Eigenvalues and distance-regularity of graphs

Edwin van Dam

Dept. Econometrics and Operations Research
Tilburg University

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Dedication

EvDRG
Dedication
Spectrum
Two many
Distance-regular
Walks
Central equation
Structure
Twisted and odd
Good conditions
Polynomials
Projection
Spectral Excess
Desargues
Partial linear space
$q$-ary Desargues
Ugly DRGs
Perturbations
Remove vertices
Remove edges
Adding edges
Amalgamate
Generalized Odd
Proof

David Gregory

Durham, July 20, 2013
A (finite simple) graph $\Gamma$ on $n$ vertices

The spectrum (of eigenvalues) $\lambda_1 \geq \ldots \geq \lambda_n$ of the (a) 01-adjacency matrix $A$ of $\Gamma$
There are 2 graphs on 30 vertices with spectrum $12, 2 \ (9 \times), 0 \ (15 \times), -6 \ (5 \times)$.

There are more than 60,000 graphs on 30 vertices with spectrum $12, 3 \ (10 \times), 0 \ (5 \times), -3 \ (14 \times)$.
Distance-regular: there are $c_i, a_i, b_i, i = 0, 1, \ldots, d$ such that for every pair of vertices $u$ and $w$ at distance $i$:

- # neighbors $z$ of $w$ at distance $i - 1$ from $u$ equals $c_i$
- # neighbors $z$ of $w$ at distance $i$ from $u$ equals $a_i$
- # neighbors $z$ of $w$ at distance $i + 1$ from $u$ equals $b_i$
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Complete graphs, Strongly regular graphs (among which are regular complete multipartite graphs), Cycles,

Hamming graphs, Johnson graphs, Grassmann graphs, Odd graphs ....
Distance-regularity: there are $c_i, a_i, b_i$, $i = 0, 1, \ldots, d$ such that for every pair of vertices $u$ and $w$ at distance $i$:

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Fon-Der-Flaass (2002) ⇒ Almost all distance-regular graphs are not determined by the spectrum.

cf. EvD & Haemers (2003) ‘would bet’ that almost all graphs are determined by the spectrum.
$A_i$ is the distance-$i$ adjacency matrix, $A = A_1$:

$$AA_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1}, \quad i = 0, 1, \ldots, d,$$

$A_i = p_i(A)$ for a polynomial $p_i$ of degree $i$

Rowlinson (1997): A graph is a DRG iff the number of walks of length $\ell$ from $x$ to $y$ depends only on $\ell$ and the distance between $x$ and $y$
Walks

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Distance-regular graphs: intersection numbers $\leftrightarrow$ eigenvalues

Intersection numbers do not determine the graph (in general)

Do the eigenvalues determine distance-regularity?
Central equation

\[ \sum_{u} (A^\ell)_{uu} = \text{tr } A^\ell = \sum_{i} \lambda_i^\ell \]

\[ \sum_{u} p(A)_{uu} = \text{tr } p(A) = \sum_{i} p(\lambda_i) \]

for every polynomial \( p \)

All spectral information is in these equations
The following can be derived from the spectrum:

- number of vertices
- number of edges
- number of triangles
- number of closed walks of length $\ell$
- bipartiteness
- regularity
- regularity + connectedness
- regularity + girth
- odd-girth
Distance-regularity is not determined by the spectrum

The (‘almost’ dr) twisted Desargues graph  
(Bussemaker & Cvetković 1976, Schwenk 1978)

Note: Desargues is Doubled Petersen
Theorem. If $\Gamma$ is distance-regular, diameter $d$, valency $k$, girth $g$, distinct eigenvalues $k = \theta_0, \theta_1, \ldots, \theta_d$, satisfying one of the following properties, then every graph cospectral with $\Gamma$ is also distance-regular:
Theorem. If $\Gamma$ is distance-regular, diameter $d$, valency $k$, girth $g$, distinct eigenvalues $k = \theta_0, \theta_1, \ldots, \theta_d$, satisfying one of the following properties, then every graph cospectral with $\Gamma$ is also distance-regular:

1. $g \geq 2d - 1$ (Brouwer&Haemers),
2. $g \geq 2d - 2$ and $\Gamma$ is bipartite (EvD&Haemers),
3. $g \geq 2d - 2$ and $c_{d-1}c_d < -(c_{d-1} + 1)(\theta_1 + \ldots + \theta_d)$ (EvD&Haemers),
4. $c_1 = \ldots = c_{d-1} = 1$ (EvD&Haemers),
5. $\Gamma =$ dodecahedron or icosahedron (Haemers&Spence),
6. $\Gamma =$ coset graph extended ternary Golay code (EvD&Haemers),
7. $\Gamma =$ Ivanov-Ivanov-Faradjev graph (EvD&Haemers&Koolen&Spence),
8. $\Gamma =$ line graph Petersen graph or line graph Hoffman-Singleton graph (EvD&Haemers),
9. $\Gamma =$ Hamming graph $H(3, q)$, $q \geq 36$ (Bang&EvD&Koolen),
10. $\Gamma =$ generalized odd graph $(a_1 = \ldots = a_{d-1} = 0, a_d \neq 0)$ (Huang&Liu).
Consider the spectrum of a $k$-regular graph

**Inner product** $\langle p, q \rangle = \frac{1}{n} \text{tr}(p(A)q(A)) = \frac{1}{n} \sum_i p(\lambda_i)q(\lambda_i)$

on the space of polynomials mod minimal polynomial
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on the space of polynomials mod minimal polynomial

Orthogonal system of **predistance polynomials** $p_i$ of degree $i$ normalized such that $\langle p_i, p_i \rangle = p_i(k) \neq 0$

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_ip_i + \gamma_{i+1}p_{i+1}, \quad i = 0, 1, \ldots, d,$$

compare to

$$AA_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1}, \quad i = 0, 1, \ldots, d,$$

$$H = \sum_i p_i$$ is the **Hoffman** polynomial: $H(A) = J$
\[
\langle X, Y \rangle = \frac{1}{n} \text{tr}(XY): \text{ inner product on symmetric matrices of size } n
\]

\[
\langle p(A), q(A) \rangle = \langle p, q \rangle
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**Project** \( A_d \) onto the space \( A \) of polynomials in \( A \):

\[
\tilde{A}_d = \sum_{i=0}^{d} \frac{\langle A_d, p_i(A) \rangle}{\|p_i(A)\|^2} p_i(A) = \frac{\langle A_d, p_d(A) \rangle}{\|p_d(A)\|^2} p_d(A)
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\]
\[
= \frac{\langle A_d, H(A) \rangle}{\|p_d\|^2} p_d(A) = \frac{\langle A_d, J \rangle}{p_d(k)} p_d(A) = \frac{\bar{k}_d}{p_d(k)} p_d(A)
\]

where $\bar{k}_d = \frac{1}{n} \sum_u k_d(u)$
\[ \bar{k}_d = \|A_d\|^2 \geq \|\tilde{A}_d\|^2 = \frac{\bar{k}_d^2}{p_d(k)^2} \left\| p_d(A) \right\|^2 = \frac{\bar{k}_d^2}{p_d(k)} \]

hence \( \bar{k}_d \leq p_d(k) \) with equality iff \( A_d = p_d(A) \)
Spectral Excess

\[ k_d = \| A_d \|^2 \geq \| \tilde{A}_d \|^2 = \frac{k_d^2}{p_d(k)^2} \| p_d(A) \|^2 = \frac{k_d^2}{p_d(k)} \]

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Spectral Excess Theorem (Fiol & Garriga 1997):

\( \bar{k}_d \leq p_d(k) \) with equality iff the graph is distance-regular
Find graphs with spectrum \( \{3^1, 2^4, 1^5, -1^5, -2^4, -3^1\} \).
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Connected, 3-regular, bipartite on 10 + 10 vertices, girth 6.

So this is the incidence graph of a partial linear space.

Diameter at most 5, with \( \overline{k}_5 \leq 1 \).

Distance distribution diagram: \( 20 = 1_3 + _32 + _62 + ? + 3 + ? \)

\( k_4(x) = 3 \) so \( k_5(x) \leq 1 \).
The halved graphs (the point graph and line graph of the partial linear space) have spectrum \(\{6^1, 1^4, -2^5\}\). The only graph possible is \(J(5, 2)\), the complement of Petersen.
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Start from point graph, and try to construct a partial linear space: this can be done in more than one way: Desargues and twisted Desargues
Partial linear space

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**Desargues and twisted Desargues**

Neighbors of 12: 13, 14, 15, 23, 24, 25

Make lines of size 3: \(\{12, 13, 23\}, \{12, 14, 24\}, \{12, 15, 25\}\)

lines ‘123’, ‘124’, ‘125’
The halved graphs (the point graph and line graph of the partial linear space) have spectrum \( \{6^1, 1^4, -2^5\} \). The only graph possible is \( J(5, 2) \), the complement of Petersen.

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Make lines of size 3: \( \{12, 13, 23\}, \{12, 14, 24\}, \{12, 15, 25\} \)
lines ‘123’, ‘124’, ‘125’

Or (the twisted way): \( \{12, 13, 14\}, \{12, 23, 24\}, \{12, 15, 25\} \)
lines ‘1’, ‘2’, ‘125’
(2d − 1)-dimensional vector space over $GF(q)$

points: $(d − 1)$-dimensional subspaces

lines: $d$-dimensional subspaces

Incidence graph is doubled Grassmann

Point and line graph are Grassmann $J_q(2d − 1, d − 1)$
\(q\)-ary Desargues

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\textbf{ Twist: } Fix a hyperplane \(H\)

lines: \(d\)-dimensional subspaces not contained in \(H\)
twisted lines: \((d - 2)\)-dimensional subspaces contained in \(H\)
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Point graph is again $J_q(2d − 1, d − 1)$

Incidence graph is cospectral to doubled Grassmann, but not drg
**$q$-ary Desargues**

$(2d - 1)$-dimensional vector space over $GF(q)$

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Incidence graph is doubled **Grassmann**

Point and line graph are **Grassmann** $J_q(2d - 1, d - 1)$

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Incidence graph is cospectral to doubled **Grassmann**, but not drg

Line graph is cospectral to $J_q(2d - 1, d - 1)$

Spectral excess theorem: line graph is distance-regular!

....but it is UGLY!!!
Families of ‘ugly’ distance-regular graphs with unbounded diameter:

Doob, Hemmeter, Ustimenko: not distance-transitive.

Dalfó & EvD & Fiol (2011): Ugly (almost) distance-regular graphs can be used to construct cospectral graphs through perturbations:

Adding and removing vertices, edges, amalgamating vertices, etc.

The devil’s advocate (Durham, 2013): It is easy to construct cospectral graphs.
Removing vertices from the twisted Desargues graph
Removing edges from the twisted Desargues graph
Adding edges to the twisted Desargues graph
Amalgamate vertices in the twisted Desargues graph
Generalized odd graph (drg with $a_1 = \ldots = a_{d-1} = 0$, $a_d \neq 0$)

No odd cycles of length less than $2d + 1$ (almost bipartite)
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EvD & Haemers (2011): A regular graph with \( d + 1 \) distinct eigenvalues and odd-girth \( 2d + 1 \) is a generalized odd graph

Lee & Weng (2012) extended this for non-regular graphs
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Sketch of short proof (**EvD & Fiol** 2012):

Recall \( xp_i = \beta_{i-1}p_{i-1} + \alpha_ip_i + \gamma_{i+1}p_{i+1}, \ i = 0, 1, \ldots, d, \)

Here \( \alpha_i = 0, i < d; \ p_i \) is an even/odd polynomial if \( i \) is even/odd
**Generalized Odd**

**Generalized odd graph** (drg with \( a_1 = \ldots = a_{d-1} = 0, \ a_d \neq 0 \))

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If \( \theta \) is an eigenvalue, then \( -\theta \) is not
Proof

\[ A^\ell = \sum_i \theta_i^\ell E_i \] (spectral decomposition)

Odd powers \((\ell = 1, 3, \ldots, 2d - 1)\) have zero diagonal \(E_i\)s and hence \(A^2\) have constant diagonal, so the graph is regular.
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Hoffman polynomial: \(H(A) = \sum_i p_i(A) = J\)

\(u, v\) at distance \(d\): \(p_d(A)_{uv} = 1\)
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\( u, v \) at distance \( d \): \( p_d(A)_{uv} = 1 \)

If dist(\( u, v \)) and \( d \) have different parity: \( p_d(A)_{uv} = 0 \)

If dist(\( u, v \)) < \( d \) and \( d \) have same parity: \( \alpha_d p_d(A)_{uv} = \beta_{d-1} p_{d-1}(A)_{uv} + \alpha_d p_d(A)_{uv} = (Ap_d(A))_{uv} = \sum_{w \sim u} p_d(A)_{wv} = 0 \)
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If \(\text{dist}(u, v)\) and \(d\) have different parity: \(p_d(A)_{uv} = 0\)

If \(\text{dist}(u, v) < d\) and \(d\) have same parity: \(\alpha_d p_d(A)_{uv} = \beta_{d-1} p_{d-1}(A)_{uv} + \alpha_d p_d(A)_{uv} = (A p_d(A))_{uv} = \sum_{w \sim u} p_d(A)_{wv} = 0\)

\(A_d = p_d(A)\) so by the spectral excess theorem the graph is distance-regular.

THE END