NVFEM: a Galerkin method for (fully) nonlinear elliptic equations

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based on joint work
with
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Outline

Fully nonlinear elliptic PDE's

Definition and notation

Given a real-valued nonlinear function F of matrices

(FNFun) $$F : \operatorname{Sym}(\mathbb{R}^{d\times d}) \to \mathbb{R}.$$

Consider the equation

(FNE) $$\mathfrak{N}[u] := F(D^2 u) - f = 0 \text{ and } u|_{\partial\Omega} = 0$$

Conditional ellipticity condition, i.e.,

(NL-Ellip) $$\lambda(\boldsymbol{M}) \sup_{|\boldsymbol{\xi}|=1} |\mathbf{N}\boldsymbol{\xi}| \leq F(\boldsymbol{M} + \mathbf{N}) - F(\boldsymbol{M}) \leq \Lambda \sup_{|\boldsymbol{\xi}|=1} |\mathbf{N}\boldsymbol{\xi}|$$
$$\forall\, \boldsymbol{M} \in \mathfrak{C} \subseteq \operatorname{Sym}(\mathbb{R}^{d\times d}), \mathbf{N} \in \operatorname{Sym}(\mathbb{R}^{d\times d}).$$

for some ellipticity domain $\mathfrak{C}$ and "constants" $\lambda(\cdot), \Lambda > 0$.

Fully nonlinear elliptic PDE's

The ellipticity fauna

$$\mathfrak{N}[u] := F(D^2 u) - f = 0$$

- Conditionally elliptic

$$\exists\, \mathfrak{C} \subseteq \mathrm{Sym}\,(\mathbb{R}^{d\times d}), \lambda(\cdot), \Lambda > 0 :$$
$$\lambda(\boldsymbol{M}) \sup_{|\xi|=1} |\mathbf{N}\xi| \leq F(\boldsymbol{M}+\mathbf{N}) - F(\boldsymbol{M}) \leq \Lambda \sup_{|\xi|=1} |\mathbf{N}\xi|$$
$$\forall\, \boldsymbol{M} \in \mathfrak{C} \subseteq \mathrm{Sym}\,(\mathbb{R}^{d\times d}), \mathbf{N} \in \mathrm{Sym}\,(\mathbb{R}^{d\times d}).$$

- Unconditionally elliptic if $\mathfrak{C} = \mathrm{Sym}\,(\mathbb{R}^{d\times d})$.
- Uniformly elliptic $\inf \lambda > 0$.

Characterisation of the ellipticity condition

in the smooth case

Ellipticity condition, i.e.,

(NL-Ellip)
$$\lambda(\boldsymbol{M}) \sup_{|\boldsymbol{\xi}|=1} |\mathbf{N}\boldsymbol{\xi}| \leq F(\boldsymbol{M}+\mathbf{N}) - F(\boldsymbol{M}) \leq \Lambda \sup_{|\boldsymbol{\xi}|=1} |\mathbf{N}\boldsymbol{\xi}|$$
$$\forall\, \boldsymbol{M} \in \mathfrak{C} \subseteq \operatorname{Sym}(\mathbb{R}^{d\times d}), \mathbf{N} \in \operatorname{Sym}(\mathbb{R}^{d\times d}).$$

for some ellipticity "constants" $\lambda(\cdot), \Lambda > 0$. If F is differentiable then (NL-Ellip) is satisfied if and only if for each $\boldsymbol{M} \in \mathfrak{C}$ there exists $\mu > 0$ such that

(5.1)
$$\boldsymbol{\xi}^{\mathsf{T}} F'(\boldsymbol{M})\boldsymbol{\xi} \geq \mu\, |\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d.$$

Furthermore $\mathfrak{C} = \operatorname{Sym}(\mathbb{R}^{d\times d})$ and μ is independent of $\boldsymbol{M}$ if and only if F is uniformly elliptic.

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Boundary value problem

(MAD)
$$\begin{aligned} \det D^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

admits a unique solution in the cone of convex functions when $f > 0$.[Caffarelli and Cabré, 1995]
Derivative of nonlinear function $F(\mathbf{X}) = \det \mathbf{X}$ yields

$$F'(\mathbf{X}) = \operatorname{Cof} \mathbf{X}.$$

Problem elliptic if and only if

$$\boldsymbol{\xi}^{\intercal} \operatorname{Cof} D^2 u \, \boldsymbol{\xi} \geq \lambda |\boldsymbol{\xi}|^2 \qquad \forall \, \boldsymbol{\xi} \in \mathbb{R}^d$$

for some $\lambda > 0$.

Conotonic constraint

Restriction on unknown functions u: they must be **globally either convex or concave** (**conotonic**).

A simple fully nonlinear elliptic PDE

Consider problem

$$\mathfrak{N}[u] := \sin(\Delta u) + 2\Delta u - f = 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

Differentiating, we see that

$$\mathrm{D}\,\mathfrak{N}[v]w = (\cos(\Delta v) + 2)\,\mathbf{I}{:}\mathrm{D}^2 w = (\cos(\Delta v) + 2)\,\Delta w.$$

Hence problem uniformly elliptic.

A Krylov-type cubic elliptic PDE

The problem is for $d = 2$

$$
\text{(Krylov)} \qquad \begin{aligned} \mathfrak{N}[u] := (\partial_{11}u)^3 + (\partial_{22}u)^3 + \partial_{11}u + \partial_{22}u - f &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}
$$

Problem is uniformly elliptic since its differentiation gives:

$$
F'(\mathbf{X}) = \begin{bmatrix} 3x_{22}^2 + 1 & 0 \\ 0 & 3x_{11}^2 + 1 \end{bmatrix}.
$$

Pucci's equation

Consider $F : \mathrm{Sym}(\mathbb{R}^{d\times d}) \to \mathbb{R}$ to be the extremal function

(Pucci) $$F(\mathbf{N}) = \sum_{i=1}^{d} \alpha_i \lambda_i(\mathbf{N}) \text{ where } \lambda_i(\mathbf{N}) \text{ eigenvalues of } \mathbf{N}$$

for some given $\alpha_1, \dots, \alpha_d \in \mathbb{R}$.

Special case when $d = 2$, $\alpha_1 = \alpha \geq 1$ and $\alpha_2 = 1$ yields equation

($\mathbb{R}^2$ Pucci) $$0 = (\alpha + 1)\,\Delta u + (\alpha - 1)\left((\Delta u)^2 - 4 \det D^2 u\right)^{1/2}.$$

The problem is unconditionally elliptic.

Classes of fully nonlinear equations

A rough guide

See Caffarelli and Cabré, 1995 for a more systematic classification.

- Isaacs form: $\inf_\beta \sup_\alpha L_{\alpha\beta} u = 0$.
  - Bellman type: Isaacs with only one β ($\Leftrightarrow$ no inf). Related to Hamilton–Jacobi–Bellman, stochastic control and differential game theory.
    - Isaacs form is very general: "non-algebraic" and harder to treat numerically. We don't, yet.[Jensen and Smears, 2012; Lio and Ley, 2010, e.g.].
- Hessian invariants (algebraic): Monge–Ampère, Pucci, Laplace (!). Subdivided into unconeditionally elliptic (Pucci, Laplace) and coneditionally elliptic (Monge–Ampère).
- Other algebraic FNE's (Krylov, algebraic nonlinearities, etc.)
- Aronson equations and infinite-harmonic functions, nicely reviewed in Barron, Evans, and Jensen, 2008. (These aren't proper FNE's, as they are quasilinear, nevertheless, Hessian recovery applies well.)

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- consider densities f and $g \geq 0$
- supports $\operatorname{spt} f =: \Omega$ and $\operatorname{spt} g =: \Upsilon$ convex
- $f, g > 0$ on $\operatorname{int} \Omega$ and $\operatorname{int} \Upsilon$.

Look for $\psi : \Omega \to \Upsilon$ that transports the mass density f into the mass density g.
Mass conservation:

$$\int_A f(\boldsymbol{x}) \, \mathrm{d}\, \boldsymbol{x} = \int_{\boldsymbol{\psi}(A)} g(\boldsymbol{y}) \, \mathrm{d}\, \boldsymbol{y} \quad \forall\, A \text{ (Borel) } \subseteq \Omega. \tag{11.1}$$

Then

$$\int_{\boldsymbol{\psi}(A)} g(\boldsymbol{y}) \, \mathrm{d}\, \boldsymbol{y} = \int_A g(\boldsymbol{\psi}(\boldsymbol{x})) \, |\det \mathrm{D}\, \psi(\boldsymbol{x})| \, \mathrm{d}\, \boldsymbol{x} \quad \forall\, A \text{ (Borel) } \subseteq \Omega. \tag{11.2}$$

Hence (assuming $\det \mathrm{D}\, \psi > 0$)

$$g(\boldsymbol{\psi}(\boldsymbol{x})) \det \mathrm{D}\, \psi(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \forall\, \boldsymbol{x} \in \Omega. \tag{11.3}$$

Following Caffarelli, 1990a; Caffarelli, 1990b,c; Caffarelli and Cabré, 1995 Evans, 2001 Urbas, 1997 under convexity and regularity assumptions, the Monge-Ampere equation

$$\det \mathrm{D}^2 u(\boldsymbol{x}) = k(\boldsymbol{x}, u(\boldsymbol{x}), \nabla u(\boldsymbol{x}))$$

coupled to the second boundary condition second boundary condition

$$\nabla u(\Omega) = \Upsilon, \tag{12.1}$$

provides a solution to the Monge problem and the right-hand side

$$\frac{f(x)}{g(\nabla u(\boldsymbol{x}))} \tag{12.2}$$

Finite difference approaches

1. Earliest known provided approximations of the Monge–Ampère (and other equations) by Oliker and Prussner, 1988.
2. Kuo and Trudinger, 1992 gave mostly theoretical work introduced the concept of wide stencils and proving convergence for wide enough stencils.
3. Benamou and Brenier, 2000 proposed an approach based on the Brenier-solution concept related to fluid-dynamics and mass-trasportation.
4. Oberman, 2008 introduced more practically effective work working out the details, proiding a bound on the wide stencil's width. See also Froese, 2011 and Benamou, Froese, and Oberman, 2012 for second boundary conditions.

Galerkin (mainly finite element) methods I

- Dean and Glowinski, 2006 (and earlier work) introduced a FE least square method to solve Monge–Ampère equation.
- Böhmer, 2010 (and earlier papers) advocates (mostly theoretically, proving convergence) the use of C^1/spline finite elements to directly compute the Hessian. Practical schemes have been constructed recently by Davydov and Saeed, 2013.
- Feng and Neilan, 2009 introduce the vanishing moment method a fourth order semilinear approximation: $\epsilon\Delta^2 u + F[u] = 0$ and take $\epsilon \to 0$. (Similar to early vanishing viscosity methods for first order PDE's.)
- More recently, related work by Brenner et al., 2011 introduces a penalty method to deal with the convexity.
- Neilan, 2012 considers a generalization of scheme in Lakkis and Pryer, 2011 and proves convergence rates for MAD in 2d.

Galerkin (mainly finite element) methods II

- Awanou, 2011 uses a pseudo time [sic] approach.
- Jensen and Smears, 2012 provide and analyze a FEM for a special class of Hamilton–Jacobi–Bellman equation. Further work in Smears and Süli, 2013, 2014 for a DGFEM approach.

A fixed-point solution

Nonlinear PDE

$$\mathfrak{N}[u] := F(D^2 u) - f = 0$$

can be rewritten as follows

$$\mathfrak{N}[u] = \left[\int_0^1 F'(t\, D^2 u)\, \mathrm{d}\, t\right] : D^2 u + F(\mathbf{0}) - f = 0.$$

Define

$$\mathbf{N}(D^2 u) := \int_0^1 F'(t\, D^2 u)\, \mathrm{d}\, t,$$
$$g := f - F(\mathbf{0}),$$

then if u solves (FNE), it also solves

$$\mathbf{N}(D^2 u) : D^2 u = g.$$

Fixed point iteration: given u^0 find

$$\mathbf{N}(D^2 u^n) : D^2 u^{n+1} = g, \text{ for } n = 1, 2, \dots .$$

Crucial remark

Note that solving

$$\mathbf{N}(\mathrm{D}^2 u^n):\mathrm{D}^2 u^{n+1} = g$$

involves a linear elliptic equation in non-divergence form.

Big fat note

Standard variational FEM's do not apply.

Newton's method

Given an initial guess u^0, let

$$\mathrm{D}\,\mathfrak{N}[u^n]\left(u^{n+1}-u^n\right)=-\mathfrak{N}[u^n],\ \text{for } n=0,1,2,\ldots,$$

where

$$\mathrm{D}\,\mathfrak{N}[u]v=F'(\mathrm{D}^2 u):\mathrm{D}^2 v.$$

I.e.,

$$F'(\mathrm{D}^2 u^n):\mathrm{D}^2\left(u^{n+1}-u^n\right)=f-F(\mathrm{D}^2 u^n).$$

Big fat note (repeated)

Equation in nondivergence form, standard FEM's will not apply.

The need for Hessian recovery

Detailed in Lakkis and Pryer, 2013

Fixed point iteration

$$\mathbf{N}(\mathrm{D}^2 u^n) : \mathrm{D}^2 u^{n+1} = g$$

and Newton's iteration

$$F'(\mathrm{D}^2 u^n) : \mathrm{D}^2 \left(u^{n+1} - u^n \right) = f - F(\mathrm{D}^2 u^n).$$

besides being nonvariational, like fixed-point, requires the suitable approximation of a Hessian's function.

Big fat note (a variation)

Hence the use of the **recovered Hessian** introduced by Lakkis and Pryer, 2011.

Introduce Galerkin finite element spaces

$$\mathbb{V}_h := \left\{ \Phi \in \mathrm{H}^1(\Omega) : \; \Phi|_K \in \mathbb{P}^p \; \forall\, K \in \mathfrak{T} \text{ and } \Phi \in \mathrm{C}^0(\Omega) \right\},$$
$$\mathbb{V}_0 := \mathbb{V} \cap \mathrm{H}^1_0(\Omega),$$

Unbalanced mixed problem:

Find $(U, \mathbf{H}) \in \mathbb{V}_0 \times \mathbb{V}^{d \times d}$ satisfying

$$\langle \mathbf{H}, \Phi \rangle + \int_\Omega \nabla U \otimes \nabla \Phi - \int_{\partial\Omega} \nabla U \otimes \mathbf{n}\, \Phi = 0$$
$$\langle \mathbf{A}{:}\mathbf{H}, \Psi \rangle = \langle f, \Psi \rangle \qquad \forall\, (\Phi, \Psi) \in \mathbb{V} \times \mathbb{V}_0.$$

A (sometimes) commutative diagram

discretization are often possible (e.g., when the nonlinearity is algebraic in the Hessian):

$$
\begin{tikzcd}
\text{(exact) nonlinear PDE} \arrow[r, "\text{Newton}"] \arrow[d, "\text{FNFEM}"] & \text{(exact) nonvariational linear PDE's} \arrow[d, "\text{NVFEM}"] \\
\text{FE fully nonlinear PDE} \arrow[r, "\text{Newton}"] & \begin{matrix}\text{discrete linear 1}\\ \text{discrete linear 2}\end{matrix}
\end{tikzcd}
$$

A priori estimates for the error

$$\left\|\mathbf{A} : (D^2 u - \mathbf{H}[u_h])\right\|_{H^{-1}(\Omega)}.$$

A posteriori error estimate for the error

$$\|u - u_h\|_{L_2(\Omega)}^2 \leq \sum_{K \in \mathfrak{T}} \left(h_K^2 \|f - \mathbf{A}{:}D^2 U\|_{L_2(K)}^2 + h_K \|\mathbf{A}{:}[\![\nabla U \otimes]\!]\|_{L_2(\partial K)}^2 \right)$$

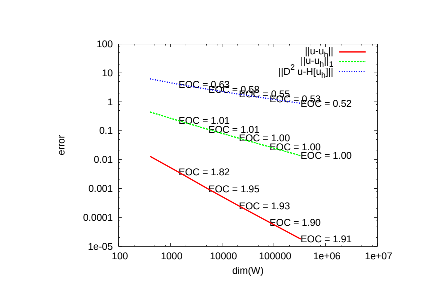
where the tensor jump of a field $\boldsymbol{v}$ across an edge $E = \overline{K} \cap \overline{K'}$ is given by

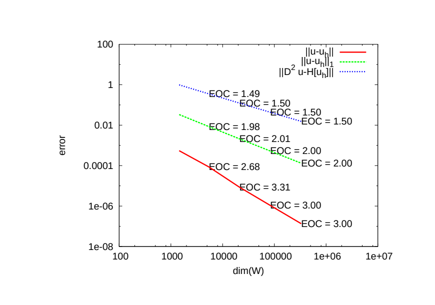
$$[\![\boldsymbol{v} \otimes]\!]_E := \lim_{\epsilon \to 0} \left(\boldsymbol{v}(x + \epsilon \mathbf{n}_K) \otimes \mathbf{n}_K + \boldsymbol{v}(x - \epsilon \mathbf{n}_{K'}) \otimes \mathbf{n}_{K'} \right)$$

A nonlinear function of Δu

$$\mathfrak{N}[u] := \sin(\Delta u) + 2\Delta u - f = 0 \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

P1 elements (left) and P2 elements (right)

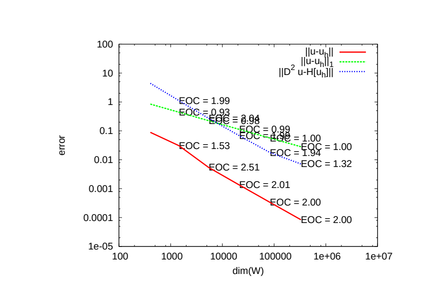


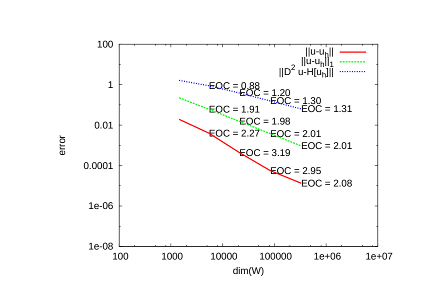


Krylov's equation

$$\mathfrak{N}[u] := (\partial_{11}u)^3 + (\partial_{22}u)^3 + \partial_{11}u + \partial_{22}u - f = 0 \quad \text{in } \Omega \qquad (24.1)$$
$$u = 0 \quad \text{on } \partial\Omega.$$

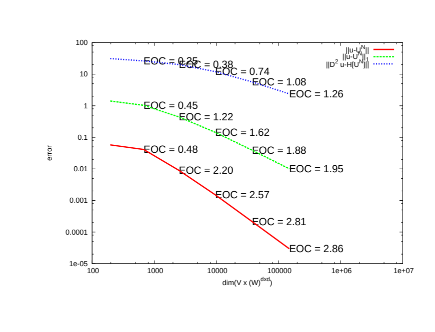
P1 elements (left) and P2 elements (right)

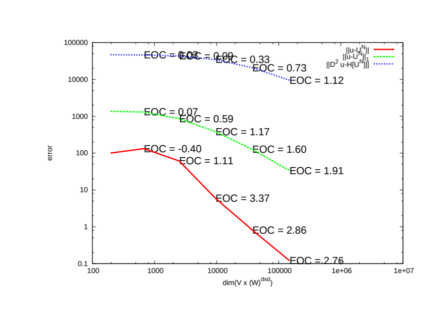




$$0 = (\alpha + 1)\,\Delta u + (\alpha - 1)\left((\Delta u)^2 - 4\det \mathrm{D}^2 u\right)^{1/2}.$$

$\mathbb{P}^2, \alpha = 2$ (left) and $\mathbb{P}^2, \alpha = 5$ (right)



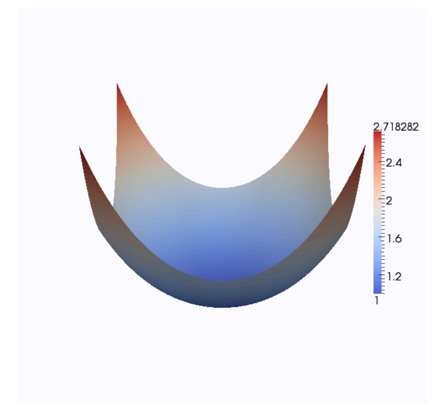


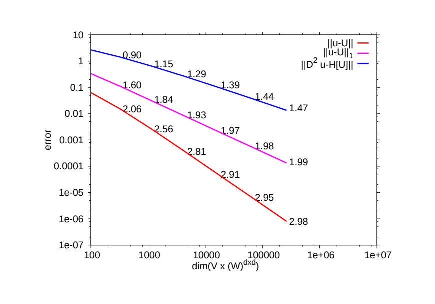
MAD stuff I

reminder: MAD = Monge–Ampère–Dirichlet

FE-convexity check inspired from Aguilera and Morin, 2009.

Exact solution and EOC's for $\mathbb{P}^2$ elements (suboptimal for $\mathbb{P}^1$)

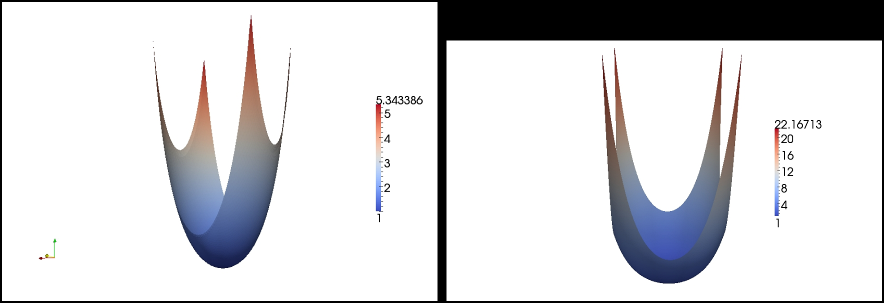




MAD stuff II

reminder: MAD = Monge–Ampère–Dirichlet

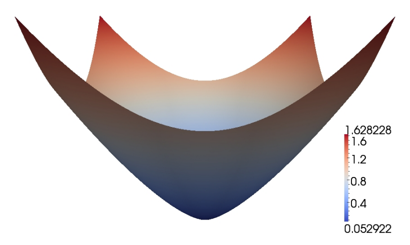
principal minor and determinant instances

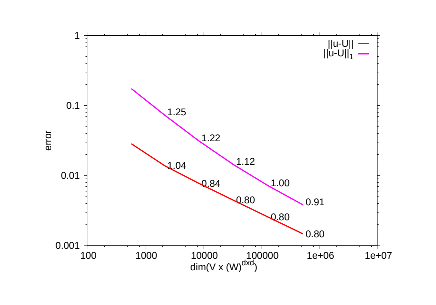


Nonclassical solutions

Viscosity or Alexandrov[Evans, 2001]

Singular solution $u(\boldsymbol{x}) = |\boldsymbol{x}|^{2\alpha}$

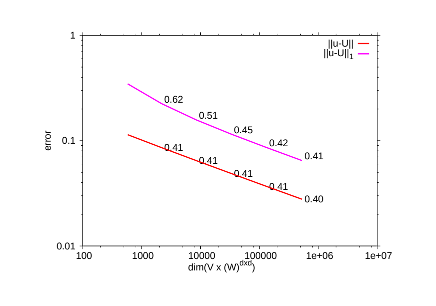


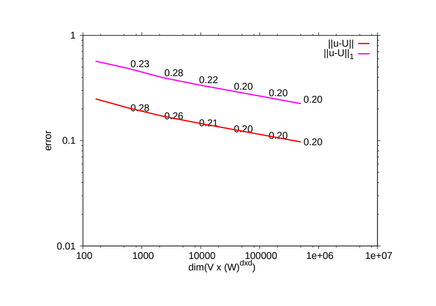


Nonclassical solutions

Viscosity or Alexandrov[Evans, 2001]

More singular, $\alpha = 0.6$, $\alpha = 0.55, \ldots$





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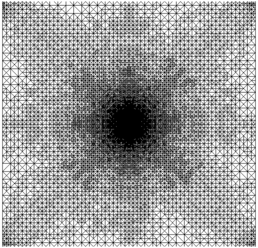
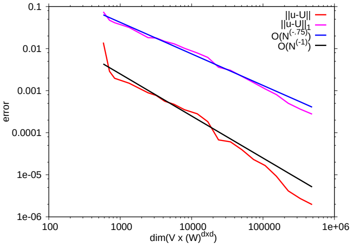
Singular solution $u(\boldsymbol{x}) = |\boldsymbol{x}|^{1.1}$ (empirical ZZ-estimators)





Conclusions and outlook

- Obtained and tested a practical and "easy" Netwon scheme based on nonvariational FEM (NVFEM) via Hessian recovery.
- Convergence rates optimal in all examples.
- A posteriori error esimates for very weak norms in the linear problem, provide an elementary way to do adaptivity.
- In progress: prove apriori convergence for stronger norms in linear problems.
- In progress: embed second boundary condition $(\nabla u(\Omega) = \Upsilon$ with prescribed $\Upsilon)$. (This was achieved for wide-stencils but on structured grids by Benamou, Froese, and Oberman, 2012.)
- Open problem: prove conservation of conotonicity for MAD/MAS.
- Open problem: apriori and aposteriori analysis for nonlinear problem.

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