An introduction to the Virtual Element Method

L. Beirão da Veiga
Department of Mathematics
University of Milan

in collaboration with:
F. Brezzi, A. Cangiani, D. Marini, G. Manzini, A. Russo

DURHAM, July 2014
The Virtual Element Method (VEM) is a generalization of the Finite Element Method that takes inspiration from modern Mimetic Finite Difference schemes.

- VEM allow to use very general **polygonal** and **polyhedral** meshes, also for high polynomial degrees and guaranteeing the patch test.
- The **flexibility** of VEM is not limited to the mesh: an example will be shown later.
Why polygons/polyhedrons?

The interest (and use in commercial codes*) for polygons/polyhedra is recently growing.

- Immediate combination of tets and hexahedrons
- Easier/better meshing of domain (and data) features
- Automatic inclusion of “hanging nodes”
- Adaptivity: more efficient mesh refinement/coarsening
- Generate meshes with more local rotational symmetries
- Robustness to distortion
- ......

* for example CD-ADAPCO and ANSYS.
Some polytopal methods

- **Mimetic F.D.** Shashkov, Lipnikov, Brezzi, Manzini, BdV, ....
- **HMM**: Eymard, Droniou, ...
- **Polygonal FEM**: Sukumar, Paulino, ...
- **Weak Galerkin FEM**: Wang, ....
- **HHO**: Ern, di Pietro
- **Polygonal DG**: Cangiani, Houston, Georgoulis, ...
- **VEM**: this talk !!!
- ..........
The model problem

We consider the **Poisson problem** in two dimensions

\[-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,\]

where

- $\Omega \subset \mathbb{R}^2$ is a polygonal domain;
- the loading $f$ is assumed in $L^2(\Omega)$.

**Variational formulation:**

\[
\begin{cases}
\text{find } u \in V := H^1_0(\Omega) \text{ such that } \\
 a(u, v) = \int_\Omega f \, v \, dx \quad \forall v \in V,
\end{cases}
\]

where

\[a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx, \quad \forall v, w \in V.\]
We will build a discrete problem in the following form

\[
\left\{ \begin{array}{l}
\text{find } u_h \in V_h \text{ such that } \\
a_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V_h,
\end{array} \right.
\]

where

- \( V_h \subset V \) is a finite dimensional space;
- \( a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R} \) is a discrete bilinear form approximating the continuous form \( a(\cdot, \cdot) \);
- \( \langle f_h, v_h \rangle \) is a right hand side term approximating the load.
A Virtual Element Method

We will build a discrete problem in following form

\[
\begin{aligned}
\text{find } u_h & \in V_h \text{ such that } \\
a_h(u_h, v_h) &= \langle f_h, v_h \rangle \quad \forall v_h \in V_h,
\end{aligned}
\]

where

- \(V_h \subset V\) is a finite dimensional space;
- \(a_h(\cdot, \cdot) : V_h \times V_h \rightarrow \mathbb{R}\) is a discrete bilinear form approximating the continuous form \(a(\cdot, \cdot)\);
- \(\langle f_h, v_h \rangle\) is a right hand side term approximating the load.

Let \(m \geq 1\) be a fixed integer index. Such index will represent the degree of accuracy of the method.
Let $\Omega_h$ be a simple polygonal mesh on $\Omega$. This can be any decomposition of $\Omega$ in non overlapping polygons $E$ with straight faces.

The space $V_h$ will be defined element-wise, by introducing:

- local spaces $V_{h|E}$;
- the associated local degrees of freedom.
The local spaces $V_{h|E}$

Let $\Omega_h$ be a simple polygonal mesh on $\Omega$. This can be any decomposition of $\Omega$ in non overlapping polygons $E$ with straight faces.

The space $V_h$ will be defined element-wise, by introducing

- local spaces $V_{h|E}$;
- the associated local degrees of freedom.

For all $E \in \Omega_h$:

$$V_{h|E} = \{ v \in H^1(E) : -\Delta v \in P_{m-2}(E), v|_e \in P_m(e) \quad \forall e \in \partial E \}.$$
The local spaces $V_{h|E}$

Let $\Omega_h$ be a simple polygonal mesh on $\Omega$. This can be any decomposition of $\Omega$ in non overlapping polygons $E$ with straight faces.

The space $V_h$ will be defined element-wise, by introducing

- local spaces $V_{h|E}$;
- the associated local degrees of freedom.

For all $E \in \Omega_h$:

$$V_{h|E} = \left\{ v \in H^1(E) : -\Delta v \in \mathbb{P}_{m-2}(E),
\quad v|_e \in \mathbb{P}_m(e) \quad \forall e \in \partial E \right\}.$$  

- the functions $v \in V_{h|E}$ are continuous (and known) on $\partial E$;
- the functions $v \in V_{h|E}$ are unknown inside the element $E$!
- it holds $\mathbb{P}_m(E) \subseteq V_{h|E}$
The dimension of the space $V_{h|E}$ is clearly

$$\dim(V_{h|E}) = N_e m + m(m - 1)/2,$$

with $N_e$ number of edges of $E$. 

---

L. Beirão da Veiga (Univ. of Milan) The Virtual Element Method DURHAM - 2014 8 / 35
Degrees of freedom for $V_{h|E}$

The dimension of the space $V_{h|E}$ is clearly

$$\dim(V_{h|E}) = N_e m + m(m - 1)/2,$$

with $N_e$ number of edges of $E$.

- pointwise values $v_h(\nu)$ at all corners $\nu$ of $E$;
- $(m - 1)$ pointwise values on each edge:
  $$v_h(x^e_i), \quad \{x^e_i\}_{i=1}^{m-1} \text{ distinct points on edge } e;$$
- volume moments:
  $$\int_E v_h \cdot p_{m-2} \quad \forall p_{m-2} \in \mathbb{P}_{m-2}(E).$$
Depiction of the degrees of freedom for $V_{h|E}$

Green dots stand for vertex pointwise values
Red squares represent edge pointwise values
Blue squares represent internal (volume) moments

L. Beirão da Veiga (Univ. of Milan)
The Virtual Element Method
DURHAM - 2014
Degrees of freedom for $V_h|E$

The following holds
[BdV, Brezzi, Cangiani, Manzini, Marini, Russo, M3AS 2013].

**Proposition**

The proposed collection of operators $V_h|E \rightarrow \mathbb{R}$ constitutes a set of degrees of freedom for $V_h|E$, $\forall E \in V_h|E$.

- We already know $\#\text{dofs} = \dim(V_h|E)$.
- If $v_h \in V_h|E$ is null on all the d.o.f.s, then it clearly vanishes on the boundary.
- The function $\Delta v_h \in P_{m-2}$ and thus

$$0 = \int_E v_h \cdot \Delta v_h = \int_E (\nabla v_h) \cdot (\nabla v_h).$$
The global space $V_h$ is built by assembling the local spaces $V_h|_E$ as usual:

$$V_h = \{ \mathbf{v} \in H^1_0(\Omega) : \mathbf{v}|_E \in V_h|_E \ \forall \ E \in \Omega_h \}$$

The total d.o.f.s are one per internal vertex, $m - 1$ per internal edge and $m(m - 1)/2$ per element.

The choice of degrees of freedom guarantees the global continuity of the functions in $V_h$. 
The bilinear form $a_h(\cdot, \cdot)$

The bilinear form $a_h(\cdot, \cdot)$ is built element by element

$$a_h(v_h, w_h) = \sum_{E \in \Omega_h} a^E_h(v_h, w_h) \quad \forall v_h, w_h \in V_h,$$

where

$$a^E_h(\cdot, \cdot) : V_h|_E \times V_h|_E \rightarrow \mathbb{R}$$

are symmetric bilinear forms that mimic

$$a^E_h(\cdot, \cdot) \simeq a(\cdot, \cdot)|_E$$

by satisfying a stability and a consistency condition.
The stability property

There exist two positive constants $\alpha_*$ and $\alpha^*$, independent of $h$ and of $E$, such that

$$
\alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h) \quad \forall v_h \in V_h|_E.
$$

- The **stability** property guarantees that $a_h(\cdot, \cdot)$ is uniformly coercive and continuous;

- clearly, it is sufficient for the (uniform) **well posedness** of the discrete problem, but not for convergence.
The consistency property

Consistency

For all $h$ and for all $E \in \Omega_h$ it holds

$$a^E_h(p, v_h) = a^E(p, v_h) \quad \forall p \in \mathbb{P}_m(E), v_h \in V_{h|E}.$$
The consistency property

Consistency

For all \( h \) and for all \( E \in \Omega_h \) it holds

\[
a_h^E(p, v_h) = a^E(p, v_h) \quad \forall p \in \mathbb{P}_m(E), \, v_h \in V_{h|E}.
\]

**NOTE:** an integration by parts gives

\[
a^E(p, v_h) = \int_E \nabla p \cdot \nabla v_h \, dx
\]

\[
= - \int_E (\Delta p) \, v_h \, dx + \int_{\partial E} (\nabla p \cdot n_E) \, v_h \, ds.
\]

for all \( p \in \mathbb{P}_m(E), \, v_h \in V_{h|E} \).

Therefore the right hand side above is explicitly computable even if we ignore \( v_h \) inside \( E \).
The discrete load term

We consider $m \geq 2$ first. Let, for all $E \in \Omega_h$, the approximated load $f_h|_E$ be the $L_2$-projection of $f|_E$ on $\mathbb{P}_{m-2}(E)$.

Then

$$(f_h, v_h)_h := \sum_{E \in \Omega_h} \int_E f_h v_h \, dx$$

that is computable due to the internal dofs of $V_h|_E$. 

Note: for $m = 2$ better choices can be made [Ahmad, Alsaedi, Brezzi, Marini, Russo, CMA 2013], [BdV, Brezzi, Marini, SINUM 2013].
We consider $m \geq 2$ first. Let, for all $E \in \Omega_h$, the approximated load $f_h|_E$ be the $L_2$-projection of $f|_E$ on $P_{m-2}(E)$.

Then

$$\langle f_h, v_h \rangle_h := \sum_{E \in \Omega_h} \int_E f_h v_h \, dx$$

that is computable due to the internal dofs of $V_h|_E$.

In the case $m = 1$ a simple integration rule based on the vertex values of the polygon can be used, for instance

$$\langle f_h, v_h \rangle_h := \sum_{E \in \Omega_h} \left( \int_E f \, dx \right) \frac{1}{N_E} \sum_{\nu \in \partial E} v_h(\nu) \, dx.$$ 

Note: for $m = 2$ better choices can be made [Ahmad, Alsaedi, Brezzi, Marini, Russo, CMA 2013], [BdV, Brezzi, Marini, SINUM 2013].
A convergence result

Let the sequence \( \{ \Omega_h \}_h \) satisfy the following mesh assumptions:

- each element \( E \) in \( \Omega_h \) is star-shaped with respect to a ball of uniform radius (or suitable union of);
- for each element \( E \) in \( \Omega_h \), the length of all edges is comparable with its diameter \( h_E \) (not needed by paying \( |\log(h_e)| \)).

Then the following holds [... volley team ..., M3AS 2013].

**Theorem**

Let the stability and consistency assumptions hold. Then, if \( f \in H^s(\Omega_h) \) and \( u \in H^{s+1}(\Omega_h) \), we have

\[
|u - u_h|_{H^1(\Omega)} \leq C h^s \left( |u|_{H^{s+1}(\Omega_h)} + |f|_{H^s(\Omega_h)} \right)
\]

for \( 0 \leq s \leq m \) and with \( C \) independent of \( h \).
k-plain

\[-\text{div}(K \nabla u) + \alpha u = f\]

exact solution:

\[u_e(x, y) = y - x + \log(y^3 + x + 1) - xy - x y^2 + x^2 y + x^2 + x^3 + \sin(5x) \sin(7y) - 1\]

diffusion:

\[K(x, y) = 1\]

zero-order term:

\[\alpha(x, y) = 1\]

right-hand-side:

\[f(x, y) = \log(y^3 + x + 1) - y - 5x + \frac{1}{(y^3 + x + 1)^2} + \frac{9y^4}{(y^3 + x + 1)^2} - x y - x y^2 + x^2 y + x^2 + x^3 - \frac{6y}{y^3 + x + 1} + 75 \sin(5x) \sin(7y) - 3\]

L2 norm of the exact solution:

\[\|u_e\|_{0, \Omega} = 0.6431084584\]

H1 seminorm of the exact solution:

\[|u_e|_{1, \Omega} = 5.031264492\]
$L^2$ error $k = 3$ $H^3$ error

**h=hmean slope:** $||Π_0 u_h - u_e||_{0,Ω} = 4.011, ||Π_0 u_h - u_e||_{0,Ω} = 4.014$

**h=hmean slope:** $||Π_0 u_h - u_e||_{0,Ω} = 3.010, ||Π_0 u_h - u_e||_{0,Ω} = 3.013$

**$L^2$ error**  

**$H^3$ error**
\[ h = \text{mean} \quad \text{slope: } ||\nabla \Pi^0 u_h - \nabla u_e||_0,\Omega = 5.009, ||\nabla \Pi^1 u_h - \nabla u_e||_0,\Omega = 5.017 \]

\[ h_1 \]

\[ h_2 \]

\[ h_3 \]

\[ h_4 \]

\[ h_5 \]

\[ L^2 \text{ error} \]

\[ k = 4 \]

\[ H^3 \text{ error} \]

\[ \begin{align*}
\text{triangle}0.01 & \quad 88 \text{ polygons}, \ h_{\text{max}} = 0.174116, h_{\text{mean}} = 0.149291 \\
\text{triangle}0.005 & \quad 177 \text{ polygons}, \ h_{\text{max}} = 0.133616, h_{\text{mean}} = 0.107452 \\
\text{triangle}0.001 & \quad 809 \text{ polygons}, \ h_{\text{max}} = 0.063418, h_{\text{mean}} = 0.049083 \\
\text{triangle}0.0005 & \quad 1627 \text{ polygons}, \ h_{\text{max}} = 0.044872, h_{\text{mean}} = 0.034516 \\
\text{triangle}0.0001 & \quad 7921 \text{ polygons}, \ h_{\text{max}} = 0.019754, h_{\text{mean}} = 0.015641 
\end{align*} \]
$h = h_{\text{mean}}$ slope: $||\Pi^0 u^h - u^e||_{0,\Omega} = 5.928$, $||\Pi^\nabla u^h - u^e||_{0,\Omega} = 5.871$

$L^2$ error

$k = 5$

$H^1$ error

$L^2$ error for $k = 5$

$H^1$ error for $k = 5$

$\text{triangle} 0.01$ polygons, $h_{\text{max}} = 0.174116$, $h_{\text{mean}} = 0.149291$

$\text{triangle} 0.005$ polygons, $h_{\text{max}} = 0.133616$, $h_{\text{mean}} = 0.107452$

$\text{triangle} 0.001$ polygons, $h_{\text{max}} = 0.063418$, $h_{\text{mean}} = 0.049083$

$\text{triangle} 0.0005$ polygons, $h_{\text{max}} = 0.044872$, $h_{\text{mean}} = 0.034516$

$\text{triangle} 0.0001$ polygons, $h_{\text{max}} = 0.019754$, $h_{\text{mean}} = 0.015641$
Just for fun ...

We consider a family of meshes based on the following pattern:

7 polygons

Courtesy of A. Russo !!
\[-\Delta u = f\]
\[-\Delta u = f\]
Construction of the stiffness matrix

We introduce the following energy projector

$$\Pi : V_{h|E} \rightarrow \mathbb{P}_m(E)$$

defined by, for all $v_h \in V_{h|E}$,

$$\begin{cases} 
  a^E(\Pi v_h, p) = a^E(v_h, p) & \forall p \in \mathbb{P}_m(E)/\mathbb{R} \\
  P_0(\Pi v_h) = P_0(v_h)
\end{cases}$$

The operator $P_0$ is a projection on constants, that for $m \geq 2$ is simply the average, and for $m = 1$ the vertex value average.

**NOTE:** due to the consistency assumption, the projection operator above is computable (more later).
Construction of the stiffness matrix

It is immediate to check that \( \forall v_h, w_h \in V_{h|E} \)

\[
a^E(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + a^E((I - \Pi)v_h, (I - \Pi)w_h).
\]

Then the bilinear form

\[
a^E_h(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + s^E((I - \Pi)v_h, (I - \Pi)w_h)
\]

is consistent and stable, provided the positive bilinear form

\( s^E : V_{h|E} \times V_{h|E} \rightarrow \mathbb{R} \) scales like the original bilinear form \( a \).
Construction of the stiffness matrix

It is immediate to check that $\forall \ v_h, w_h \in V_{h|E}$

$$a^E(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + a^E((I - \Pi)v_h, (I - \Pi)w_h).$$

Then the bilinear form

$$a_h^E(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + s^E((I - \Pi)v_h, (I - \Pi)w_h)$$

is consistent and stable, provided the positive bilinear form $s^E : V_{h|E} \times V_{h|E} \to \mathbb{R}$ scales like the original bilinear form $a$.

Let now the **local stiffness matrix** $M^E$

$$M^E_{ij} := a_h^E(\phi_i, \phi_j) \quad \forall i, j = 1, 2, ..., N$$

with $\{\phi_i\}$ the canonical basis associated to the degrees of freedom.
We introduce also \( \{ m_\alpha \}_{\alpha=1}^n \) a basis for the polynomial space

\[
P_m = \text{span}\{ m_\alpha \}_{\alpha=1}^n.
\]

Since \( P_m \subseteq \mathcal{V}_h|_E \), the matrix

\[
D = \begin{bmatrix}
dof_1(m_1) & dof_1(m_2) & \ldots & dof_1(m_n) \\
dof_2(m_1) & dof_2(m_2) & \ldots & dof_2(m_n) \\
\vdots & \vdots & \ddots & \vdots \\
dof_N(m_1) & dof_N(m_2) & \ldots & dof_N(m_n)
\end{bmatrix}
\]

expresses the \( \{ m_\alpha \} \) in terms of the \( \mathcal{V}_h|_E \) basis.
Construction of the stiffness matrix

We have also a matrix

\[
B = \begin{bmatrix}
P_0\phi_1 & \cdots & P_0\phi_N \\
a^E(\phi_1, m_2) & \cdots & a^E(\phi_N, m_2) \\
\vdots & \ddots & \vdots \\
a^E(\phi_1, m_n) & \cdots & a^E(\phi_N, m_n)
\end{bmatrix}
\]

expressing (in terms of the bases) the right hand side in the definition

\[
\begin{aligned}
a^E(\Pi v_h, p) &= a^E(v_h, p) & \forall p \in \mathbb{P}_m(E)/\mathbb{R} \\
P_0(\Pi v_h) &= P_0(v_h)
\end{aligned}
\]

**NOTE:** such matrix is computable! (integration by parts and d.o.f.s definition)
Construction of the stiffness matrix

Let

\[ G = BD = \begin{bmatrix}
0 & P_0 m_1 & P_0 m_2 & \cdots & P_0 m_n \\
(\nabla m_2, \nabla m_2)_{0, P} & 0 & (\nabla m_2, \nabla m_2)_{0, P} & \cdots & (\nabla m_2, \nabla m_n)_{0, P} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & (\nabla m_n, \nabla m_2)_{0, P} & \cdots & (\nabla m_n, \nabla m_n)_{0, P} & 0
\end{bmatrix}. \]
Construction of the stiffness matrix

Let

\[
G = BD = \begin{bmatrix}
P_0 m_1 & P_0 m_2 & \ldots & P_0 m_n \\
0 & (\nabla m_2, \nabla m_2)_{0,P} & \ldots & (\nabla m_2, \nabla m_n)_{0,P} \\
\vdots & \vdots & \ddots & \vdots \\
0 & (\nabla m_n, \nabla m_2)_{0,P} & \ldots & (\nabla m_n, \nabla m_n)_{0,P}
\end{bmatrix}.
\]

Compute the matrices corresponding to the projection operator:

\[
\Pi_* = (G)^{-1} B, \quad \Pi = D \Pi_*.
\]
Construction of the stiffness matrix

Let

\[
G = BD = \begin{bmatrix}
P_0 m_1 & P_0 m_2 & \cdots & P_0 m_n \\
0 & (\nabla m_2, \nabla m_2)_{0,P} & \cdots & (\nabla m_2, \nabla m_n)_{0,P} \\
\vdots & \vdots & \ddots & \vdots \\
0 & (\nabla m_n, \nabla m_2)_{0,P} & \cdots & (\nabla m_n, \nabla m_n)_{0,P}
\end{bmatrix}.
\]

Compute the matrices corresponding to the projection operator:

\[
\Pi_\star = (G)^{-1}B, \quad \Pi = D\Pi_\star.
\]

Finally compute the local stiffness matrix

\[
M^E = \Pi_\star^T \tilde{G} \Pi_\star + (I - \Pi)^T(I - \Pi).
\]
In principle the 3D case is analogous; the degrees of freedom are

- one point value per vertex
- \((m - 1)\) point values per edge
- \(M^f_0, M^f_1, \ldots, M^f_{m-1}\) moments per face
- \(M^E_0, M^E_1, \ldots, M^E_{m-2}\) moments per element.
In principle the 3D case is analogous; the degrees of freedom are

- one point value per vertex
- \((m - 1)\) point values per edge
- \(M^f_0, M^f_1, \ldots, M^f_{m-1}\) moments per face
- \(M^E_0, M^E_1, \ldots, M^E_{m-2}\) moments per element.

A more efficient formulation with less degrees of freedom per face

- \(M^f_0, M^f_1, \ldots, M^f_{m-2}\) moments per face

can be built using the \(L^2\) projector in [Ahmed, Alsaedi, Brezzi, Marini, Russo, CMA 2013].
Virtual Elements for $H_{\text{div}}$: introduction

Consider the diffusion problem in mixed form

$$\begin{align*}
\begin{cases}
\text{Find } F \in V := H_{\text{div}}(\Omega), p \in Q := L^2(\Omega) : \\
\int_{\Omega} F \cdot G + \int_{\Omega} (\text{div} G) p = 0 & \forall G \in V, \\
\int_{\Omega} (\text{div} F) q = -\int_{\Omega} f q & \forall q \in Q.
\end{cases}
\end{align*}$$

We introduce the VEM spaces:

$$Q_h = \{ q \in L^2(\Omega) : q|_E \in \mathbb{P}_{k-1}(E) \forall E \in \Omega_h \} \subset Q,$$

$$V_h = \{ G \in H_{\text{div}}(\Omega) : G|_E \in V_h|_E \forall E \in \Omega_h \} \subset V.$$ 

What follows taken from [BdV, Brezzi, Marini, Russo, submitted]. (se also [Brezzi, Falk, Marini, M2AN, 2014])
Let $E \in \Omega_h$. We introduce the local VEM space

$$V_{h|E} = \left\{ G \in H_{\text{div}}(E) \cap H_{\text{rot}}(E) : \text{div} G \in \mathbb{P}_{k-1}(E), \text{rot} G \in \mathbb{P}_{k-1}(E),
G\mid_e \cdot n^e_E \in \mathbb{P}_k(e) \forall e \in \partial E \right\}.$$
Virtual Elements for $H_{\text{div}}$: local spaces

Let $E \in \Omega_h$. We introduce the local VEM space

$$V_{h|E} = \left\{ G \in H_{\text{div}}(E) \cap H_{\text{rot}}(E) : \text{div} G \in \mathbb{P}_{k-1}(E), \text{rot} G \in \mathbb{P}_{k-1}(E), \right.$$

$$G|_e \cdot n_e^E \in \mathbb{P}_k(e) \ \forall e \in \partial E \left. \right\}.$$  

This space is associated to the problem

$$\begin{cases}
\text{div} G = f_1 \quad & \text{on } E, \\
\text{rot} G = f_2 \quad & \text{on } E, \\
G \cdot n = f_{\partial} \quad & \text{on } \partial E.
\end{cases}$$

that is well posed if $\int_E f_1 = \int_{\partial E} f_{\partial}$.

Thus

$$\dim(V_{h|E}) = 2 \dim(\mathbb{P}_{k-1}(E)) + N_e \dim(\mathbb{P}_k(e)) - 1.$$
Virtual Elements for $H_{\text{div}}$: local degrees of freedom

- **edge moments**

  \[ \int_e (G \cdot n_E^e)p_k \quad \forall p_k \in \mathbb{P}_k(e), \ \forall e \in \partial E. \]

- **div volume moments:**

  \[ \int_E (\text{div} G)p_{k-1} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(E)/\mathbb{R}. \]
Virtual Elements for $H_{\text{div}}$: local degrees of freedom

- **edge moments**
  \[
  \int_{e} (G \cdot n^e_E) p_k \quad \forall p_k \in \mathbb{P}_k(e), \forall e \in \partial E.
  \]

- **div volume moments**:
  \[
  \int_{E} (\text{div} G) p_{k-1} \quad \forall p_{k-1} \in \mathbb{P}_{k-1}(E)/\mathbb{R}.
  \]

- **additional volume moments**:
  \[
  \int_{E} G \cdot p_k \quad \forall p_k \in G_k^\perp(E).
  \]

The space

\[
G_k^\perp = \{ p \in \mathbb{P}_k : \int_{E} p \cdot \nabla q = 0 \ \forall q \in \mathbb{P}_{k+1}(E) \}.
\]

has dimension equal to $\mathbb{P}_{k-1}(E)$. 
Unisolvenence. Let $G \in V_{h|E}$, null on all dofs. Since

- $\text{rot} : G^\perp_k \rightarrow \mathbb{P}_{k-1}$ is a bijection,
- $\text{rot}G \in \mathbb{P}_{k-1}$,

it exists $\varphi \in G^\perp_k$ such that

$$0 = \text{rot}(G - \varphi) \implies G = \nabla \psi + \varphi \text{ with } \psi \in H^1(E).$$
**Unisolvence.** Let $G \in V_{h|E}$, null on all dofs. Since

- $\text{rot} : G_{k} \perp \rightarrow \mathbb{P}_{k-1}$ is a bijection,
- $\text{rot}G \in \mathbb{P}_{k-1}$,

it exists $\varphi \in G_{k} \perp$ such that

$$0 = \text{rot}(G - \varphi) \implies G = \nabla \psi + \varphi \text{ with } \psi \in H^{1}(E).$$

Thus

$$\|G\|_{L^{2}(E)}^{2} = \int_{E} G (\nabla \psi + \varphi)$$

$$= -\int_{E} (\text{div}G)\psi + \int_{\partial E} (G \cdot n_{E})\psi + \int_{E} G \varphi = 0.$$
the first set of dofs determines $G \cdot n$ on $\partial E$; 

since $\text{div} G \in \mathbb{P}_{k-1}(E)$, the second set of dofs determines $\text{div} G$. 

therefore we can compute

$$
\int_E G \nabla \psi = - \int_E (\text{div} G) \psi + \int_{\partial E} (G \cdot n_E) \psi \quad \forall \psi \text{ polynomial};
$$

L. Beirão da Veiga (Univ. of Milan)
Virtual Elements for $H_{\text{div}}$: computing the $L^2$ projection

- the first set of dofs determines $G \cdot \mathbf{n}$ on $\partial E$;
- since $\text{div} G \in \mathbb{P}_{k-1}(E)$, the second set of dofs determines $\text{div} G$.
- therefore we can compute

$$
\int_E G \nabla \psi = - \int_E (\text{div} G) \psi + \int_{\partial E} (G \cdot \mathbf{n}_E) \psi \quad \forall \psi \text{ polynomial};
$$

- any $q \in \mathbb{P}_k(E)^2$ can be written as

$$
q = p + \nabla \psi, \quad p \in G_k^\perp, \quad \psi \in \mathbb{P}_{k+1}(E).
$$

Thus we can compute

$$
\int_E G \cdot q = \int_E G \cdot p + \int_E G \cdot \nabla \psi.
$$
The proposed Virtual spaces \((V_h, Q_h)\) satisfy a commuting diagram property.

Thus are suitable for the approximation of the problem:

\[
\begin{align*}
\text{Find } F_h & \in V_h, \ p \in Q_h : \\
\int_{\Omega} F_h \cdot G_h + \int_{\Omega} (\text{div} \ G_h) p_h &= 0 \quad \forall G_h \in V_h, \\
\int_{\Omega} (\text{div} F_h) q_h &= \int_{\Omega} f \ q_h \quad \forall q_h \in Q_h.
\end{align*}
\]

**NOTE:** with the choices that we made, everything above is computable (up to the usual VEM construction and using the local \(L^2\) projections).
The proposed Virtual spaces \((V_h, Q_h)\) satisfy a commuting diagram property.

Thus are suitable for the approximation of the problem:

\[
\begin{align*}
\text{Find } F_h & \in V_h, \ p \in Q_h : \\
\int_{\Omega} F_h \cdot G_h + \int_{\Omega} (\text{div} G_h) p_h &= 0 \quad \forall G_h \in V_h, \\
\int_{\Omega} (\text{div} F_h) q_h &= \int_{\Omega} f q_h \quad \forall q_h \in Q_h.
\end{align*}
\]

**NOTE:** with the choices that we made, everything above is computable (up to the usual VEM construction and using the local \(L^2\) projections).

**VEM exact sequences**

A full “Safari” of VEM to appear in [BdV, Brezzi, Marini, Russo].
An application: the Cahn-Hilliard equation

- With **standard finite elements** it is very complicated to build spaces with global regularity higher than $C^0$.

- With VEM, this is instead easy to achieve. We can build elements with arbitrary $C^k$ regularity:
  
  - [Brezzi and Marini, CMAME, 2013]: $C^1$ VEM for Kirchhoff plates

- We will here show some “spoiler” from a paper in collaboration with Antonietti, Scacchi, Verani for applications to the Cahn-Hilliard equation for phase transition.
The Cahn-Hilliard equation

We search for \( u : \Omega \times [0, T] \rightarrow \mathbb{R} \) such that:

\[
\begin{align*}
\partial_t u - \Delta (\phi'(u) - \gamma^2 \Delta u(t)) &= 0 \quad \text{in } \Omega \times [0, T] \\
u(\cdot, 0) &= u_0(\cdot) \quad \text{in } \Omega \\
\partial_n u &= \partial_n (\phi'(u) - \gamma^2 \Delta u(t)) = 0 \quad \text{on } \partial\Omega \times [0, T],
\end{align*}
\]

where the function \( \phi(x) = (1 - x^2)^2/4 \) and \( \gamma \in \mathbb{R}^+ \) “small”.

- Mixed DG (Kay, Styles, Suli)
- Morley element (Elliott)
- Isogeometric Analysis (Hughes and co-workers)
- ....

L. Beirão da Veiga (Univ. of Milan)

The Virtual Element Method

DURHAM - 2014 31 / 35
The Cahn-Hilliard equation

We search for $u : \Omega \times [0, T] \rightarrow \mathbb{R}$ such that:

$$
\begin{aligned}
\begin{cases}
\partial_t u - \Delta (\phi'(u) - \gamma^2 \Delta u(t)) = 0 & \text{in } \Omega \times [0, T] \\
u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega \\
\partial_n u = \partial_n (\phi'(u) - \gamma^2 \Delta u(t)) = 0 & \text{on } \partial \Omega \times [0, T],
\end{cases}
\end{aligned}
$$

where the function $\phi(x) = (1 - x^2)^2/4$ and $\gamma \in \mathbb{R}^+ \text{ “small”}$.

The natural variational space is $H^2(\Omega)$, thus a $C^1$ regularity is needed for a conforming method.

- Mixed DG (Kay, Styles, Suli)
- Morley element (Elliott)
- Isogeometric Analysis (Hughes and co-workers)
- ....
$C^1$ VEM elements (of minimal degree)

$$
V_{h|E} = \left\{ v \in H^2(E) : -\Delta^2 v \in P_0(E), \right. \\
\left. v|_e \in P_3(e), \, \partial_n v|_e \in P_1(e) \quad \forall e \in \partial E \right\}.
$$

Degrees of freedom:

$k=1$, $p=2$
Some numerical result

We apply a primal VEM $C^1$ discretization to the problem:

- it involves $\Pi^0$, $\Pi^\nabla$ and $\Pi^\Delta$ projections;
- it grants a conforming solution and accepts general polygons;
- theoretical convergence estimates hold;
- initial numerical tests are encouraging.
More VEM literature not mentioned in the previous slides:

- Virtual Elements for linear elasticity problems
  [BdV, Brezzi, and Marini, SINUM, 2013]
- A stream function formulation for Stokes
  [Antonietti, BdV, Mora, Verani, SINUM, 2014]
- Three dimensional compressible elasticity
  [A.L. Gain, C. Talischi, G.H. Paulino, CMAME, 2014]
- IN PROGRESS: nonconforming elements (Ayuso, Lipnikov, Manzini), eigenvalue problems (Mora, Rodriguez), discrete fracture network (Berrone at al.), contact problems (Wriggers et al.), topology optimization (Paulino et al.), etc..
More VEM literature not mentioned in the previous slides:

- Virtual Elements for linear elasticity problems
  [BdV, Brezzi, and Marini, SINUM, 2013]
- A stream function formulation for Stokes
  [Antonietti, BdV, Mora, Verani, SINUM, 2014]
- Three dimensional compressible elasticity
  [A.L. Gain, C. Talischi, G.H. Paulino, CMAME, 2014]
- IN PROGRESS: nonconforming elements (Ayuso, Lipnikov, Manzini), eigenvalue problems (Mora, Rodriguez), discrete fracture network (Berrone at al.), contact problems (Wriggers et al.), topology optimization (Paulino et al.), etc..

Regarding VEM implementation:

On M3AS: The hitchhikers guide to VEM, a paper all about VEM implementation.
Conclusions

- The Virtual Element Method is a generalization of FEM that takes inspiration from modern mimetic schemes.

- The freedom that is left to the local spaces allows for a large flexibility, for instance in terms of meshes (polygons, “hanging nodes”), global regularity of the discrete space, definition of the local matrixes (M-optimization), etc..

- A lot of development is still to be done in VEM, and we believe it can be a very interesting new field of research.

- Moreover, more complex coding and problems need to be challenged in order to assess the impact of VEM in applications (various nonlinear problems already under development ...).