \cdot)_{\mathfrak{g}}$.
  2. The anchor $\rho' : E' \to TM$ is the projection.
  3. The bracket is a combination of H_0-twisted Dorfman and the Atiyah-Lie algebroid bracket on $TM \oplus \mathfrak{g}_P$.
- It is an actual Courant algebroid, if H_0 is a potential for the first Pontryagin class of P with respect to $(\cdot, \cdot)_{\mathfrak{g}}$:

$$dH_0 + \frac{1}{2}(F \wedge F)_{\mathfrak{g}} = 0. \tag{14}$$

- Generalized metric $V'_+ \subset E'$ corresponds to (g_0, B_0, ϑ), where g_0 is a metric on M, $B_0 \in \Omega^2(M)$ and $\vartheta \in \Omega^1(M, \mathfrak{g}_P)$.

- A direct analogue of the theorem can be used to describe the equations of motion of the effective action

$$S_0[g_0, B_0, \phi_0, \vartheta] = \int_M e^{-2\phi_0}\{\mathcal{R}(g_0) + \frac{1}{2}\langle\langle F', F'\rangle\rangle \\ -\frac{1}{2}\langle H_0', H_0'\rangle_{g_0} + 4\langle d\phi_0, d\phi_0\rangle_{g_0} - 2\Lambda_0\} \cdot d\,\mathrm{vol}_{g_0}, \tag{15}$$

where $F' = F'(\vartheta)$ and $H' = H'(B_0, \vartheta)$ and $\Lambda_0 \in \mathbb{R}$ is a kind of a cosmological constant.
- This is sometimes called the **Einstein-Yang-Mills gravity**.
- By the choice of $P = P_{\mathbf{YM}} \times_M P_{\mathbf{Spin}}$ where $P_{\mathbf{YM}}$ is a principal $\mathrm{SO}(32)$ or $E(8) \times E(8)$ bundle and $P_{\mathbf{Spin}}$ is the $\mathrm{Spin}(9,1)$-bundle and by some minor fiddling, one may fit this onto the **heterotic supergravity**. The above condition on the Pontryagin class leads to the anomaly cancellation condition

$$[(F_{\mathbf{YM}} \wedge F_{\mathbf{YM}})_{\mathfrak{k}}]_{dR} = [(F_{\mathbf{Spin}} \wedge F_{\mathbf{Spin}})_{\mathfrak{so}}]_{dR}. \tag{16}$$

- Every heterotic Courant algebroid E' can be obtained by the reduction procedure from the Courant algebroid $E = \mathbb{T}P$ equipped by the H-twisted Dorfman, where

$$H = \pi^*(H_0) + \frac{1}{2}\mathrm{CS}_3(A) \tag{17}$$

- It resembles the symplectic reduction. There must exist a map $\Re : \mathfrak{g} \to \Gamma(E)$, such that $x \triangleright \psi = [\Re(x), \psi]_E$ defines a Lie algebra action, integrating to a (certain) Lie group action on E. Then set $K = \Re(P \times \mathfrak{g})$ and define

$$E' = \frac{K^{\perp}/G}{(K \cap K^{\perp})/G} \tag{18}$$

All CA structures on E' are naturally inherited from those of E.
- The generalized metric $V_+ \subset E$ under some conditions reduces to the generalized metric $V'_+ \subset E$. We have $V_+ \approx (g, B)$ and $V'_+ \approx (g_0, B_0, \vartheta)$. This "for free" provides some Kaluza-Klein like conditions on (g, B)!

Proposition

$V_+ \approx (g, B)$ can be reduced to $V'_+ \approx (g_0, B_0, \vartheta)$ iff (g, B) are G-invariant tensor fields and with respect to the decomposition $\Gamma_G(TP) \cong TM \oplus \mathfrak{g}_P$, they have the block form

$$g = \begin{pmatrix} 1 & \vartheta^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_0 & 0 \\ 0 & g_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \vartheta & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & \frac{1}{2}\vartheta^T c \\ -\frac{1}{2} c\vartheta & 0 \end{pmatrix}. \tag{19}$$

- Having the LC connections theorems in mind, one may examine the spaces of Levi-Civita connections on two Courant algebroids E and E', respectively. This can be done.

Proposition

Let $\phi = \phi_0 \circ \pi$. Then there is a pair of connections $\nabla \in \mathrm{LC}(E, V_+, \phi)$ and $\nabla' \in \mathrm{LC}(E', V'_+, \phi')$, such that ∇ reduces to ∇'.

In particular, ∇ is Ricci compatible with V_+ iff ∇' is Ricci compatible with V'_+ and

$$\mathcal{R}^+_\nabla = \mathcal{R}_{\nabla'} \circ \pi + \frac{1}{6}\dim(\mathfrak{g}). \tag{20}$$

Theorem (**Kaluza-Klein reduction**)

Fir (g, B, ϕ) and $(g_0, B_0, \vartheta, \phi_0)$ related as above and cosmological constants fulfilling $\Lambda = \Lambda_0 + \frac{1}{6}\dim(\mathfrak{g})$, the EOM for the action S are equivalent to those of S_0.

- In particular, the heterotic supergravity can be obtained from the ordinary type II supergravity (no fermions and RR fields) on $P = P_{\mathbf{YM}} \times_M P_{\mathbf{Spin}}$, if we impose some symmetry on (g, B, ϕ) and compare the cosmological constants.
- Reduction of CA was discussed in detail by (Bursztyn, Cavalcanti, Gualtieri 2005), (Baraglia, Hekmati 2013) or (Ševera 2015).
- For details see the paper

  Jan Vysoký: **Kaluza-Klein Reduction of Low-Energy Effective Actions: Geometrical Approach**, arXiv:1704.01123.

Applications: Poisson-Lie T-dual sigma models

- It is nice idea of (Ševera 2015, 2017) that the old (1994-ish) idea of Poisson-Lie T-duality (PLT duality) can be described in terms of reductions of CA.
- The simplest setting is the following. A **Manin pair** $(\mathfrak{d}, \mathfrak{g})$ is a pair of a quadratic LA $(\mathfrak{d}, \langle \cdot, \cdot \rangle_{\mathfrak{d}}, [\cdot, \cdot]_{\mathfrak{d}})$ together with its Lagrangian subalgebra $\mathfrak{g} \subset \mathfrak{d}$. Suppose it integrates to a pair (D, G) of a Lie group and its (closed) subgroup $G \subset D$.
- D can be viewed as a principal D-bundle over the point $\{*\}$ or a principal G-bundle $\pi_0 : P \to N$ over left cosets $N = D/G$.
- The CA $E = \mathbb{T}D$ with H-twisted Dorfman, where $H = \frac{1}{2}\mathrm{CS}_3(\theta_L)$ can be reduced in two ways. We get
  1. Reducing by D, we obtain $E'_{\mathfrak{d}} = (\mathfrak{d}, 0, \langle \cdot, \cdot \rangle_{\mathfrak{d}}, -[\cdot, \cdot]_{\mathfrak{d}})$.
  2. Reducing by G, we obtain $E'_{\mathfrak{g}} = N \times \mathfrak{d}$, where the anchor is the extension of the generator $\#^{\triangleright} : \mathfrak{d} \to \mathfrak{X}(N)$ of the left dressing action of D on N, the rest is a fiber-wise extension.

We can now do a following procedure:

1. Choose a generalized metric $\mathcal{E}_+ \subset E'_{\mathfrak{d}} = \mathfrak{d}$, that is a maximal positive subspace with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$.
2. The subbundle $V'_+ := N \times \mathcal{E}_+$ forms a GM in $E'_{\mathfrak{g}} = N \times \mathfrak{d}$.
3. $E'_{\mathfrak{g}}$ is so called **exact CA** over N. Those are always isomorphic to the standard CA on $\mathbb{T}N$ with H-twisted Dorfman bracket for H in a unique de Rham class $[H]_{dR}$.
4. Fix one of these isomorphism $\mathbf{\Psi} : \mathbb{T}N \to E'_{\mathfrak{g}}$ and use it to induce a generalized metric $V_+ \subseteq \mathbb{T}N$. We know that $V_+ \approx (g, B)$
5. One can now consider a sigma model (with WZW term) targeted in $N = D/G$ with backgrounds (g, B, H).

Proposition (Ševera 2017)

For fixed $\mathcal{E}_+ \subset \mathfrak{d}$, all so constructed (for any G) sigma models are (in some sense, under some technical conditions) equivalent.

In particular, if $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple integrating to (D, G, G^*), interchanging of G and G^* leads to the standard PLT duality.

- One should confirm this on the "quantum level" by comparing the corresponding low-energy effective actions. This is where our machinery enters.
- Fix a a connection $\nabla^0 \in \mathrm{LC}(\mathfrak{d}, \mathcal{E}_+)$. By moving it in the same fashion as before, one finds $\nabla \in \mathrm{LC}(\mathbb{T}N, V_+)$. By construction: $\mathcal{R}^+_{\nabla^0} = \mathcal{R}^+_{\nabla}$ **and ∇^0 is Ricci compatible with $\mathcal{E}_+$ if and only if ∇ is Ricci compatible with V_+.**

Tiny little catch

We do not know how the remaining background $\phi \in C^\infty(N)$ should like. We can find it by **enforcing** the condition $\nabla \in \mathrm{LC}(\mathbb{T}N, V_+, \phi)$.

- Not every Manin triple $(\mathfrak{d}, \mathfrak{g})$ allows for such solution. It turns out that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ must be a **unimodular** Lie algebra, that is $\mathrm{Tr}(\mathrm{ad}_x) = 0$ for all $x \in \mathfrak{g}$.
- In turn, the connection ∇^0 has to be divergence-free. This fixes it uniquely for the purposes of EOM.

- We had to make a technical assumption - (D, G) is a so called **complete group double**. There must exist a splitting of

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \arrow[r, "i"] & \mathfrak{d} \arrow[r, "i'"] & \mathfrak{g}^* \arrow[l, bend left, dashed, "j"] \arrow[r] & 0, \end{tikzcd} \tag{21}$$

such that $\xi \mapsto \#_s^{\triangleright}(j(\xi)) \equiv \xi_s^{\triangleright}$ is an isomorphism for all $s \in N$.

- We are then able to find an explicit formula for ϕ, unique up to an additive constant in terms of $\mathcal{E}_+$, blocks of Ad and a quasi-Poisson structure $\Pi_N \in \mathfrak{X}^2(N)$.
- For Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ we recover the dilaton formulas obtained from path integral formulation of PLT (von Unge 2002).

Theorem (Jurčo, V. 2017)

(g, B, ϕ) satisfy the equations of motion on N iff $\nabla^0 \in \mathrm{LC}(\mathfrak{d}, \mathcal{E}_+)$ is Ricci compatible with $\mathcal{E}_+$ and $\mathcal{R}^+_{\nabla^0} = 0$.
This is a system of algebraic equations for $\mathcal{E}_+$. By solving them, we obtain solutions of EOM on any such constructed coset space $N = D/G$.

Outlooks (a.k.a. dreams)

- It is hard to find solutions $\mathcal{E}_+$ for non-trivial examples of Manin pairs $(\mathfrak{d}, \mathfrak{g})$. Maybe adding the RR fields could save the day. Six-dimensional Manin triples are classified - in principle, one can find all solutions.
- Is there a "generalized geometry" to describe the fermionic fields? Courant algebroids on supermanifolds?
- Suppose $\pi : P \to M$ is any principal D-bundle, (D, G) still integrates a Manin pair $(\mathfrak{d}, \mathfrak{g})$. There is an intriguing geometry in the diagram

$$\begin{tikzcd} & P \arrow[dl, "\pi_0"'] \arrow[d, "\pi"] \\ P/G \arrow[r, dashed] & M \end{tikzcd} . \tag{22}$$

  In particular, reductions of CA provide relations of characteristic classes. Is this some kind of "topological" T-duality?

Branislav Jurčo, Jan Vysoký: **Poisson-Lie T-duality of String Effective Actions: A New Approach to the Dilaton Puzzle**, arXiv:1708.04079,

Branislav Jurčo, Jan Vysoký: **Courant Algebroid Connections and String Effective Actions**, arXiv:1612.0154.

Thank you for your attention!