Eigenvalues and distance-regularity of graphs

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Dedication

David Gregory

Spectrum

A (finite simple) graph Γ on n vertices

$\Downarrow$ $\Uparrow$ **?**

The spectrum (of eigenvalues) $\lambda_1 \geq \ldots \geq \lambda_n$
of the (a) 01-adjacency matrix A of Γ

Two many

There are 2 graphs on 30 vertices with spectrum

$$12,\ 2\ (9\times),\ 0\ (15\times),\ -6\ (5\times).$$

There are more than 60,000 graphs on 30 vertices with spectrum

$$12,\ 3\ (10\times),\ 0\ (5\times),\ -3\ (14\times).$$

Distance-regular

Distance-regularity: there are c_i, a_i, b_i, $i = 0, 1, \ldots, d$ such that for every pair of vertices u and w at distance i:

\# neighbors z of w at distance $i-1$ from u equals c_i
\# neighbors z of w at distance i from u equals a_i
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Hamming graphs, Johnson graphs, Grassmann graphs, Odd graphs ....

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Fon-Der-Flaass (2002) $\Rightarrow$ Almost all distance-regular graphs are not determined by the spectrum.

cf. EvD & Haemers (2003) 'would bet' that almost all graphs are determined by the spectrum.

Walks

A_i is the distance-i adjacency matrix, $A = A_1$:

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad i = 0, 1, \ldots, d,$$

$A_i = p_i(A)$ for a polynomial p_i of degree i

Rowlinson (1997): A graph is a DRG iff the number of walks of length ℓ from x to y depends only on ℓ and the distance between x and y

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Distance-regular graphs: intersection numbers $\leftrightarrow$ eigenvalues

Intersection numbers do not determine the graph (in general)

Do the eigenvalues determine distance-regularity ?

Central equation

$$\sum_u (A^\ell)_{uu} = \operatorname{tr} A^\ell = \sum_i \lambda_i^\ell$$

$$\sum_u p(A)_{uu} = \operatorname{tr} p(A) = \sum_i p(\lambda_i)$$

for every polynomial p

All spectral information is in these equations

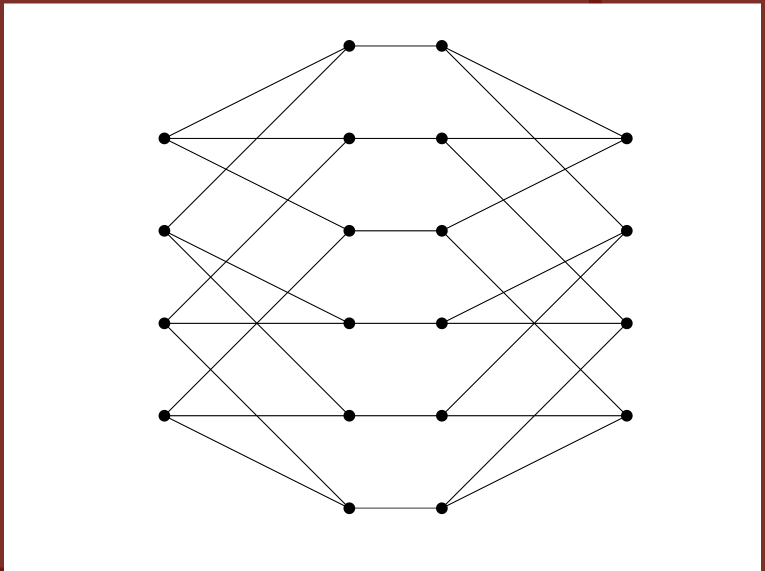
Structure

The following can be derived from the spectrum:

- number of vertices
- number of edges
- number of triangles
- number of closed walks of length ℓ
- bipartiteness
- regularity
- regularity + connectedness
- regularity + girth
- odd-girth

Twisted and odd

Distance-regularity is not determined by the spectrum



The ('almost' dr) twisted Desargues graph
(Bussemaker & Cvetković 1976, Schwenk 1978)

Note: Desargues is Doubled Petersen

Good conditions

Theorem. If Γ is distance-regular, diameter d, valency k, girth g, distinct eigenvalues $k = \theta_0, \theta_1, \ldots, \theta_d$, satisfying one of the following properties, then every graph cospectral with Γ is also distance-regular:

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1. $g \geq 2d - 1$ (Brouwer&Haemers),
2. $g \geq 2d - 2$ and Γ is bipartite (EvD&Haemers),
3. $g \geq 2d - 2$ and $c_{d-1}c_d < -(c_{d-1} + 1)(\theta_1 + \ldots + \theta_d)$ (EvD&Haemers),
4. $c_1 = \ldots = c_{d-1} = 1$ (EvD&Haemers),
5. $\Gamma =$ dodecahedron or icosahedron (Haemers&Spence),
6. $\Gamma =$ coset graph extended ternary Golay code (EvD&Haemers),
7. $\Gamma =$ Ivanov-Ivanov-Faradjev graph (EvD&Haemers&Koolen&Spence),
8. $\Gamma =$ line graph Petersen graph or line graph Hoffman-Singleton graph (EvD&Haemers),
9. $\Gamma =$ Hamming graph $H(3, q)$, $q \geq 36$ (Bang&EvD&Koolen),
10. $\Gamma =$ generalized odd graph ($a_1 = \ldots = a_{d-1} = 0$, $a_d \neq 0$) (Huang&Liu).

Polynomials

Consider the spectrum of a k-regular graph

Inner product $\langle p, q\rangle = \frac{1}{n}\operatorname{tr}(p(A)q(A)) = \frac{1}{n}\sum_i p(\lambda_i)q(\lambda_i)$

on the space of polynomials mod minimal polynomial

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Orthogonal system of predistance polynomials p_i of degree i normalized such that $\langle p_i, p_i \rangle = p_i(k) \neq 0$

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1}, \quad i = 0, 1, \ldots, d,$$

compare to

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}, \quad i = 0, 1, \ldots, d,$$

$H = \sum_i p_i$ is the Hoffman polynomial: $H(A) = J$

Projection

$\langle X, Y\rangle = \frac{1}{n}\,\mathrm{tr}(XY)$: inner product on symmetric matrices of size n

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Project A_d onto the space $\mathcal{A}$ of polynomials in A:

$$\widetilde{A}_d = \sum_{i=0}^{d} \frac{\langle A_d, p_i(A) \rangle}{\|p_i(A)\|^2} p_i(A) = \frac{\langle A_d, p_d(A) \rangle}{\|p_d(A)\|^2} p_d(A)$$

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$$\begin{aligned}\widetilde{A_d} &= \sum_{i=0}^{d} \frac{\langle A_d, p_i(A)\rangle}{\|p_i(A)\|^2} p_i(A) = \frac{\langle A_d, p_d(A)\rangle}{\|p_d(A)\|^2} p_d(A) \\ &= \frac{\langle A_d, H(A)\rangle}{\|p_d\|^2} p_d(A) = \frac{\langle A_d, J\rangle}{p_d(k)} p_d(A) = \frac{\overline{k}_d}{p_d(k)} p_d(A)\end{aligned}$$

where $\overline{k}_d = \frac{1}{n}\sum_u k_d(u)$

Spectral Excess

$$\overline{k}_d = \|A_d\|^2 \geq \|\widetilde{A_d}\|^2 = \frac{\overline{k}_d^2}{p_d(k)^2}\|p_d(A)\|^2 = \frac{\overline{k}_d^2}{p_d(k)}$$

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Spectral Excess Theorem (Fiol & Garriga 1997):

$\overline{k}_d \leq p_d(k)$ with equality iff the graph is distance-regular

Desargues

Find graphs with spectrum $\{3^1, 2^4, 1^5, -1^5, -2^4, -3^1\}$.

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Connected, 3-regular, bipartite on $10 + 10$ vertices, girth 6.

So this is the incidence graph of a partial linear space.

Diameter at most 5, with $\overline{k}_5 \leq 1$.

Distance distribution diagram: $20 = 1_3 + {}_1 3_2 + {}_1 6_2 + ? + 3 + ?$

$k_4(x) = 3$ so $k_5(x) \leq 1$.

Partial linear space

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Make lines of size 3: $\{12, 13, 23\}, \{12, 14, 24\}, \{12, 15, 25\}$
lines '123', '124', '125'

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Or (the twisted way): $\{12, 13, 14\}, \{12, 23, 24\}, \{12, 15, 25\}$
lines '1', '2', '125'

q-ary Desargues

$(2d-1)$-dimensional vector space over $GF(q)$

points: $(d-1)$-dimensional subspaces
lines: d-dimensional subspaces

Incidence graph is doubled **Grassmann**
Point and line graph are **Grassmann** $J_q(2d-1, d-1)$

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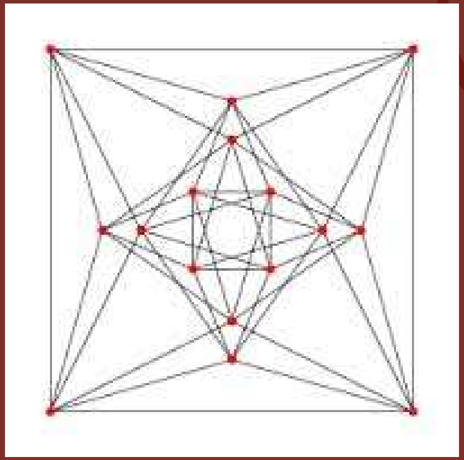
Line graph is cospectral to $J_q(2d-1, d-1)$
Spectral excess theorem: line graph is distance-regular!

....but it is UGLY!!!

Families of ‘ugly’ distance-regular graphs with unbounded diameter:

Doob, Hemmeter, Ustimenko: not distance-transitive.



twisted Grassmann (aka vD-Koolen 2005): not even vertex-transitive.

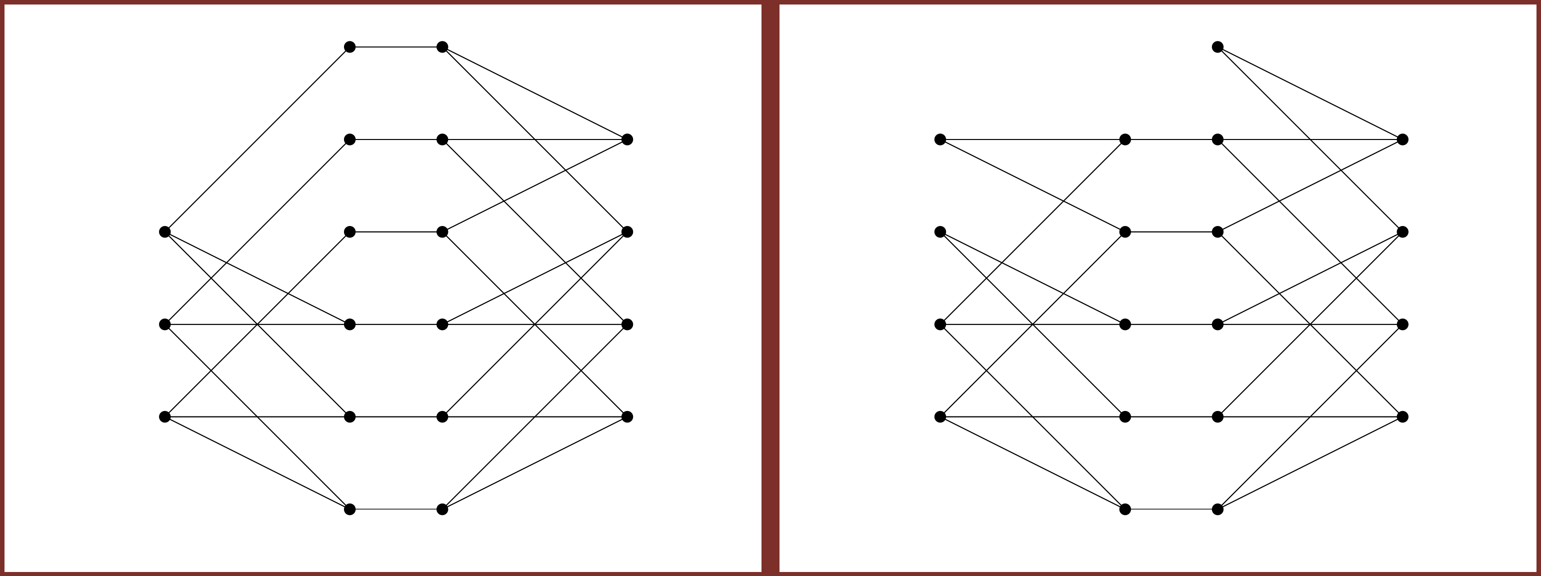
Perturbations

Dalfó & EvD & Fiol (2011): Ugly (almost) distance-regular graphs can be used to construct cospectral graphs through perturbations:

Adding and removing vertices, edges, amalgamating vertices, etc.

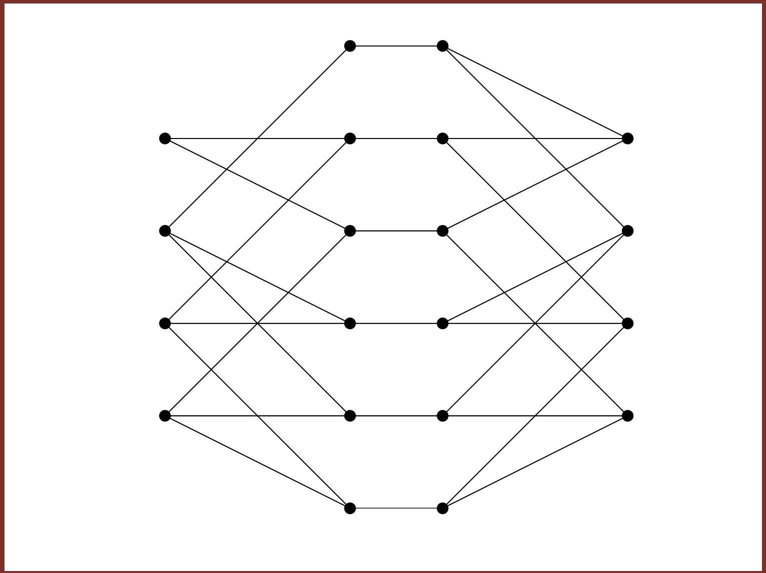
The devil's advocate (Durham, 2013): It is easy to construct cospectral graphs

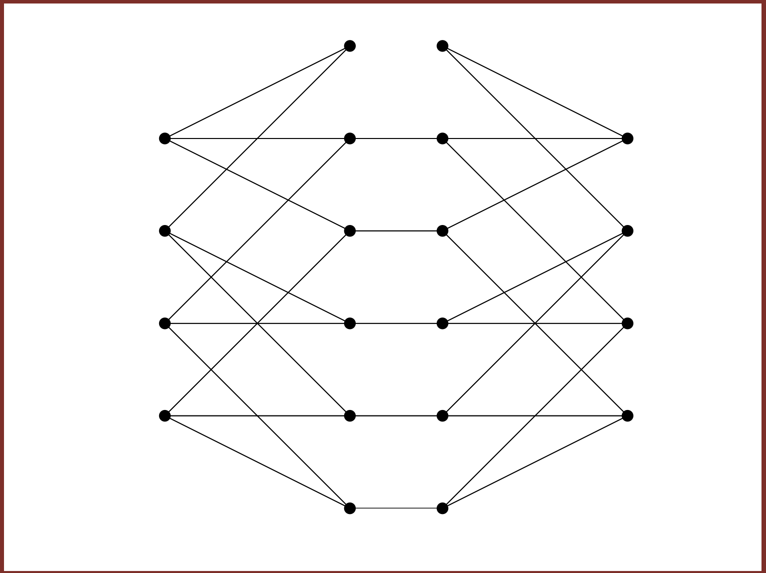
Remove vertices



Removing vertices from the twisted Desargues graph

Remove edges





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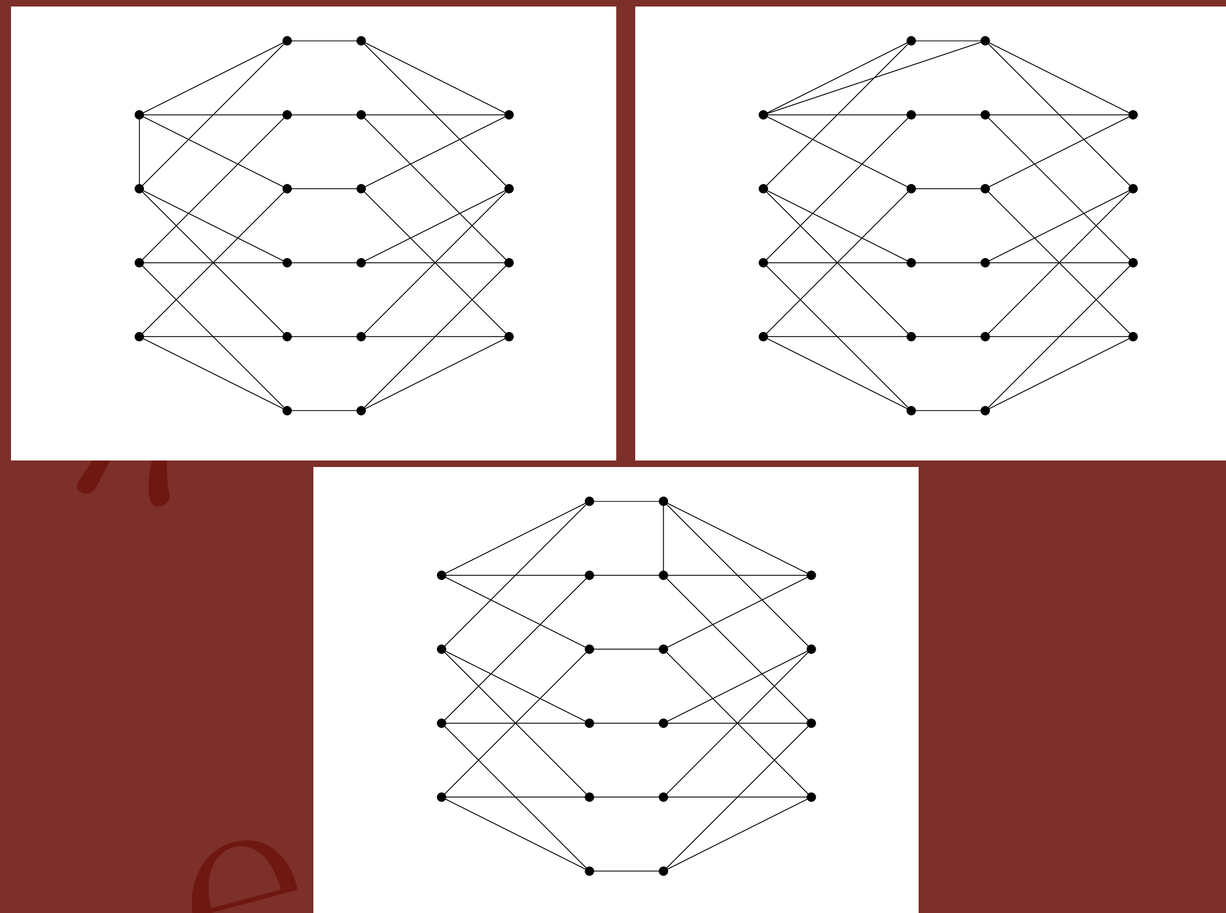
Adding edges



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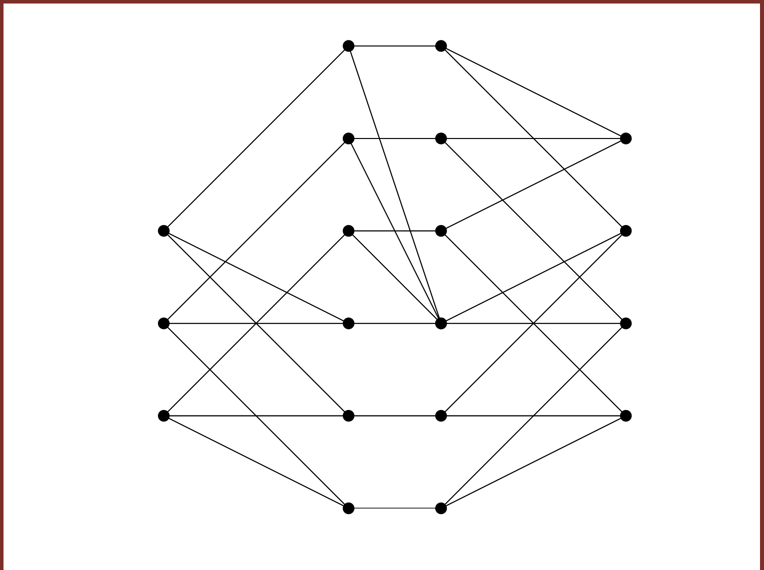
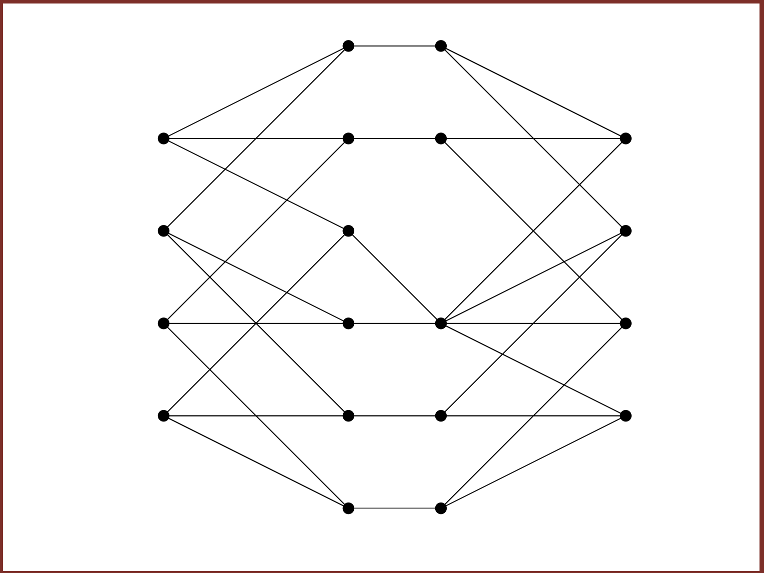
Amalgamate





Amalgamate vertices in the twisted Desargues graph

Generalized Odd

Generalized odd graph (drg with $a_1 = \ldots = a_{d-1} = 0,\ a_d \neq 0$)

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Sketch of short proof (EvD & Fiol 2012):

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If θ is an eigenvalue, then $-\theta$ is not

Proof

$A^\ell = \sum_i \theta_i^\ell E_i$ (spectral decomposition)

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If $dist(u,v)$ and d have different parity: $p_d(A)_{uv} = 0$

If $dist(u,v) < d$ and d have same parity: $\alpha_d p_d(A)_{uv} = \beta_{d-1} p_{d-1}(A)_{uv} + \alpha_d p_d(A)_{uv} = (A p_d(A))_{uv} = \sum_{w \sim u} p_d(A)_{wv} = 0$

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$A_d = p_d(A)$ so by the spectral excess theorem the graph is distance-regular

THE END