Multiple polylogarithms and Feynman integrals

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joint work with
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Outline:

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Part 1:
Multiple polylogarithms in several variables

Let

- $k$ be a field (either $\mathbb{R}$ or $\mathbb{C}$),
- $M$ a smooth manifold over $k$,
- $\gamma : [0, 1] \to M$ a smooth path on $M$,
- $\omega_1, \ldots, \omega_n$ smooth differential 1-forms on $M$,
- $\gamma^*(\omega_i) = f_i(t)dt$ the pull-back of $\omega_i$ to $[0, 1]$

**Def.**: The *iterated integral* of $\omega_1, \ldots, \omega_n$ along $\gamma$ is

$$\int_\gamma \omega_n \ldots \omega_1 = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} f_n(t_n) dt_n \ldots f_1(t_1) dt_1.$$

We use the term *iterated integral* for $k$-linear combinations of such integrals.
We obtain different classes of functions by choosing different finite sets of 1-forms $\Omega$.

- $\Omega_1 = \left\{ \frac{dt}{t}, \frac{dt}{t-1} \right\}$, $\omega_0 \equiv \frac{dt}{t}$, $\omega_1 \equiv \frac{dt}{t-1}$

  - classical polylogarithms: $\text{Li}_n(z) = \int_{\gamma} \underbrace{\omega_0 \ldots \omega_0}_{n-1 \text{ times}} \omega_1 = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} \frac{dt_n}{t_n} \ldots \frac{dt_2}{t_2} \frac{zdt_1}{1-zt_1}$

  - multiple polylogarithms in one variable: $\text{Li}_{n_1, \ldots, n_r}(z) = (-1)^r \int_{\gamma} \underbrace{\omega_0 \ldots \omega_0}_{n_r-1 \text{ times}} \underbrace{\omega_0 \ldots \omega_0}_{n_1-1} \omega_1$, where $\gamma$ a smooth path in $\mathbb{C}\backslash\{0, 1\}$ with end-point $z$

- $\Omega_H^{\text{hyp}} = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1-1}, \frac{t_2dt_1}{t_1t_2-1}, \ldots, \frac{(\prod_{i=2}^{n} t_i)dt_1}{\prod_{i=1}^{n} t_i-1} \right\}$: hyperlogarithms (Poincaré, Kummer 1840, Lappo-Danilevsky 1911)

including harmonic polylogarithms (Remiddi, Vermaisen 1999), two-dimensional harmonic polylogarithms (Gehrmann, Remiddi '01)
Let $\Omega_n$ be the set of differential 1-forms $\frac{df}{f}$ with $f \in \left\{ t_1, \ldots, t_n, \prod_{a \leq i \leq b} t_i - 1 \right\}$, for $1 \leq a \leq b \leq n$:

$$\Omega_n = \left\{ \frac{dt_1}{t_1}, \ldots, \frac{dt_n}{t_n}, \frac{d \left( \prod_{a \leq i \leq b} t_i \right)}{\prod_{a \leq i \leq b} t_i - 1} \text{ where } 1 \leq a \leq b \leq n \right\}$$

Examples:

$$\Omega_1 = \left\{ \frac{dt_1}{t_1}, \frac{dt_1}{t_1 - 1} \right\} \quad \text{(→ multiple polylogs in one variable)}$$

$$\Omega_2 = \left\{ \frac{dt_1}{t_1}, \frac{dt_2}{t_2}, \frac{dt_1}{t_1 - 1}, \frac{dt_2}{t_2 - 1}, \frac{t_1 dt_2 + t_2 dt_1}{t_1 t_2 - 1} \right\}$$
From this $\Omega_n$ we want to construct iterated integrals which are \textit{homotopy invariant}, i.e.

$$\int_{\gamma_1} \omega_n \ldots \omega_1 = \int_{\gamma_2} \omega_n \ldots \omega_1$$ for homotopic paths $\gamma_1, \gamma_2$.

Consider tensor products $\omega_1 \otimes \ldots \otimes \omega_m \equiv [\omega_1 \ldots | \omega_m]$ over $\mathbb{Q}$.

Define an operator $D$ by

$$D ([\omega_1 \ldots | \omega_m]) = \sum_{i=1}^{m} [\omega_1 \ldots | \omega_{i-1} | d \omega_i | \omega_{i+1} \ldots \omega_m] + \sum_{i=1}^{m-1} [\omega_1 \ldots | \omega_{i-1} | \omega_i \wedge \omega_{i+1} \ldots | \omega_m].$$

\textbf{Def.:} A $\mathbb{Q}$–linear combination of tensor products

$$\xi = \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} [\omega_{i_1} \ldots | \omega_{i_l}], \quad c_{i_1, \ldots, i_l} \in \mathbb{Q}$$

is called \textit{integrable word} if

$$D(\xi) = 0.$$
Consider the integration map

\[
\sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} [\omega_{i_1} | \ldots | \omega_{i_l}] \mapsto \sum_{l=0}^{m} \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} \int_{\gamma} \omega_{i_1} \ldots \omega_{i_l}
\]

**Theorem (Chen '77):** Under certain conditions on \( \Omega \) this map is an isomorphism from integrable words to homotopy invariant iterated integrals.

(also see Lemma 1.1.3 of Zhao's lecture)

Our class of homotopy invariant functions:

- Construct the integrable words of 1-forms in \( \Omega_n \).
  
  (for an explicit construction see CB, Brown '12 and cf. Duhr, Gangl, Rhodes '11, Goncharov et al '10)

- By the integration map obtain the set of multiple polylogarithms in several variables \( B(\Omega_n) \).
Properties of $B(\Omega_n)$ (Brown '05):

- They are well-defined functions of $n$ variables, corresponding to end-points of paths.
- On these functions, functional relations are algebraic identities.
- They can be decomposed to an explicit basis.
- $B(\Omega_n)$ is closed under taking primitives.
- Let $\mathcal{Z}$ be the $\mathbb{Q}$-vector space of multiple zeta values. The limits at 0 and 1 of functions in $B(\Omega_n)$ are $\mathcal{Z}$-linear combinations of elements in $B(\Omega_{n-1})$.
Consequence:

Let $F_n$ be the vector space of rational functions with denominators in
$\{t_1, ..., t_n, \prod_{a \leq i \leq b} t_i - 1\}, \ 1 \leq a \leq b \leq n$.

Consider integrals of the type MPL

$$\int_0^1 dt_n \sum_j f_j \beta_j \text{ with } f_j \in F_n, \ \beta_j \in B(\Omega_n).$$

We can compute such integrals. The results are $\mathbb{Z}$-linear combinations of elements in $B(\Omega_{n-1})$, multiplied by elements in $F_{n-1}$.

Concept: Map Feynman integrals to integrals of this type and evaluate them.

When is this possible?
**Scalar Feynman integrals**

For a generic Feynman graph $G$ with $N$ edges and loop-number (first Betti number) $L$ we consider the scalar Feynman integral

$$I(\Lambda) = \int \prod_{i=1}^{L} \frac{d^D k_i}{i \pi^{D/2}} \prod_{j=1}^{N} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}, \quad N, L, \nu_j \in \mathbb{Z}, \ D \in \mathbb{C},$$

$\Lambda$ : external parameters, i.e. kinematical invariants and masses $m_j$; $q_j$ : momenta

Using the "Feynman trick" we can re-write this as

$$I(\Lambda) = \frac{\Gamma(\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \ldots \int_0^\infty \left( \prod_{i=1}^{N} dx_i x_i^{\nu_i - 1} \right) \delta \left( 1 - \sum_{i=1}^{N} x_i \right) \frac{U^{\nu - (L+1)D/2}}{(\mathcal{F}(\Lambda))^{\nu - LD/2}},$$

where $\nu = \sum_{j=1}^{N} \nu_j$, $\epsilon = (4 - D)/2$.

$U$ and $\mathcal{F}$ are the first and the second Symanzik polynomial.
Labelling the edges of $G$ with Feynman parameters $x_1, \ldots, x_N$, we obtain the Symanzik polynomials as:

\[
\mathcal{U} = \sum_{\text{spanning trees } T \text{ of } G \text{ edges } \notin T} \prod x_i
\]

\[
\mathcal{F}_0 = -\sum_{\text{spanning 2-forests } (T_1, T_2)} \left( \prod_{\text{edges } \notin (T_1, T_2)} x_i \right) \left( \sum_{\text{edges } \notin (T_1, T_2)} q_i \right)^2,
\]

\[
\mathcal{F} = \mathcal{F}_0 + \mathcal{U} \sum_{i=1}^N x_i m_i^2.
\]
Assumption: We are interested in integrands with $U$ and/or $F$ in the denominator and arguments of polylogs:

\[
\text{multiple polylogs of } \{U, F\} \quad \{U, F\}
\]

i.e. a Feynman integral which is finite from the beginning or appropriately renormalized (see Kreimer’s talk)

Approach: Try to integrate out all Feynman parameters:

- After integration over $x_i$, consider the set of polynomials in the denominator and in arguments of (possible) multiple polylogs in the integrand.
  
Condition: If there is a next Feynman parameter $x_j$ in which all of these polynomials are linear, we can continue.

- Map the integral over $x_j$ to an integral over $t_n$ of the type MPL and integrate over $t_n$.
  
Alternatively: Integrate over $x_j$ directly, using an appropriate class of iterated integrals.
  (see recent work by E. Panzer, C. Duhr, F. Wissbrock, ...)

Question: For which polynomials (i.e. which graph) does this approach succeed?
Linear reduction algorithm (Brown '08)

- If the polynomials $S = \{f_1, \ldots, f_n\}$ are linear in a Feynman parameter $x_{r_1}$, consider:
  
  $$f_i = g_i x_{r_1} + h_i, \quad g_i = \frac{\partial f_i}{\partial x_{r_1}}, \quad h_i = f_i|_{x_{r_1}=0}$$

- $S(r_1)$ = irreducible factors of $\{g_i\}_{1 \leq i \leq n}$, $\{h_i\}_{1 \leq i \leq n}$, $\{h_ig_j - g_ih_j\}_{1 \leq i < j \leq n}$

- iterate for a sequence $(x_{r_1}, x_{r_2}, \ldots, x_{r_n}) \Rightarrow S(r_1), S(r_1, r_2), \ldots, S(r_1, \ldots, r_n)$

- take intersections:
  
  $$S_{[r_1, r_2]} = S(r_1, r_2) \cap S(r_2, r_1)$$
  $$S_{[r_1, r_2, \ldots, r_k]} = \cap_{1 \leq i \leq k} S_{[r_1, \ldots, \hat{r_i}, \ldots, r_k]}(r_i), \; k > 3$$

  $$x_{r_1}, x_{r_2}, \ldots, x_{r_n} \Rightarrow S(r_1), S_{[r_1, r_2]}, \ldots, S_{[r_1, \ldots, r_n]}$$

**Def.** A Feynman graph $G$ is called linearly reducible, if the set $\{\mathcal{U}_G, \mathcal{F}_G\}$ is linearly reducible, i.e. there is a $(x_{r_1}, x_{r_2}, \ldots, x_{r_n})$ such that for all $1 \leq k \leq n$ every polynomial in $S_{[r_1, \ldots, r_k]}$ is linear in $x_{r_{k+1}}$. 
For an edge of \( G \) consider the deletion \((G \setminus e)\) and contraction \((G/e)\) of \( e \)

The deletion and contraction of different edges is commutative.

\[ \Rightarrow \text{If } C, D \text{ are disjoint sets of edges of } G \text{ then } G \setminus D /e C \text{ is a unique graph.} \]

Any such graph is called minor of \( G \).

**Def.:** A set \( \mathcal{G} \) of graphs is called *minor-closed* if for each \( G \in \mathcal{G} \) all minors belong to \( \mathcal{G} \) as well.

**Example:** The set of all planar graphs is minor-closed.
B is a minor of A
B is a minor of A
B is a minor of A
Let $\mathcal{H}$ be a finite set of graphs. Define $\mathcal{G}_{\mathcal{H}}$ to be the set of graphs whose minors do not belong to $\mathcal{H}$. Then the graphs in $\mathcal{H}$ are called forbidden minors of $\mathcal{G}_{\mathcal{H}}$. The set $\mathcal{G}_{\mathcal{H}}$ is minor-closed.

**Theorem (Robertson and Seymour):** Any minor-closed set of graphs can be defined by a finite set of forbidden minors.

**Example:**
The set of planar graphs is the set of all graphs which have neither $K_5$ nor $K_{3,3}$ as a minor. (Wagner’s theorem)
Theorem (Brown ’09, CB and Lüders ’13): The set of linearly reducible Feynman graphs is minor-closed.

We should search for the forbidden minors!

A first case study (with M. Lüders):

- Let $\Lambda$ be the set of massless Feynman graphs with four on-shell legs. (On-shell condition: $p_i^2 = 0$, $i = 1, \ldots, 4$)
- At two loops we find all graphs to be linearly reducible.
- At three loops we find first forbidden minors.
- Four loops are running on our computers and confirm the forbidden three-loop minors so far.
Part 2: 
The sunrise graph - a case beyond multiple polylogarithms
Consider the sunrise graph with arbitrary masses:

In $D = 2$ dimensions we obtain the finite Feynman integral

$$S_{D=2}(t) = \int_{\sigma} \frac{\omega}{F_G},$$

with

$$\omega = x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2$$

$$F_G (t, m_1^2, m_2^2, m_3^2) = -x_1 x_2 x_3 t + (x_1 x_2 + x_2 x_3 + x_1 x_3)(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2), \quad t = p^2,$$

$$\sigma = \{ [x_1 : x_2 : x_3] \in \mathbb{P}^2 | x_i \geq 0, \ i = 1, 2, 3 \}$$

Simple observation: As $F_G$ is not linear in any $x_i$, the graph is not linearly reducible.
(Incomplete) History of sunrises:

Equal mass case:

- Broadhurst, Fleischer, Tarasov (1993): result with hypergeometric functions
- Groote, Pivovarov (2000): Cutkosky rules $\Rightarrow$ imaginary part expressed by elliptic integrals
- Laporta, Remiddi (2004): solving a second-order differential equation $\Rightarrow$ result by integrals over elliptic integrals
- Bloch, Vanhove (in progress): a new result involving the elliptic dilogarithm (see talks by Broadhurst and Kerr)

Arbitrary mass case:

- Berends, Buza, Böhm, Scharf (1994): result with Lauricella functions
- Groote, Körner, Pivovarov (2005): integral representations involving Bessel functions
- Müller-Stach, Weinzierl, Zayadeh (2012): one second-order differential equation

Our goal: Solve the new differential equation (as Laporta and Remiddi did for equal masses) and obtain a result involving elliptic integrals
A result for $D$ dimensions is known from Berends, Buza, Böhm and Scharf (1994):

$$S_D(t) = (-t)^{D-3} \left( \frac{\Gamma(3-D)\Gamma(D/2-1)^3}{\Gamma(D/2-3)} F_C \left( 3-D, 4 - \frac{3}{2}D; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \right)$$

$$+ \frac{\Gamma(2-D/2)\Gamma(1-D/2)\Gamma(D/2-1)^2}{\Gamma(D-2)} \left( F_C \left( 3-D, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( -\frac{m_1^2}{t} \right)^{D/2-1} \right)$$

$$+ F_C \left( 3-D, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( -\frac{m_2^2}{t} \right)^{D/2-1}$$

$$+ F_C \left( 3-D, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( -\frac{m_3^2}{t} \right)^{D/2-1}$$

$$+ \Gamma(1-D/2)^2 \left( F_C \left( 1, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( \frac{m_1^2 m_2^2}{t^2} \right)^{D/2-1} \right)$$

$$+ F_C \left( 1, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( \frac{m_1^2 m_3^2}{t^2} \right)^{D/2-1}$$

$$+ F_C \left( 1, 2 - \frac{D}{2}; 2 - \frac{D}{2}, 2 - \frac{D}{2}, 2 - \frac{D}{2}; \frac{m_1^2}{t}, \frac{m_2^2}{t}, \frac{m_3^2}{t} \right) \left( \frac{m_2^2 m_3^2}{t^2} \right)^{D/2-1} \right)$$

with the Lauricella function

$$F_C(a_1, a_2; b_1, b_2, b_3; x_1, x_2, x_3) = \sum_{j_1=0}^\infty \sum_{j_2=0}^\infty \sum_{j_3=0}^\infty \frac{\Gamma(a_1+j_1+a_2+j_2+b_1)\Gamma(j_1)\Gamma(j_2+b_2)\Gamma(j_3+b_3)\Gamma(j_1+j_2+j_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(j_1)\Gamma(j_2)\Gamma(j_3)!}$$

and the Pochhammer symbol $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$
Using Euler-Zagier sums $Z_1(n) = \sum_{j=1}^{n} \frac{1}{j}$, $Z_{11}(n) = \sum_{j=1}^{n} \frac{1}{j} Z_1(j - 1)$ we can expand this result in $D = 2$ and obtain:

$$S_{D=2}(t) = -\frac{1}{t} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \left( \frac{j_{123}!}{j_1! j_2! j_3!} \right)^2 \left( \frac{m_1^2}{t} \right)^{j_1} \left( \frac{m_2^2}{t} \right)^{j_2} \left( \frac{m_3^2}{t} \right)^{j_3}$$

$$(12Z_{11}(j_{123}) + 6Z_1(j_{123})Z_1(j_{123}) - 8Z_1(j_{123})(Z_1(j_1) + Z_1(j_2) + Z_1(j_3))) +$$

$$4(Z_1(j_1)Z_1(j_2) + Z_1(j_2)Z_1(j_3) + Z_1(j_3)Z_1(j_1)) +$$

$$2(2Z_1(j_{123}) - Z_1(j_2) - Z_1(j_3)) \ln \left( -\frac{m_1^2}{t} \right) + 2(2Z_1(j_{123}) - Z_1(j_3) - Z_1(j_1)) \ln \left( -\frac{m_2^2}{t} \right) +$$

$$2(2Z_1(j_{123}) - Z_1(j_1) - Z_1(j_2)) \ln \left( -\frac{m_3^2}{t} \right) + \ln \left( -\frac{m_1^2}{t} \right) \ln \left( -\frac{m_2^2}{t} \right) + \ln \left( -\frac{m_2^2}{t} \right) \ln \left( -\frac{m_3^2}{t} \right) + \ln \left( -\frac{m_3^2}{t} \right) \ln \left( -\frac{m_1^2}{t} \right)$$

We obtain a five-fold nested sum.
Can we obtain a result avoiding multiple nested sums?
Start from the second order differential equation (Müller-Stach, Weinzierl, Zayadeh '12):

\[
\left( p_0(t) \frac{d^2}{dt^2} + p_1(t) \frac{d}{dt} + p_2(t) \right) S(t) = p_3(t)
\]

\( p_0, p_1, p_2, p_3 \) are polynomials in \( t \) (of degrees 7, 6, 5, 4) and in \( m_1^2, m_2^2, m_3^2 \) and \( p_3 \) involves \( \ln \left( \frac{m_i^2}{\mu^2} \right) \)

Ansatz for the solution:

\[
S(t) = C_1 \psi_1(t) + C_2 \psi_2(t) + \int_0^t dt_1 \frac{p_3(t_1)}{p_0(t_1) W(t_1)} (-\psi_1(t) \psi_2(t_1) + \psi_2(t) \psi_1(t_1))
\]

with the solutions of the homogeneous equation \( \psi_1, \psi_2 \), constants \( C_1, C_2 \),

Wronski determinant \( W(t) = \psi_1(t) \frac{d}{dt} \psi_2(t) - \psi_2(t) \frac{d}{dt} \psi_1(t) \)
We will use

- complete elliptic integral of the first kind:

\[ K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \]

- complete elliptic integral of the second kind:

\[ E(k) = \int_0^1 \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}} \, dx \]

- the moduli \( k(t), k'(t) \) satisfy \( k(t)^2 + k'(t)^2 = 1 \)
Introduce the notation
\[ x_1 = (m_1 - m_2)^2, \quad x_2 = (m_3 - \sqrt{t})^2, \quad x_3 = (m_3 + \sqrt{t})^2, \quad x_4 = (m_1 + m_2)^2 \]

Consider the **auxiliary elliptic curve** given by the equation
\[ y^2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4). \]

By the associated holomorphic 1-form \( \frac{dx}{y} \) one obtains the period integrals
\[
\psi_1(t) = 2 \int_{x_2}^{x_3} \frac{dx}{y} = \frac{4}{\xi(t)} K(k(t)),
\]
\[
\psi_2(t) = 2 \int_{x_4}^{x_3} \frac{dx}{y} = \frac{4i}{\xi(t)} K(k'(t))
\]

with \( \xi(t) = \sqrt{(x_3 - x_1)(x_4 - x_2)} \),
\[
k(t) = \sqrt{\frac{(x_3 - x_2)(x_4 - x_1)}{(x_3 - x_1)(x_4 - x_2)}}, \quad k'(t) = \sqrt{\frac{(x_2 - x_1)(x_4 - x_3)}{(x_3 - x_1)(x_4 - x_2)}}, \quad k(t)^2 + k'(t)^2 = 1
\]

\( \psi_1(t) \) and \( \psi_2(t) \) solve the **homogeneous differential equation** for \( S(t) \).
Furthermore, from integrating over \( \frac{x \, dx}{y} \) we obtain

\[
\phi_1(t) = \frac{4}{\xi(t)} \left( K(k(t)) - E(k(t)) \right)
\]

\[
\phi_2(t) = \frac{4i}{\xi(t)} E(k'(t))
\]

The period matrix of the elliptic curve is

\[
\begin{pmatrix}
\psi_1(t) & \psi_2(t) \\
\phi_1(t) & \phi_2(t)
\end{pmatrix}
\]

and we have the Legendre relation

\[
\psi_1(t)\phi_2(t) - \psi_2(t)\phi_1(t) = \frac{8\pi i}{\xi(t)}.
\]

These are appropriate functions to express the full solution in a compact way.
Full solution (Adams, CB, Weinzierl '13):

\[ S(t) = \frac{1}{\pi} \left( \sum_{i=1}^{3} \text{Cl}_2(\alpha_i) \right) \psi_1(t) + \frac{1}{i\pi} \int_0^t dt_1 \left( \eta_1(t_1) - \frac{b_1 t_1 - b_0}{3(x_2 - x_1)(x_4 - x_3)} \eta_2(t_1) - \eta_1(t_1) \right) \]

where

\[ \eta_1(t_1) = \psi_2(t) \psi_1(t_1) - \psi_1(t) \psi_2(t_1) \]

\[ \eta_2(t_1) = \psi_2(t) \phi_1(t_1) - \psi_1(t) \phi_2(t_1) \]

Clausen function: \( \text{Cl}_2(x) = \frac{1}{2i} \left( \text{Li}_2(e^{ix}) - \text{Li}_2(e^{-ix}) \right) \)

\( \alpha_i = 2\arctan \left( \frac{\sqrt{\Delta}}{\delta_i} \right), \Delta, \delta_i : \text{polynomials in } m_1, m_2, m_3 \text{ of degrees } 4 \text{ and } 2 \text{ resp.} \)

\( b_i = d_i(m_1, m_2, m_3) \ln(m_i^2) + d_i(m_2, m_3, m_1) \ln(m_i^2) + d_i(m_3, m_1, m_2) \ln(m_i^2), \)

\( d_1(m_1, m_2, m_3) = 2m_1^2 - m_2^2 - m_3^2, \)

\( d_0(m_1, m_2, m_3) = 2m_1^4 - m_2^4 - m_3^4 - m_1^2 m_2^2 - m_1^2 m_3^2 + 2m_2^2 m_3^2 \)
Conclusions:

- Multiple polylogarithms in several variables are homotopy invariant iterated integrals with particularly good properties. We want to use them to iteratively integrate out Feynman parameters.

- To decide whether the approach can succeed there is a criterion of linear reducibility on the graphs. The class of linearly reducible graphs is minor-closed. This allows for a convenient classification by forbidden minors.

- The sunrise integral with arbitrary masses is a case where we can express the result by integrals over elliptic integrals. This result can be built up from the period integrals of an (auxiliary) elliptic curve.