DISCONTINUOUS PETROV-GALERKIN (DPG) METHOD
WITH OPTIMAL TEST FUNCTIONS
Fundamentals

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Act 1: The Big (Functional Analysis) Picture
Act 2: Broken Test Spaces and Primal DPG Method
Act 3: Robust Primal DPG Method: Controlling the Convergence (Trial) Norm (new!)
Act 4: Ultraweak Variational Formulation
Act One

The Big (Functional Analysis) Picture
Three Interpretations of DPG

Optimal Test Functions

Mixed Method

Step 1
Step 2
Step 2a
Step 2b

Minimum Residual Method
Abstract variational problem

$U, V$ - Hilbert spaces,

$b(u, v)$ - bilinear (sesquilinear) continuous form on $U \times V$,

$$|b(u, v)| \leq \|b\| \|u\|_U \|v\|_V,$$

$$=: M$$

$l(v)$ - linear (antilinear) continuous functional on $V$,

$$|l(v)| \leq \|l\|_V, \|v\|$$

The abstract variational problem:

$$\begin{cases} u \in U \\ b(u, v) = l(v) \ \forall v \in V \end{cases} \Leftrightarrow Bu = l \quad B : U \to V'$$

$$< Bu, v > = b(u, v) \quad v \in V$$
If $b$ satisfies the inf-sup condition ($\Leftrightarrow B$ is bounded below),

$$\inf_{\|u\|_{U}=1} \sup_{\|v\|_{V}=1} |b(u,v)| =: \gamma > 0 \Leftrightarrow \sup_{v \in V} \frac{|b(u,v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

and $l$ satisfies the compatibility condition:

$$l(v) = 0 \quad \forall v \in V_0$$

where

$$V_0 := \mathcal{N}(B') = \{ v \in V : b(u,v) = 0 \quad \forall u \in U \}$$

then the variational problem has a unique solution $u$ that satisfies the stability estimate:

$$\|u\| \leq \frac{1}{\gamma} \|l\|_{V'}.$$  

**Proof:** Direct interpretation of Banach Closed Range Theorem*.  

---  

*see e.g. Oden, D, *Functional Analysis*, Chapman & Hall, 2nd ed., 2010, p.518
Petrov-Galerkin Method and Babuška Theorem

$U_h \subset U, V_h \subset V$, $\dim U_h = \dim V_h$ - finite-dimensional trial and test (sub)spaces

\[
\begin{cases}
\ u_h \in U_h \\
\ b(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h
\end{cases}
\]

**Theorem** (Babuška†).

The *discrete inf-sup condition*

\[
\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U
\]

implies existence, uniqueness and discrete stability

\[
\|u_h\|_U \leq \gamma_h^{-1} \|l\|_{V_h'}
\]

Petrov-Galerkin Method and Babuška Theorem

\[ U_h \subset U, V_h \subset V, \dim U_h = \dim V_h - \text{finite-dimensional trial and test (sub)spaces} \]

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  u_h \in U_h \\
  b(u_h, v_h) = l(v_h), \quad \forall v_h \in V_h 
\end{cases} \]

**Theorem** (Babuška\(^\dagger\)).

The *discrete inf-sup condition*

\[ \sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U \]

implies existence, uniqueness and discrete stability

\[ \|u_h\|_U \leq \gamma_h^{-1} \|l\|_{V_h} \]

and convergence

\[ \|u - u_h\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|u - w_h\|_U \]

*(Uniform) discrete stability and approximability imply convergence.*

Optimal test functions

The main trouble:

continuous inf-sup condition $\iff$ discrete inf-sup condition
Optimal test functions

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\[
\text{continuous inf-sup condition} \not\Rightarrow \text{discrete inf-sup condition}
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\text{continuous inf-sup condition } \not\iff \text{discrete inf-sup condition}
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unless we employ special test functions that realize the supremum in the inf-sup condition:

\[
v_h = \arg \max_{v \in V} \frac{|b(u_h, v)|}{\|v\|}
\]

Optimal test functions

The main trouble:

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unless \‡ we employ special test functions that \textit{realize} the supremum in the inf-sup condition:

\[
v_h = \arg \max_{v \in V} \frac{|b(u_h, v)|}{\|v\|}
\]

Recall that the Riesz operator \( R_V : V \to V' \) is an isometry. Then:

\[
\sup_v \frac{|b(u_h, v)|}{\|v_h\|} = \| Bu_h \|_{V'} = \left\| R_V^{-1} Bu_h \right\|_V = \frac{(R_V^{-1} Bu_h, v_h)_V}{\|v\|_V}
\]

\[
= \frac{\langle Bu_h, v_h \rangle}{\|v\|_V} = \frac{b(u_h, v_h)}{\|v\|_V}
\]

\[
\text{Variational definition of } v_h: \quad \begin{cases} v_h \in V \\ (v, \delta v)_V = b(u_h, \delta v) \quad \forall \delta v \in V \end{cases}
\]

The operator \( T := R_V^{-1} B : U_h \to V \) will be called the \textit{trial to test operator}.

DPG is a Minimum Residual Method

With the optimal test functions in place, \( \gamma_h \geq \gamma \), and the Galerkin method is automatically stable. Trade now the original norm in \( U \) for an energy norm:\n
\[ \| u \|_E := \| R_V^{-1} Bu \|_V = \| Bu \|_V = \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \]

Two points:

- With respect to the new, energy norm, both continuity constant \( M \) and inf-sup constant \( \gamma \) are unity.

- The use of optimal test functions (their construction is independent of the choice of trial norm) implies that \( \gamma_h \geq \gamma = 1 \).

Thus, by the Babuška Theorem,

\[ \| u - u_h \|_E \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \| u - w_h \|_E = 1 \]

In other words, FE solution \( u_h \) is the best approximation of the exact solution \( u \) in the energy norm. We have arrived through a back door at a Minimum Residual Method.

---

\( \S \)Residual norm really...
Moral of the story

The minimum residual method, with the residual measured in the dual test norm, is the most stable Petrov-Galerkin method you can have.
DPG is a minimum residual method

\[ \begin{aligned}
\{ & u \in U \\
& b(u, v) = l(v) \quad v \in V \end{aligned} \iff
\begin{aligned}
Bu = l & \quad B : U \to V' \\
\langle Bu, v \rangle &= b(u, v)
\end{aligned} \]

---

DPG is a minimum residual method

\[
\begin{align*}
\begin{cases}
  u \in U \\
  b(u, v) = l(v)
\end{cases} &\iff Bu = l & B : U \to V' \\
  v \in V &\iff \langle Bu, v \rangle = b(u, v)
\end{align*}
\]

- **Minimum residual method:** \( U_h \subset U \),

\[
\frac{1}{2} \| Bu_h - l \|^2_{V'} \rightarrow \min_{u_h \in U_h}
\]


DPG is a minimum residual method

\[
\begin{aligned}
\begin{cases}
  u \in U \\
  b(u, v) = l(v)
\end{cases}
\quad \Leftrightarrow 
\quad Bu = l \\
\quad v \in V \\
\quad \langle Bu, v \rangle = b(u, v)
\end{aligned}
\]

- **Minimum residual method:** \( U_h \subset U, \)

\[
\frac{1}{2} \| Bu_h - l \|_{V'}^2 \to \min_{u_h \in U_h}
\]

- **Riesz operator:**

\[
R_V : V \to V', \quad \langle R_V v, \delta v \rangle = (v, \delta v)_V
\]

is an isometry, \( \| R_V v \|_{V'} = \| v \|_V. \)

---


DPG is a minimum residual method

\[
\begin{aligned}
\begin{cases}
  u \in U \\
  b(u, v) = l(v)
\end{cases}
\iff
Bu = l \quad B : U \to V' \\
\langle Bu, v \rangle = b(u, v)
\end{aligned}
\]

- **Minimum residual method:** \( U_h \subset U, \)

\[
\frac{1}{2} \| Bu_h - l \|_{V'} \to \min_{u_h \in U_h}
\]

- **Riesz operator:**

\[
R_V : V \to V', \quad \langle R_V v, \delta v \rangle = (v, \delta v)_V
\]

is an *isometry*, \( \| R_V v \|_{V'} = \| v \|_V. \)

- **Minimum residual method reformulated:**

\[
\frac{1}{2} \| Bu_h - l \|_{V'}^2 = \frac{1}{2} \| R_V^{-1}(Bu_h - l) \|_V^2 \to \min_{u_h \in U_h}
\]

---


Taking Gâteaux derivative,

\[(R_V^{-1}(B u_h - l), R_V^{-1} B \delta u_h)_V = 0 \quad \delta u_h \in U_h\]
Taking Gâteaux derivative,

\[(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \delta u_h \in U_h\]
Taking Gâteaux derivative,

\[
(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h
\]

or

\[
\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle_{v_h} = 0 \quad \delta u_h \in U_h
\]
Taking Gâteaux derivative,

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(R_V^{-1}(B u_h - l), R_V^{-1} B \delta u_h)_V = 0 \quad \delta u_h \in U_h
\]

or

\[
\langle B u_h - l, v_h \rangle = 0 \quad v_h = R_V^{-1} B \delta u_h
\]
Taking Gâteaux derivative,

\[(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[\langle Bu_h, v_h \rangle = \langle l, v_h \rangle \quad v_h = R_V^{-1}B\delta u_h\]
Taking Gâteaux derivative,

\[
(R_V^{-1}(Bu_h - l), R_V^{-1} B \delta u_h)_V = 0 \quad \delta u_h \in U_h
\]

or

\[
b(u_h, v_h) = l(v_h)
\]

where

\[
\begin{cases}
    v_h \in V \\
    (v_h, \delta v)_V = b(\delta u_h, \delta v) \quad \delta v \in V
\end{cases}
\]
DPG is a mixed method

An alternate route $\parallel$, 

$$
\left( R_{V}^{-1}(B\delta u_{h} - l), R_{V}^{-1}B\delta u_{h}\right)_{V} = 0 \quad \delta u_{h} \in U_{h}
$$

$=:\psi$(error representation function)

---

DPG is a mixed method

An alternate route \( \parallel \),

\[
\begin{align*}
\left( R_V^{-1}(Bu_h - l) \right)_V, R_V^{-1} B \delta u_h = 0 \quad \delta u_h \in U_h
\end{align*}
\]

\( =: \psi \) (error representation function)

or

\[
\begin{align*}
\psi &= R_V^{-1}(Bu_h - l) \\
(\psi, R_V^{-1} B \delta u_h)_V = 0 \quad \delta u_h \in U_h
\end{align*}
\]

DPG is a mixed method

An alternate route $\parallel$

$$
\left( R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h \right)_V = 0 \quad \delta u_h \in U_h
$$

$=:\psi$ (error representation function)

or

$$
\begin{cases}
(\psi, \delta v)_V - b(u_h, \delta v) &= -l(\delta v) \quad \forall \delta v \in V \\
b(\delta u_h, \psi) &= 0 \quad \forall \delta u_h \in U_h
\end{cases}
$$

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DPG method, a summary so far

- Stiffness matrix is always hermitian and positive-definite (it is a generalization of the least squares method...).

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The method delivers the best approximation error (BAE) in the “energy norm”:

\[ \|u\|_E := \|Bu\|_V' = \sup_{v \in V} \frac{b(u,v)}{\|v\|_V} \]

Stiffness matrix is always hermitian and positive-definite (it is a generalization of the least squares method...).

The method delivers the *best approximation error* (BAE) in the “energy norm”:

\[ \|u\|_E := \|Bu\|_V = \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \]

The energy norm of the FE error \(u - u_h\) equals the residual and can be computed,

\[ \|u - u_h\|_E = \|Bu - Bu_h\|_V = \|l - Bu_h\|_V, \quad \|R_V^{-1}(l - Bu_h)\|_V = \|\psi\|_V \]

where the *error representation function* \(\psi\) comes from

\[
\begin{cases}
\psi \in V \\
(\psi, \delta v)_V = \langle l - Bu_h, \delta v \rangle = l(\delta v) - b(u_h, \delta v), \quad \delta v \in V
\end{cases}
\]

No need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques **

A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.
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How to choose the test norm in a systematic way?
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How to choose the test norm in a systematic way?

Is the inversion of Riesz operator (computation of the optimal test functions, energy error) feasible?
A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.

How to choose the test norm in a systematic way?

Is the inversion of Riesz operator (computation of the optimal test functions, energy error) feasible?

Being a Ritz method, DPG does not experience any preasymptotic limitations.
You cannot compute the optimal test functions!
Approximate optimal test functions

Take a finite-dimensional *enriched* test space: $\tilde{V} \subset V$, $\dim \tilde{V} \gg \dim U_h$, and invert the Riesz operator approximately,

$$
\begin{cases}
\tilde{v}_h \in \tilde{V} \\
(\tilde{v}_h, \delta v)_V = b(u_h, \delta v) \quad \forall \delta v \in \tilde{V}.
\end{cases}
$$

This leads to an *approximate trial to test operator*:

$$
\tilde{T} : U_h \rightarrow \tilde{V} \quad \tilde{T}u_h := \tilde{v}_h
$$

and *approximate optimal test space*:

$$
\tilde{V}_h := \tilde{T}U_h.
$$

Some stability must be lost. How much?
Approximate mixed problem

\[
\begin{cases}
\tilde{\psi} \in \tilde{V}, \tilde{u}_h \in U_h \\
(\tilde{\psi}, \delta \tilde{\psi})_V - b(\tilde{u}_h, \delta \tilde{\psi}) = -l(\delta \tilde{\psi}) \quad \tilde{\psi} \in \tilde{V} \\
b(\delta u_h, \tilde{\psi}) = 0 \quad \delta u_h \in U_h
\end{cases}
\]

The (discrete) inf sup condition must be satisfied:

\[
\sup_{\tilde{\psi} \in \tilde{V}} \frac{|b(u_h, \delta \tilde{\psi})|}{\|\delta \tilde{\psi}\|} \geq \gamma_h \|u_h\|
\]

Back to square one??
Coming up with a Fortin operator

\[ \tilde{\Pi} : V \to \tilde{V} \]

such that

\[ \|\tilde{\Pi} v\|_V \leq C \|v\|_V \]

and

\[ b(u_h, \tilde{\Pi} v - v) = 0 \quad \forall u_h \in U_h \]

solves the problem ††

Act Two

Broken Test Spaces and
Primal DPG Method
Standard assumptions: $\Omega \subset \mathbb{R}^N$ Lipschitz domain.

Elements: $K$

Edges: $e$

Skeleton: $\Gamma_h = \bigcup_K \partial K$

Internal skeleton: $\Gamma^0_h = \Gamma_h - \partial \Omega$
Primal DPG method

Given \( f \in L^2(\Omega) \), consider the model problem,

\[
\begin{aligned}
&\left\{ \begin{array}{l}
  u = 0 \quad \text{on } \Gamma := \partial \Omega \\
  -\Delta u = f \quad \text{in } \Omega
\end{array} \right.
\end{aligned}
\]

Multiply the PDE with a test function \( v \), integrate over each element \( K \), integrate by parts and sum up over all elements,

\[
\sum_K \int_K \nabla u \cdot \nabla v + \sum_K \int_{\partial K} \frac{\partial u}{\partial n} v = \sum_K \int f v
\]

The boundary term represents jumps,

\[
\sum_K \int_{\partial K} \frac{\partial u}{\partial n} v = \sum_e \int_e \frac{\partial u}{\partial n_e} [v]_e
\]

where

\[
[v]_e = \begin{cases} 
  v_+ - v_- & e \subset \Omega \\
  v & e \subset \Gamma
\end{cases}
\]
This leads to the variational problem:

\[
\begin{cases}
    u \in H^1(\Omega), \hat{t} \in H^{-1/2}(\Gamma_h) \\
    (\nabla u, \nabla_h v) - \langle \hat{t}, v \rangle_{\Gamma_h} = (f, v) \quad v \in H^1(\Omega_h)
\end{cases}
\]

where

\[H^{-1/2}(\Gamma_h) = \text{trace of } H(\text{div}, \Omega) \text{ on } \Gamma_h\]
equipped with the quotient norm.

**Theorem ‡‡**
The variational problem above is well posed with a mesh independent inf-sup constant \(\gamma\).

---

The main point

The test norm is *localizable*, i.e.

\[
\|v\|^2_{H^1(\Omega_h)} = \sum_K \|v|_{K}|^2_{H^1(K)} \cdot
\]

The (approximate) inversion of the Riesz operator is done locally (elementwise)
DPG element stiffness matrix and load vector

\[ u_h = \sum_{i=1}^{N} u_i e_i, \quad v_h \approx \sum_{j=1}^{M} v_j g_j, \quad M >> N \]

Computation of (approximate) optimal test function \( v = T e_i \),

\[ \sum_j \underbrace{(g_j, g_l)}_{\text{Gram matrix } G} v_j^i = \underbrace{b(e_i, g_l)}_{\text{expanded stiffness matrix } B}, \quad l = 1, \ldots, M \]

or

\[ v = G^{-1} B \delta u \]

The DPG stiffness matrix and load vector:

\[ v^T B u = (G^{-1} B \delta u)^T B u = (\delta u)^T B^T G^{-1} B u \]

\[ v^T b = (G^{-1} B \delta u)^T b = (\delta u)^T B^T G^{-1} b \]
Same result with the mixed method interpretation

\[
\begin{pmatrix}
G & -B \\
B^T & 
\end{pmatrix}
\begin{pmatrix}
\psi \\
u
\end{pmatrix} =
\begin{pmatrix}
-b \\
0
\end{pmatrix}
\]

Condensing out error indication function \( \psi \),

\[
\psi = G^{-1}(Bu - b)
\]

we get again,

\[
B^T G^{-1} Bu = G^{-1} B b
\]
Primal DPG Formulation for the Poisson problem

Group unknown (watch for the overloaded symbol):

$$u_h := \left( u_h, \hat{t}_h \right)$$

Mixed system:

$$
\begin{pmatrix}
G & -B_1 & -B_2 \\
B_1^T & 0 & 0 \\
B_2^T & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\psi \\
u \\
\hat{t}
\end{pmatrix}
= 
\begin{pmatrix}
-b \\
0 \\
0
\end{pmatrix}
$$

where $B_1, B_2$ correspond to $(\nabla u_h, \nabla_h \tilde{v})$ and $-\langle \hat{t}_h, \tilde{v} \rangle$, resp.

Eliminate $\psi$ to get the DPG system:

$$
\begin{pmatrix}
B_1^T G^{-1} B_1 & B_1^T G^{-1} B_2 \\
B_2^T G^{-1} B_1 & B_2^T G^{-1} B_2
\end{pmatrix}
\begin{pmatrix}
u \\
\hat{t}
\end{pmatrix}
= 
\begin{pmatrix}
B_1^T G^{-1} b \\
B_2^T G^{-1} b
\end{pmatrix}
$$
Neglecting the error stemming from the approximation of optimal test function (computation of residual), we have,

\[
\left\{ \begin{align*}
\left( \| u - u_h \|_{H^1(\Omega)}^2 + \| \hat{t} - \hat{t}_h \|_{H^{-1/2}(\Gamma_h)}^2 \right)^{1/2} \\
\leq \frac{1}{\gamma} \inf_{w_h, r_h} \left( \| u - w_h \|_{H^1(\Omega)}^2 + \| \hat{t} - r_h \|_{H^{-1/2}(\Gamma_h)}^2 \right)^{1/2}
\end{align*} \right.
\]

best approximation error

Additionally,

\[
\left\{ \begin{align*}
\left( \| u - u_h \|_{H^1(\Omega)}^2 + \| \hat{t} - \hat{t}_h \|_{H^{-1/2}(\Gamma_h)}^2 \right)^{1/2} \\
\leq \frac{1}{\gamma} \sup_{v \in H^1(\Omega_h)} \frac{\| (\nabla u_h, \nabla_h v) - \langle \hat{t}_h, v \rangle_{\Gamma_h} \|}{\| v \|_{H^1(\Omega_h)}}
\end{align*} \right.
\]

computable residual

\[
= \frac{1}{\gamma} \left( \sum_K \| \psi_K \|_{H^1(K)}^2 \right)^{1/2}
\]
2D convergence rates

Square domain: h and p convergence

L-shaped domain: h convergence
Poisson problem
Reaction-dominated diffusion
Convection-dominated diffusion
\{ \text{div-grad problems} \}

Maxwell equations - curl-curl problem

All examples have been implemented within \textit{hp3d}, a general 3D FE code supporting:

\begin{itemize}
  \item coupled problems involving $H^1$, $H(\text{curl})$ and $H(\text{div})$-conforming elements.
  \item hybrid meshes consisting of hexas, tets, prisms and pyramids,
  \item \textit{anisotropic $hp$-refinements}.
\end{itemize}

\textit{Ask me about the code...}
Hexahedral meshes

$H^1$ element for field $u_h$:

$$\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^p,$$

Trace of $H(\text{div})$ element:

$$(\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^{p-1}) \times (\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1}) \times (\mathcal{P}^{p-1} \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p)$$

for flux $\hat{t}_h$, and the enriched element:

$$\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p},$$

for test function $v_h$.

In reported experiments: $p = 1, 2, 3$, $\Delta p = 2$. 

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Poisson problem, smooth solution, uniform refinements

Rectangular domain $\Omega = (0, 1) \times (0, 2) \times (0, 1)$,
Smooth solution: $u = \sin \pi x \sin \pi y \sin \pi z$
Boundary condition: $u = 0$.

Residual versus $H^1$ error.
Poisson problem, manufactured shock solution

BC: \( u = u_0 \).
Shock solution, uniform and $h$-adaptive refinements, $p = 1$

Convergence history for the residual and $H^1$ error

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DPG Method
Shock solution, uniform and $h$-adaptive refinements, $p = 2$

Convergence history for the residual and $H^1$ error
Shock solution, uniform and $h$-adaptive refinements, $p = 3$

Convergence history for the residual and $H^1$ error

SCALES: log(nr dof), log(error)

- unif_resid
- unif_H1err
- adap_resid
- adap_H1err
Shock solution, $p = 3$, Mixed BC


Convergence history for the residual and $H^1$ error
Reaction-dominated diffusion, $p = 2$.

\[ \begin{cases} 
    u = 0 & \text{on } \Gamma \\
    -\epsilon^2 \Delta u + u = 1 & \text{in } \Omega 
\end{cases} \]

$\epsilon = 0.01$, left: solution after 7 iterations, right: convergence history
Convection-dominated diffusion, $p = 2$.

\[
\begin{aligned}
-\epsilon^2 \Delta u - u &= \sin \pi y \sin \pi z \quad \text{at } x = 0 \\
 u &= 0 \quad \text{on the rest of } \Gamma \\
-\epsilon^2 \Delta u + \frac{\partial u}{\partial x} &= 0 \quad \text{in } \Omega
\end{aligned}
\]

$\epsilon = 0.01$, left: solution after 5 iterations, right: convergence history
Maxwell equations.

Assume

\[ J_{\text{imp}}^S = n \times H^{\text{imp}} \]

and look for the unknown surface current on the skeleton also in the same form.

\[
\begin{aligned}
&E \in H(\text{curl}, \Omega), \ n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\
&\hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), \ n \times \hat{h} = n \times (-i\omega H^{\text{imp}}) \text{ on } \Gamma_2 \\
\left( \frac{1}{\mu} \nabla \times E, \nabla_h \times F \right) + \left( (-\omega^2 \epsilon + i\omega \sigma)E, F \right) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\
&\forall F \in H(\text{curl}, \Omega_h).
\end{aligned}
\]
Hexahedral meshes

$H(\text{curl})$ element for electric field $E$:

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p-1})$$

and trace of the same element for flux (surface current) $\hat{h}$.

Same element for the enriched space but with order $p + \Delta p$.

In reported experiments: $p = 2$, $\Delta p = 2$. 
DPG Supports Adaptivity with No Preasymptotic Behavior
A 3D Maxwell example

Take a cube $(0, 2)^3$
Divide it into eight smaller cubes and remove one:
Fichera corner microwave

Attach a waveguide:

\[ \epsilon = \mu = 1, \sigma = 0 \]
\[ \omega = 5(1.6 \text{ wavelengths in the cube}) \]

Cut the waveguide and use the lowest propagating mode for BC along the cut.
Fichera corner microwave, $p = 2$.

Initial mesh and real part of $E_1$
Fichera corner microwave, $p = 2$.

Mesh and real part of $E_1$ after two refinements
Fichera corner microwave, $p = 2$.

Mesh and real part of $E_1$ after four refinements
Fichera corner microwave, $p = 2$.

Mesh and real part of $E_1$ after six refinements
Fichera corner microwave, $p = 2$.

Mesh and real part of $E_1$ after eight refinements
Robust DPG Method: Controlling the Convergence (Trial) Norm
The simplest singular perturbation problem:
reaction-dominated diffusion

The simplest singular perturbation problem: Reaction-dominated diffusion

\[
\begin{aligned}
    \left\{ \begin{array}{l}
        u = 0 \quad \text{on } \Gamma \\
        -\epsilon^2 \Delta u + c(x) u = f \quad \text{in } \Omega
    \end{array} \right.
\end{aligned}
\]

where \( 0 < c_0 \leq c(x) \leq c_1 \).

Standard variational formulation:

\[
\begin{aligned}
    & \quad u \in H^1(\Omega) \\
    & \epsilon^2 (\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega)
\end{aligned}
\]

Standard Galerkin method delivers the best approximation error in the energy norm:

\[
\|u\|_{\epsilon k}^2 := \epsilon^k \|\nabla u\|^2 + \|c^{1/2} u\|^2, \quad k = 2
\]
**Fact:** Under favorable regularity conditions, the solution is *uniformly* bounded in data $f$ in a “balanced” norm:

$$\|u\|_\epsilon^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_\epsilon \lesssim \|f\|_{\text{appropriate}}$$

**Question:** Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm?

$$\|u - u_h\|_\epsilon + \|\hat{t} - \hat{t}_h\|? \lesssim \inf_{w_h} \|u - w_h\|_\epsilon + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|?$$

---

A bit of history: Optimal test functions of Barret and Morton

For each $w \in U_h$, determine the corresponding $v_w$ that solves the auxiliary variational problem:

$$
\begin{align*}
  v_w &\in H^1_0(\Omega) \\
  \mathcal{E}^2(\nabla \delta u, \nabla v_w) + (c \delta u, v_w) &\equiv \mathcal{E}(\nabla \delta u, w) + (c \delta u, w) \quad \forall \delta u \in H^1_0(\Omega)
\end{align*}
$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, “balanced” norm:

$$
\mathcal{E}^2(\nabla (u-u_h), \nabla v_w) + (c (u-u_h), v_w) = 0 \quad \Rightarrow \quad \mathcal{E}(\nabla (u-u_h), \nabla u) + (c (u-u_h), w) = 0
$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

---


Constructing optimal test norm

**Theorem**

Let $v_u$ be the Barret-Morton optimal test function corresponding to $u$. Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq BAE \text{ estimate}$$

**Proof:**

$$\|u\|^2_\epsilon = \epsilon (\nabla u, \nabla u) + (cu, u) = \epsilon^2 (\nabla u, \nabla v_u) + (cu, v_u)$$

$$= b((u, \hat{t}), v_u) \leq \frac{b((u, \hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V$$

$$\leq \sup_v \frac{b((u, \hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V$$

$$\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon$$
The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

\[ \|v\|_{V}^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2}v\|^2 \]

Then

\[ \|v_u\| \lesssim \|u\|_{\epsilon} \]

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

\[ \|v\|_{V,\text{mod}}^2 := \epsilon^3 \|\nabla v\|^3 + \min\{1, \frac{\epsilon^3}{h^2}\} \|c^{1/2}v\|^2 \]
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$

The functions exhibits strong boundary layers invisible in this scale.

Range: (-0.6, 0.6)
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$

Zoom on the north boundary layer.
Optimal mesh for $\epsilon = 10^{-1}$

Optimal $h$-adaptive mesh and numerical solution for $\epsilon = 10^{-1}$
Lin/Stynes example, $\epsilon = 1$

Residual and “balanced” error of $u$ for $h$-adaptive solution, $p = 2$
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$

Residual and “balanced” error of $u$ for $h$-adaptive solution, $p = 2$
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.

Residual and “balanced” error of $u$ for $h$-adaptive solution, $p = 2$
Lin/Stynes example, \( \epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3} \).

Residual and “balanced” error of \( u \) for \( h \)-adaptive solution, \( p = 2 \)
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.

Residual and “balanced” error of $u$ for $h$-adaptive solution, $p = 2$
Other tricks we can play: zooming on the solution

**Question:** Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0, 1)^2$?

**Answer:** Yes!

Optimal mesh and the corresponding pointwise error (range $(-0.001 \ - \ 0.001)$.
Act Four

Ultraweak Variational Formulation
2D Convection-Dominated Diffusion (Confusion) Problem

\[
\begin{aligned}
&-\epsilon \Delta u + \text{div}(\beta u) = f \quad \text{in } \Omega \\
&u = u_0 \quad \text{on } \Gamma
\end{aligned}
\]

or, equivalently,

\[
\begin{aligned}
&\frac{1}{\epsilon} \sigma - \nabla u = 0 \quad \text{in } \Omega \\
&-\text{div}(\sigma - \beta u) = f \quad \text{in } \Omega \\
&u = u_0 \quad \text{on } \partial\Omega
\end{aligned}
\]
Ultraweak (DPG) Variational Formulation

Elements: $K$

Edges: $e$

Skeleton: $\Gamma_h = \bigcup_K \partial K$

Internal skeleton: $\Gamma_h^0 = \Gamma_h - \partial \Omega$
Take an element $K$. Multiply the equations with test functions $\tau, v$:

\[
\begin{align*}
\frac{1}{\varepsilon} \sigma \cdot \tau - \nabla u \cdot \tau &= 0 \\
-\text{div}(\sigma - \beta u)v &= f v
\end{align*}
\]
Integrate over the element $K$:

\[
\begin{cases}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau - \nabla u \cdot \tau &= 0 \\
- \int_K \text{div}(\sigma - \beta u) v &= f v
\end{cases}
\]
Integrate by parts (relax) \textit{both} equations:

\[
\begin{align*}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau + \int_K u \ \text{div} \tau - \int_{\partial K} u \tau_n &= 0 \\
\int_K (\sigma - \beta u) \cdot \nabla v - \int_{\partial K} q \ \text{sgn}(n) \ v &= \int_K f \ v
\end{align*}
\]

where \( q = (\sigma - \beta u) \cdot n_e \) and

\[
\text{sgn}(n) = \begin{cases} 
1 & \text{if } n = n_e \\
-1 & \text{if } n = -n_e 
\end{cases}
\]
Declare traces and fluxes to be independent unknowns, common for adjacent elements:

\[
\begin{align*}
\int_{K} \frac{1}{\varepsilon} \sigma \cdot \tau + \int_{K} u \, \text{div} \tau - \int_{\partial K} \hat{u} \, \tau_n &= 0 \\
- \int_{K} (\sigma - \beta u) \cdot \nabla v + \int_{\partial K} \hat{q} \, \text{sgn}(n)v &= \int_{K} f v
\end{align*}
\]
Use BC to eliminate the known traces

\[
\begin{aligned}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau + \int_K u \, \text{div} \tau - \int_{\partial K - \partial \Omega} \hat{u} \, \tau_n &= \int_{\partial K \cap \partial \Omega} u_0 \, \tau_n \\
- \int_K (\sigma - \beta u) \cdot \nabla v + \int_{\partial K} \hat{q} \, \text{sgn}(n) v &= \int_K f v
\end{aligned}
\]
Abstract Notation

Integration by parts:

\[(Au, v) = (u, A^*_h v) - \langle \hat{u}, v \rangle_{\Gamma_h}\]

where (watch for overloaded symbols...)

\[u = (\sigma, u)\]
\[v = (\tau, v)\]
\[Au = (\frac{1}{\varepsilon} \sigma - \nabla u, -\text{div}(\sigma - \beta u))\]
\[A^*_h v = (\frac{1}{\varepsilon} \tau + \nabla_h v, \text{div}_h \tau - \beta \cdot \nabla_h v)\]

\[\langle \hat{u}, v \rangle_{\Gamma_h} = \int_{\Gamma_h} (u[\tau_n] + \sigma_n - \beta_n u[v])\]

\[\hat{u} = (\hat{u}, \hat{q})\text{ with } \hat{u} = 0 \text{ on } \Gamma\]

DPG variational formulation:

\[\frac{(u, A^*_h v) - \langle \hat{u}, v \rangle_{\Gamma_h}}{b((u, \hat{u}), v)} = \frac{(f, v) + \langle \tilde{u}_0, v \rangle_{\Gamma}}{l(v)}\]
Functional Setting for the Confusion Problem

**General Functional setting:**
- \( u \in L^2(\Omega) \),
- broken graph space for \( v \),

\[
H_{A^*}(\Omega_h) := \{ v \in L^2(\Omega_h) : A^*_h v \in L^2(\Omega_h) \}
\]
- trace space for \( \hat{u} \) with minimum energy extension norm:

\[
\| \hat{u} \|^2 = \inf_{u: u|_{\Gamma_h} = \hat{u}} (\| u \|^2 + \| A u \|^2)
\]

**Confusion problem:** Group variables:
Solution \((u, \sigma, \hat{u}, \hat{q})\):

- field variables: \( u, \sigma_1, \sigma_2 \in L^2(\Omega_h) \)
- traces: \( \hat{u} \in \tilde{H}^{1/2}(\Gamma^0_h) \)
- fluxes: \( \hat{q} \in H^{-1/2}(\Gamma_h) \)

Test function \((\tau, v)\):

\[
\tau \in H(\text{div}, \Omega_h) \\
v \in H^1(\Omega_h)
\]
With broken test spaces, the inversion of Riesz operator is done element-wise.
The Point

- With broken test spaces, the inversion of Riesz operator is done element-wise.
- We still can do it only approximately, using an enriched space and standard Bubnov-Galerkin method. If trial functions $u, \hat{u} \in P^p$, we seek approximate optimal test functions by inverting the Riesz operator in an enriched space $P^p+\Delta_p$,

$$\begin{cases} 
    v_h \in P^p+\Delta_p \\
    (v_h, \delta v)_V = (u, A^* \delta v) - \langle \hat{u}, \delta v \rangle \quad \forall v \in P^p+\Delta_p 
\end{cases}$$

The error in approximating the optimal test functions is assumed to be negligible.
▶ With broken test spaces, the inversion of Riesz operator is done element-wise.
▶ We still can do it only approximately, using an enriched space and standard Bubnov-Galerkin method. If trial functions \( u, \hat{u} \in \mathcal{P}^p \), we seek approximate optimal test functions by inverting the Riesz operator in an enriched space \( \mathcal{P}^p + \Delta p \),

\[
\begin{align*}
    v_h & \in \mathcal{P}^p + \Delta p \\
    (v_h, \delta v)_V = (u, A^* \delta v) - \langle \hat{u}, \delta v \rangle & \quad \forall v \in \mathcal{P}^p + \Delta p
\end{align*}
\]

The error in approximating the optimal test functions is assumed to be negligible.

▶ As the determination of optimal test functions is done element-wise, the method fits into the standard FE technology.

**Standard FEM:** *Input:* bilinear and linear form, trial and test shape functions,  
*Output:* element stiffness matrix and load vector,  
**DPG:** *Input:* bilinear and linear form, trial shape functions, test norm,  
*Output:* element stiffness matrix and load vector
Well Posedness

**Theorem** If the original operator $A$ with homogenous BC is bounded below,

$$\|A\| \geq \gamma \|u\|$$

and the data $u_0$ comes from the trace space for the graph norm space, then the DPG formulation is well posed as well, with a **mesh-independent inf-sup** constant of order $\gamma$.

**Corollary:** If $\gamma$ is independent of the singular perturbation parameter ($\epsilon$ for the confusion problem), then the DPG method is **robust**,

$$\|u - u_h\| + \|\hat{u} - \hat{u}_h\| \lesssim \inf_{w_h, \hat{w}_h} \{\|u - w_h\| + \|\hat{u} - \hat{w}_h\|\}$$

---


Construction of an optimal test norm
Bad and good news

**Bad news:** the graph test norm may not be feasible

**Good news:** There is a systematic approach for determining alternate test norms

---


Step 1: Decide what you want

We want the $L^2$ robustness in $u$:

$$\|u\| \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E$$

($a \lesssim b$ means that there exists a constant $C$, independent of \(\epsilon\) such that $a \leq Cb$). This implies

$$\|u - u_h\| \lesssim \|(u - u_h, \sigma - \sigma_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E$$

$$= \inf_{(u_h, \sigma_h, \hat{u}_h, \hat{q}_h)} \|(u - u_h, \sigma - \sigma_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E$$

Best Approximation Error (BAE)

$$\leq C(\epsilon)h^p$$
Step 2: Select a special test function...

\[ b((u, \sigma, \hat{u}, \hat{q}), (v, \tau)) = (\sigma, \frac{1}{\epsilon} \tau + \nabla v)_{\Omega_h} + (u, \text{div} \tau - \beta \cdot \nabla v)_{\Omega_h} \]

\[ -\langle \hat{u}, \tau_n \rangle_{\Gamma_h^0} - \langle \hat{q}, v \rangle_{\Gamma_h} \]

Choose a test function \((v, \tau)\) such that

\[
\begin{align*}
    v &\in H^1_0(\Omega), \quad \tau \in H(\text{div}, \Omega) \\
    \frac{1}{\epsilon} \tau + \nabla v &= 0 \\
    \text{div} \tau - \beta \cdot \nabla v &= u
\end{align*}
\]

Then

\[
\|u\|^2 = b((u, \sigma, \hat{u}, \hat{q}), (v, \tau)) = \frac{b((u, \sigma, \hat{u}, \hat{q}), (v, \tau))}{\| (v, \tau) \|_V} \| (v, \tau) \|_V \\
\leq \sup_{(v, \tau)} \frac{b((u, \sigma, \hat{u}, \hat{q}), (v, \tau))}{\| (v, \tau) \|_V} \| (v, \tau) \|_V = \| (u, \sigma, \hat{u}, \hat{q}) \|_E \| (v, \tau) \|_V
\]
Consequently, we need to select the test norm in such a way that

\[ \|(v, \tau)\|_V \lesssim \|u\| \]

This gives,

\[ \|u\|^2 \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E \cdot \|u\| \]

Dividing by \(\|u\|\), we get what we wanted.

**The point:** Construction of a robust DPG reduces to the classical stability analysis for the adjoint equation!
Step 3: Study the stability of the adjoint equation

Theorem (Generalization of Erickson-Johnson Theorem)

\[
\begin{align*}
\| \beta \cdot \nabla v \|_w, \sqrt{\epsilon} \| \nabla v \|_w, \| \text{div} \tau \|_{w+\epsilon}, \frac{1}{\epsilon} \| \beta \cdot \tau \|_w, \frac{1}{\sqrt{\epsilon}} \| \tau \|_w \end{align*}
\]

\[\lesssim \| u \|\]

where \( w = O(1) \) is a weight vanishing on the inflow boundary that satisfies some “mild” assumptions.

The terms on the left-hand side are our “Lego” blocks with which we can build different test norms.
Step 4: Construct test norm(s)

Graph norm:

\[ \|(v, \tau)\|_{graph}^2 := \|v\|^2 + \frac{1}{\epsilon} \|\tau + \nabla v\|^2 + \|\text{div}\tau - \beta \cdot \nabla v\|^2 \]

Mesh dependent weighted norm:

\[ \|(v, \tau)\|_w^2 := \min\left\{ \frac{\epsilon}{h^2}, 1 \right\} \|v\|^2 + \|\beta \cdot \nabla v\|_w^2 + \epsilon \|\nabla v\|^2 + \min\left\{ \frac{1}{\epsilon}, \frac{1}{h^2} \right\} \|\tau\|_{w+\epsilon}^2 + \|\text{div}\tau\|_{w+\epsilon}^2 \]

Remark: Both \(u\)-robust norms are also \(L^2\)-robust in \(\sigma\), as well as in traces and fluxes measured in minimum extension energy norms.
Pros and cons for both test norms

- The quasi-optimal test norm produces strong boundary layers that need to be resolved, also in 1D,

![Graph showing components of the test function](image)

Left: $\tau$ and $v$ components of the optimal test function corresponding to trial function $u = 1$ and element size $h = 0.25$, along with the optimal $hp$ subelement mesh. Right: $10 \times$ zoom on the left end of the element. Determining optimal test functions is expensive.

- The weighted test norms produce no boundary layers. Solving for the optimal test functions is inexpensive (done with enriched space $\Delta p = 2$).

- Quasi-optimal test norm yields better estimates for the best approximation error measured in the energy norm.
2D: Model problem of Erickson and Johnson

\[ \Omega = (0, 1)^2, \quad \beta = (1, 0), \quad f = 0, \quad u_0 = \begin{cases} \sin \pi y & \text{on } x = 0 \\ 0 & \text{otherwise} \end{cases} \]

The problem can be solved analytically using separation of variables.

Velocity \( u \) and “stresses” \( \sigma_x, \sigma_y \) (using scale for \( \sigma_y \)) for \( \epsilon = 0.01 \).
**hp-adaptivity:** $h_{\text{min}} = 2\epsilon, p_{\text{max}} = 5, w = x.$

$\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$. Left: convergence in energy error. Right: convergence in relative $L^2$-error for the field variables (in percent of their $L^2$-norm).
**hp-adaptivity:** \( h_{\text{min}} = 2\epsilon, p_{\text{max}} = 5, w = x. \)

\( \epsilon = 10^{-2}, 10^{-3}, 10^{-4}. \) Ratio of \( L^2 \) and energy norms.
2D: Example II, effect of value of $\Delta p$

**hp-adaptivity:** $h_{min} = 2\epsilon, p_{max} = 5, w = x$.  

2D model problem with a “discontinuous” inflow data, $\epsilon = 0.01$. Velocity $u$ and “stresses” $\sigma_x, \sigma_y$ (using scale for $\sigma_y$).
\( \Delta p = 2, 3 \). Left: convergence in energy error. Right: convergence in relative \( L^2 \)-error for the field variables (in percent of their \( L^2 \)-norm).
$\Delta \rho = 2, 3$. Ratio of $L^2$ and energy norms.
Good Boundary Conditions are Essential

For inflow boundary condition

$$\beta_n u - \sigma_n = u_0$$

and wall outflow boundary condition,

DPG delivers

$$\|u\| + \|\sigma\| \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E$$

using test norms without the weight, e.g.,

$$\|(v, \tau)\|^2 := \epsilon \|v\|^2 + \|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \|\tau\|^2 + \|\text{div} \tau\|^2$$

Mesh/pointwise error for $\epsilon = 1e - 2$.

Confusion Revisited

(a) Convergence rates

(b) $L^2$ and energy ratio
Extrapolation to Compressible Navier-Stokes Equations: Carter’s flat plate problem

\[ M_\infty = 3, \text{Re}_L = 1000, \text{Pr} = 0.72, \gamma = 1.4, \theta_\infty = 390^\circ[R] \]

Extrapolation to Compresible NS Equations

Initial Mesh ($\rho = 2$):

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 1:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 2:

Horizontal velocity and temperature
Mesh 3:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 4:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 5:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 7:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 8:

Horizontal velocity and temperature
Extrapolation to Compressible NS Equations

Mesh 9:

Horizontal velocity and temperature
Extrapolation to Compresible NS Equations

Mesh 10:

Horizontal velocity and temperature
Normal heat flux along the boundary $y=0$

Heat flux along the plate
Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Stokes and incompressible NS equations
Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Stokes and incompressible NS equations
- Elasticity, shells (volumetric, shear, membrane locking)
Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Stokes and incompressible NS equations
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials
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