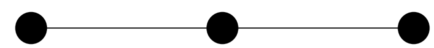
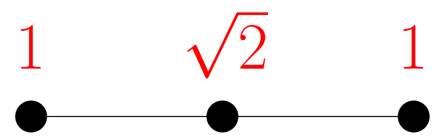
Are almost all graphs determined by their spectrum?

Willem H. Haemers

Tilburg University, The Netherlands.



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



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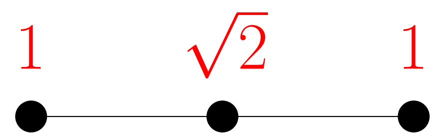
spectrum (eigenvalues): $\quad -\sqrt{2},\ 0,\ \sqrt{2}$



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A (finite simple) graph Γ on n nodes (vertices)

$$\Downarrow$$

The spectrum $\lambda_1 \geq \ldots \geq \lambda_n$ of the adjacency matrix A of Γ

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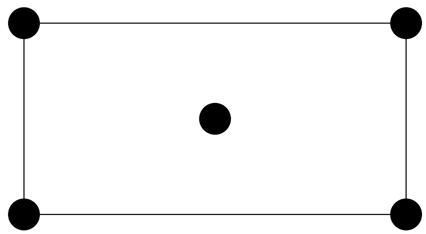
The spectrum $\lambda_1 \geq \ldots \geq \lambda_n$ of the adjacency matrix A of Γ



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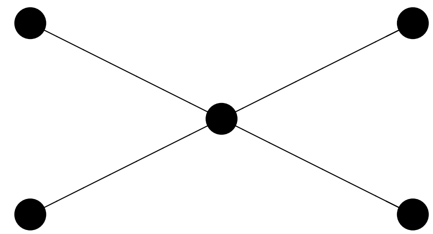

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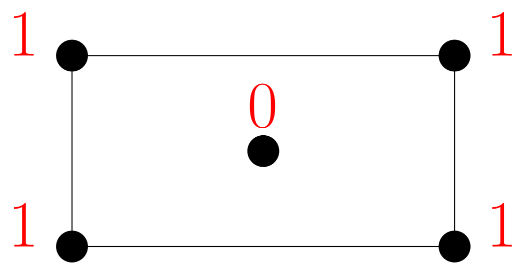

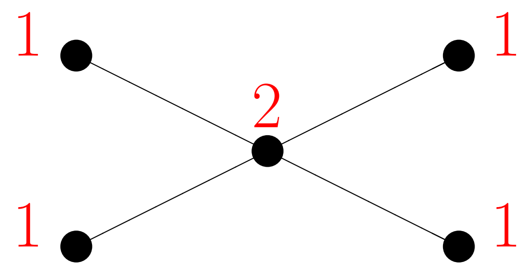

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spectrum: $-2,\ 0,\ 0,\ 0,\ 2$

Applications

- Energy of hydrocarbon molecules.
- Shape and sound of a drum.
- Graph isomorphism problem.
- Algebraic characterizations.
- Analysis of large networks.

Conjecture

Almost all graphs are determined by their spectrum (DS).

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Against

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False for trees.

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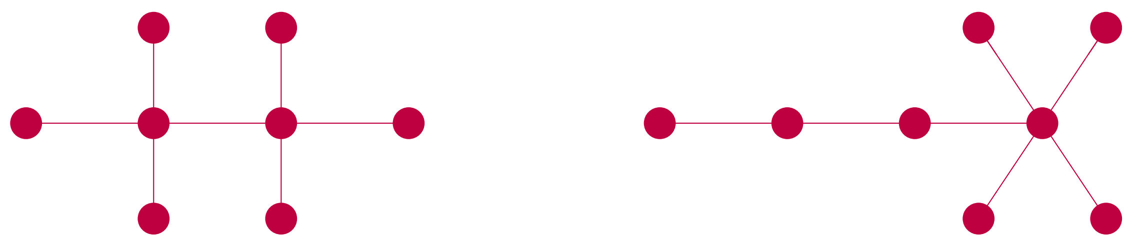
False for trees. Schwenk (1973)



Two cospectral trees

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False for strongly regular graphs.

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♡	♣	♢	♠
♠	♡	♣	♢
♢	♠	♡	♣
♣	♢	♠	♡

♡	♣	♢	♠
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Vertices: entries; adjacent: same row, column, or symbol.

Two cospectral strongly regular graphs.

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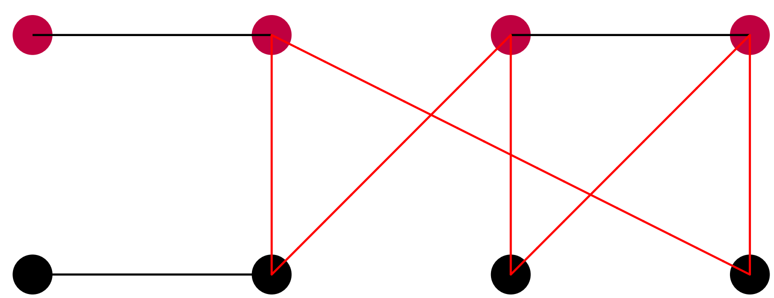
Godsil-McKay switching



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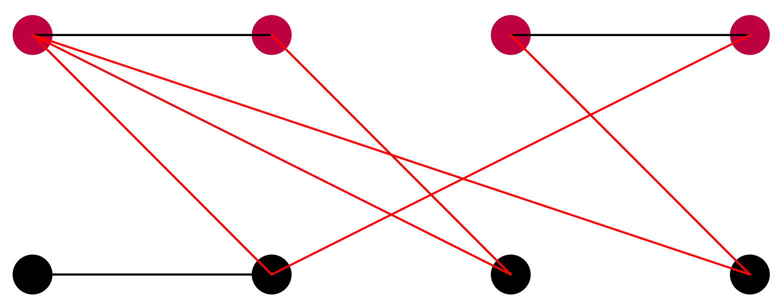

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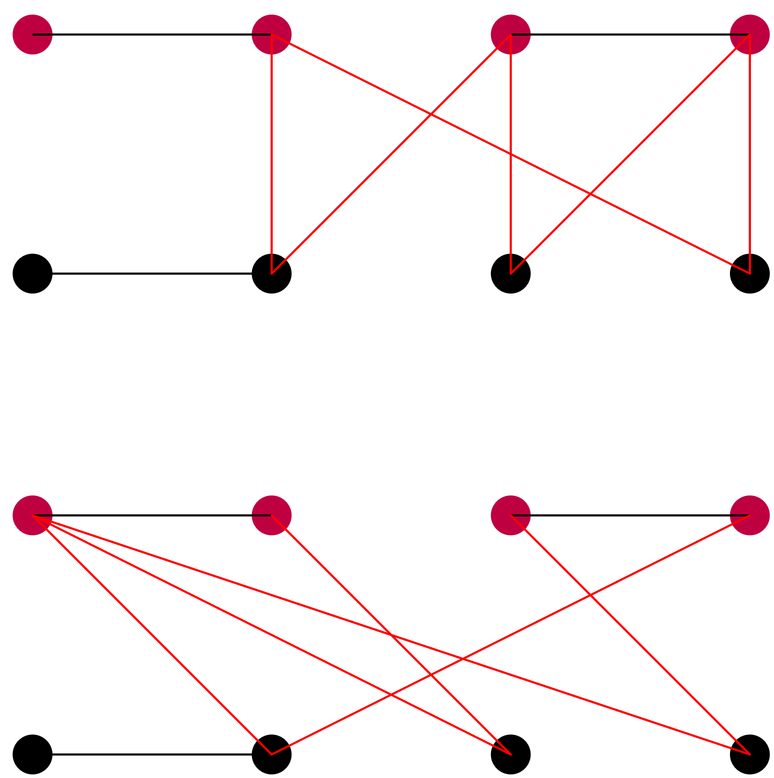
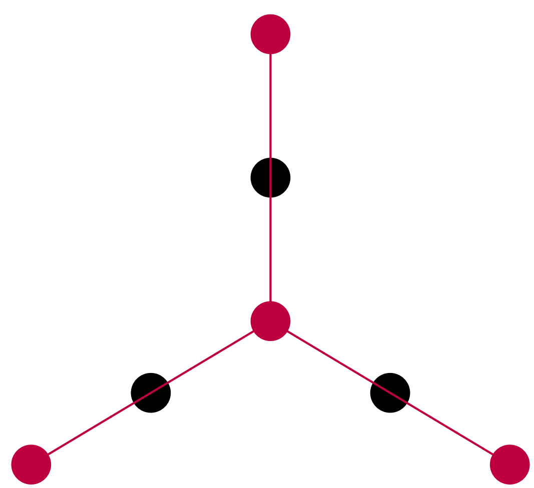
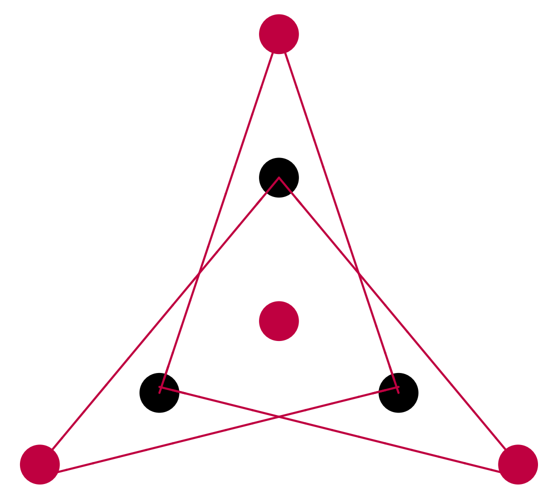


Godsil-McKay switching





Remarks

- If X is a Godsil-McKay switching set of a graph Γ, then X is also a Godsil-McKay switching set in the complement of Γ. Therefore switching also produces cospectral complements.

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- From any graph with n nodes one can make $\binom{n}{3}$ cospectral pairs on $n+1$ nodes by use of Godsil-McKay switching.

- For fixed n, the number of graphs that are not DS is at least

$$cn^3 g_{n-1}$$

for some constant c, where g_{n-1} denotes the number of non-isomorphic graphs on $n-1$ vertices. (Note that $g_{n-1} \approx 2^{-n} g_n$.)

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False for strongly regular graphs.
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Best know lower bound: $c^{\sqrt{n}}$.

Theorem

The following graph properties can be deduced from the spectrum:

- number of nodes
- number of edges
- number of triangles
- bipartite
- regular

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- number of nodes ($= n$)
- number of edges ($= \frac{1}{2}\Sigma_i \lambda_i^2$)
- number of triangles ($= \frac{1}{6}\Sigma_i \lambda_i^3$)
- bipartite ($\Leftrightarrow \lambda_i = -\lambda_{n-i+1}\ ,\ i = 1, \ldots, n$)
- regular ($\Leftrightarrow \lambda_1 = \frac{1}{n}\Sigma_i \lambda_i^2$)

Some graphs which are DS.

- The complete graph K_n
- The complete bipartite graph with equal parts $K_{k,k}$
- The cycle C_n
- The path P_n
- The complement of the path $\overline{P_n}$

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Theorem If Γ is connected, regular and DS then

- The complement of Γ is DS
- If $n+1$ is not a square then the line graph $L(\Gamma)$ of Γ is DS

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Complete computer search for small n.

Fractions of graphs on n vertices which are DS.

n	number of graphs	fraction	reference
1	1	1	
2	2	1	
3	4	1	
4	11	1	
5	34	0.941	
6	156	0.936	
7	1044	0.895	
8	12346	0.861	
9	274668	0.814	Godsil, McKay (1976)

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Complete computer search for $n = 11$ and $n = 12$.

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Several randomly generated graphs are DGS. Wang and Xu (2006)

DGS: DS w.r.t. the spectrum together with the spectrum of the complement.

Example of a graph which is DGS Wang and Xu (2006) $n = 32$.

0	1	0	1	0	1	1	0	1	0	0	1	0	1	1	0	0	0	1	1	0	1	1	1	1	1	1	0	0	1	0
1	0	1	1	0	0	0	0	1	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	0	1	0	0	1	0	0
0	1	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	1	1	1	0	0	0	1	1	1	1	0	1
1	1	1	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	1	1	0	1	0	1	1	1	0	0	0	1	0	1	0
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1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1	1	0	1	0	0	0	1	1	1	1	1	1	0
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Complete computer search for $n = 11$ and $n = 12$.

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Almost all graphs in $\mathcal{F}_n$ are DGS. Wang (and Xu) (2010, 2013)

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Method of Wang and Xu.

Definition.

Graph Γ with matrix A is *controllable* if the walk matrix

$$W = \begin{bmatrix} \mathbf{1} & A\mathbf{1} & A^2\mathbf{1} & \dots & A^{n-1}\mathbf{1} \end{bmatrix}$$

is nonsingular.

Definition.

$\Gamma \in \mathcal{F}_n$ if Γ is controllable and $\det(W)/2^{\lfloor n/2 \rfloor}$ is square free.

Theorem Wang (and Xu) (2010, 2013)

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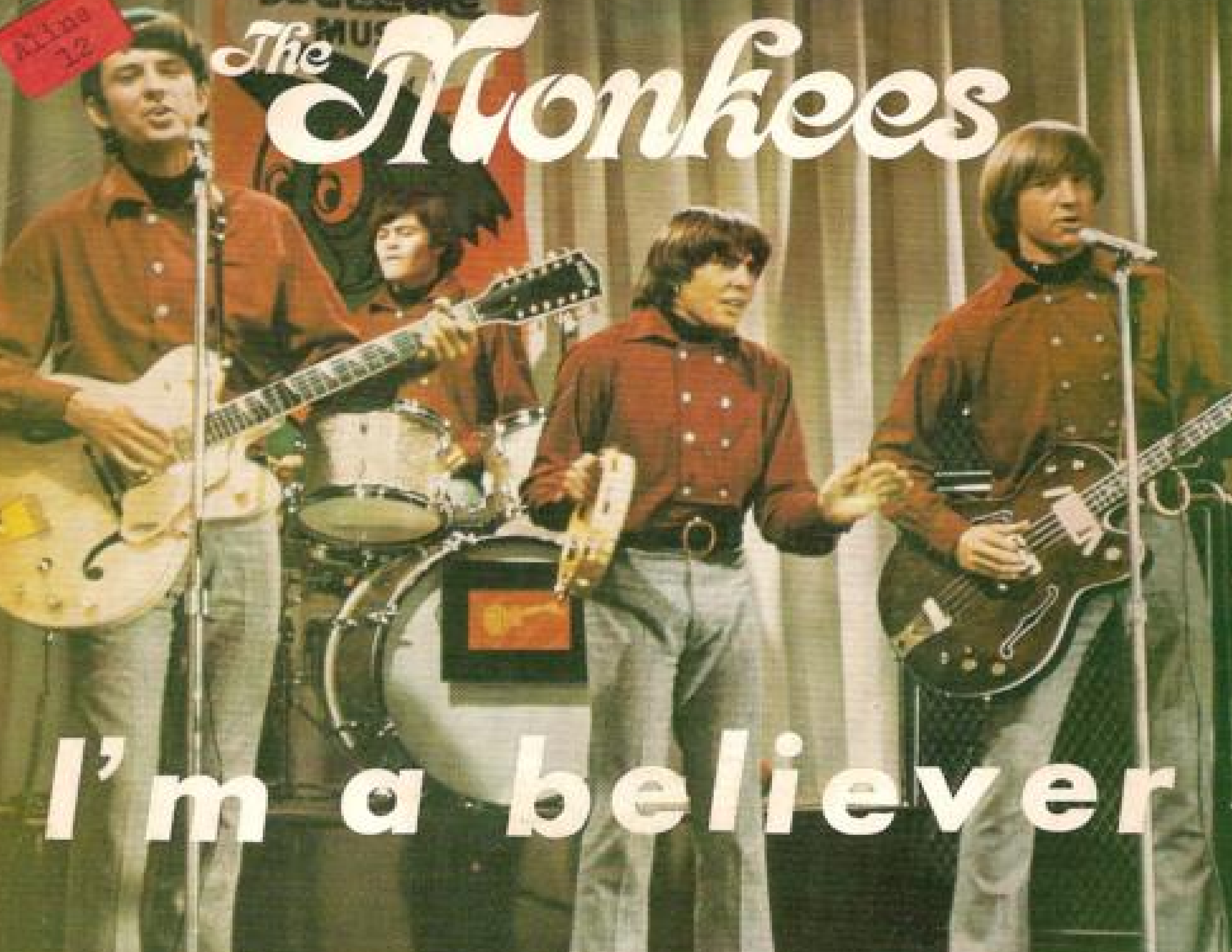
The Monkees
I'm a believer