Higher Structures in Algebraic Quantum Field Theory

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Based on joint works with Marco Benini and different subsets of
$\{$Urs Schreiber, Richard J. Szabo, Lukas Woike$\}$

Outline

1. Background on AQFT

2. Operadic formulation

3. Homotopy theory of AQFTs

4. Summary and outlook

Background on AQFT

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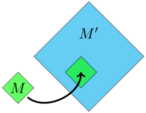

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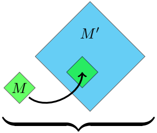

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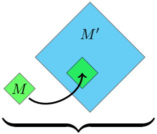

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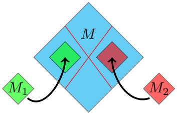

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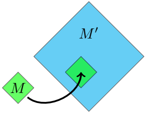

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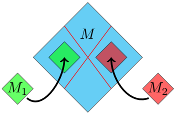

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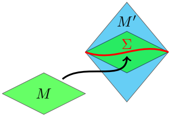
(ii) Time-Slice:



Cauchy morphism

$$\mathfrak{A}(M) \xrightarrow{\text{iso}} \mathfrak{A}(M')$$

The underlying algebraic structure

◇ **Input data:**

- A triple $(\mathbf{C}, \perp, W)$ consisting of a category $\mathbf{C}$ with orthogonality relation $\perp \subseteq \mathrm{Mor}\,\mathbf{C}\,{}_{\mathrm{t}}\!\times_{\mathrm{t}} \mathrm{Mor}\,\mathbf{C}$ (symmetric & $\circ$-stable subset) and $W \subseteq \mathrm{Mor}\,\mathbf{C}$

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NB: The relevant categories are $\mathbf{QFT}(\mathbf{C}, \perp) := \mathbf{qft}(\mathbf{C}, \perp, \emptyset)$

Operadic formulation

Motivation

- I will now show that for each orthogonal category $(\mathbf{C}, \perp)$ there exists a colored operad $\mathcal{O}_{(\mathbf{C},\perp)}$ such that

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   Homotopy theory of operads and their algebras is well understood! [Berger,Moerdijk; Hinich; Spitzweck; . . .]

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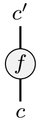

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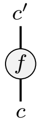

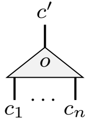

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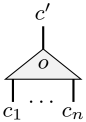
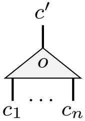


◇ $\mathcal{O}$-algebras are like "representations":



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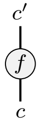

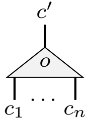

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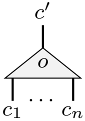

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◇ **Important construction:** For every colored operad map $\phi : \mathcal{O} \to \mathcal{P}$, there exists an adjunction

$$\phi_! \ : \ \mathbf{Alg}(\mathcal{O}) \rightleftarrows \mathbf{Alg}(\mathcal{P}) \ : \ \phi^*$$

with ϕ^* pullback of $\mathcal{P}$-algebras and $\phi_!$ operadic left Kan extension.

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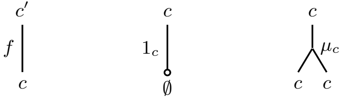

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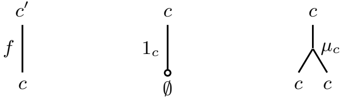

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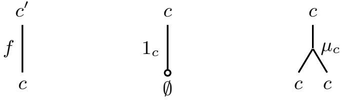

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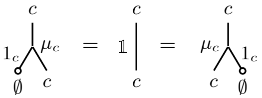

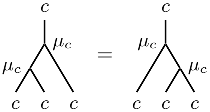

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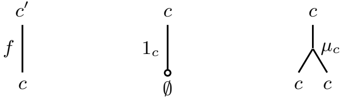

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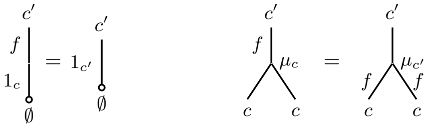

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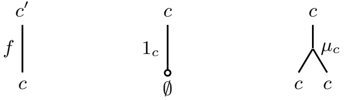

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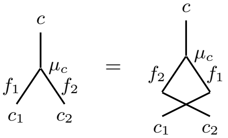

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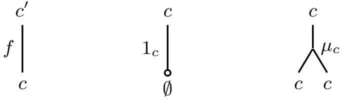
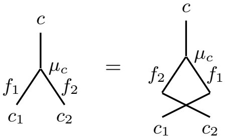


Relations: Functoriality + Monoid + Compatibility + $\perp$-commutativity



for all $f_1 \perp f_2$

Thm: The assignment $(\mathbf{C},\perp) \mapsto \mathcal{O}_{(\mathbf{C},\perp)}$ is functorial on the category of orthogonal categories. There exists a natural isomorphism of categories

$$\mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \;\cong\; \mathbf{QFT}(\mathbf{C},\perp)$$

Adjunctions between QFT categories

! Every orthogonal functor $F : (\mathbf{C}, \perp) \to (\mathbf{C}', \perp')$ defines operad map $\mathcal{O}_F : \mathcal{O}_{(\mathbf{C},\perp)} \to \mathcal{O}_{(\mathbf{C}',\perp')}$ and hence adjunction between algebra categories.

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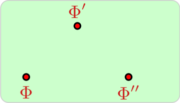
Rem: $\mathsf{ext} : \mathbf{QFT}(\mathbf{Loc}_\diamond, j^*(\perp)) \to \mathbf{QFT}(\mathbf{Loc}, \perp)$ is operadic refinement of Fredenhagen's universal algebra construction

Homotopy theory of AQFTs

Higher structures in gauge theory

◇ **Gauge theory = higher spaces of fields**



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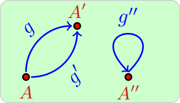

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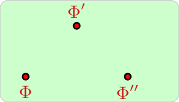

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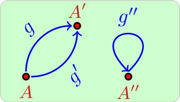

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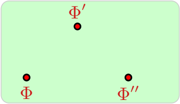

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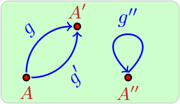

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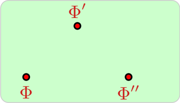

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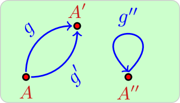

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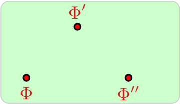

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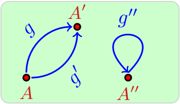

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Common feature of higher geometry and algebra

Higher spaces/algebras come with a notion of weak equivalences $X \xrightarrow{\sim} Y$

$\Rightarrow$ Need for higher category theory or model category theory!

Model structure for strict AQFTs [Benini,AS,Woike]

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   Derived extension functor $\mathbb{L}\mathrm{ext} : \mathbf{QFT}(\mathbf{Loc}_{\diamond}, j^*(\perp)) \longrightarrow \mathbf{QFT}(\mathbf{Loc}, \perp)$ is needed to obtain correct global gauge theory observables [Benini,AS,Szabo]

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- A diagram $X : \mathbf{Loc}^{\mathrm{op}} \to \mathsf{PSh}(\mathbf{Man},\mathbf{sSet})$ of stacks defines a functor $N^{\infty *}(X) : \mathbf{Loc} \to E_\infty\mathbf{Alg}$

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- Consider stack $Y \in \mathsf{PSh}(\mathbf{Man}, \mathbf{sSet})$, e.g. Yang-Mills [Benini,AS,Schreiber]
- Normalized chains $N_\ast : \mathsf{PSh}(\mathbf{Man}, \mathbf{sSet}) \to \mathsf{PSh}(\mathbf{Man}, \mathbf{Ch}(k))$ and internal hom defines E_∞-algebra $N^{\infty\ast}(Y) = [N_\ast Y, k]^\infty$ of "functions" on Y
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Resolutions and homotopy AQFTs

◇ Homotopy $\mathcal{O}$-algebras = algebras over (Σ-)cofibrant resolution $\mathcal{O}_\infty \stackrel{\sim}{\longrightarrow} \mathcal{O}$

Thm: For every $(\mathbf{C},\perp)$, the AQFT operad $\mathcal{O}_{(\mathbf{C},\perp)}$ is Σ-cofibrant.

Every (Σ-)cofib. resolution ${\mathcal{O}_{(\mathbf{C},\perp)}}_\infty \stackrel{\sim}{\longrightarrow} \mathcal{O}_{(\mathbf{C},\perp)}$ induces Quillen equivalence

$$\mathbf{QFT}_\infty(\mathbf{C},\perp) := \mathbf{Alg}({\mathcal{O}_{(\mathbf{C},\perp)}}_\infty) \underset{\sim}{\rightleftarrows} \mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)}) \cong \mathbf{QFT}(\mathbf{C},\perp)$$

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Examples via homotopy invariants (orbifoldization)

- Let $\pi : (\mathbf{D}, \pi^*(\perp)) \to (\mathbf{C}, \perp)$ be (strictified) orthogonal category fibered in groupoids and consider $\mathbf{QFT}(\mathbf{D}, \pi^*(\perp))$

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Thm: Using the typical Bousfield-Kan model

$$\mathsf{hoRan}_\pi\, \mathfrak{A}(c) \;=\; \int_{d \in \pi^{-1}(c)} \Big[N_*\big(B(\pi^{-1}(c) \downarrow d)\big), \mathfrak{A}(d) \Big] \quad ,$$

the functor $\mathsf{hoRan}_\pi\, \mathsf{U} : \mathbf{QFT}(\mathbf{D}, \pi^*(\perp)) \to \mathbf{dgAlg}(k)^{\mathbf{C}}$ admits a lift along $\mathsf{U} : \mathbf{Alg}(\mathcal{O}_{(\mathbf{C},\perp)} \otimes E_\infty) \to \mathbf{dgAlg}(k)^{\mathbf{C}}$.

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× **Open problem:** Examples of quantum gauge theories, e.g. via deformation quantization of (derived) symplectic stacks [Calaque,Pantev,Toën,Vaquié,Vezzosi]