DPG Strategies for the Helmholtz Equation

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Outline

Preliminaries

The 1D experience

The 2D and multidimensional experience

The ε-scaling approach

Dispersion analysis for the lowest order method

Conclusions

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Dream: to have a method that delivers the L^2-projection.

Theoretical ingredients

$$u \in (U, \|\cdot\|_U) \quad \text{s.t.} \quad b(u,v) = f(v), \quad \forall v \in V. \tag{1}$$

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- The test space V must be a *broken* Sobolev space.
- We numerically approach (2) using a discrete space $V_r \subset V$.

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DPG w/optimal test functions implies:

$$\|u - u_h\|_U \leq \|\Pi\| \frac{M}{\gamma} \inf_{w_h \in U_h} \|u - w_h\|_U$$

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Functional Spaces

$$\begin{cases} (u,p,\hat{u},\hat{p}) \in U := L^2(\Omega) \times L^2(\Omega) \times \mathbb{C}^n \times \mathbb{C}^n \\ (v,\eta) \in V := [H^1(\Omega_h)]^2 \end{cases}$$

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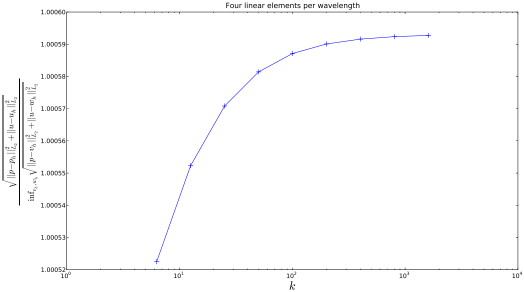
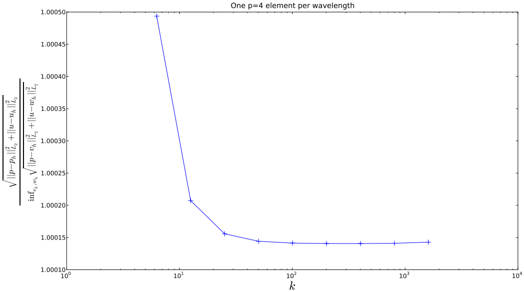
Then (3) holds with wavenumber independent $M > 0$ and $\gamma > 0$. Moreover

$$\|u-u_h\|^2_{0,\Omega}+\|p-p_h\|^2_{0,\Omega}+\|\hat{u}-\hat{u}_h\|^2_2+\|\hat{p}-\hat{p}_h\|^2_2 \le \frac{M^2}{\gamma^2}\left(\inf_{w_h}\|u-w_h\|^2_{0,\Omega} + \inf_{q_h}\|p-q_h\|^2_{0,\Omega}\right)$$

1D Numerical experiments





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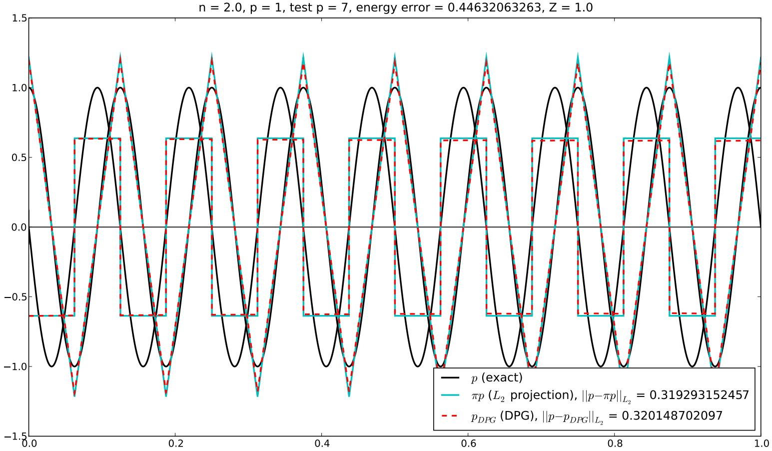

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We define the "*wave*" operator $A : H(\operatorname{div}, \Omega) \times H^1(\Omega) \to L^2(\Omega)^n \times L^2(\Omega)$ s.t.

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Functional Spaces $\begin{cases} U := L^2(\Omega)^N \times L^2(\Omega) \times \underbrace{\mathrm{Tr}_{\partial\Omega_h}\Big(H(\operatorname{div}, \Omega) \times H^1(\Omega) + \text{ B.C.}\Big)}_{=:Q} \\ V := H(\operatorname{div}, \Omega_h) \times H^1(\Omega_h), \end{cases}$

The stability result

$$\|\hat{w}_n, \hat{q}\|_Q = \inf_{\mathrm{Tr}_{\partial\Omega_h}(\mathbf{z},\varphi)=(\hat{w}_n,\hat{q})} \|A(\mathbf{z},\varphi)\|_{0,\Omega}$$

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Theorem: There are constants $M > 0$ and $\gamma > 0$, independent of wavenumbers $k > k_0$, s.t.

$$\gamma\|(\mathbf{v},\eta)\|_V \leq \|(\mathbf{v},\eta)\|_{\text{opt}} \leq M\|(\mathbf{v},\eta)\|_V$$

The stability result

$$\|\hat{w}_n, \hat{q}\|_Q = \inf_{\mathrm{Tr}_{\partial\Omega_h}(\mathbf{z},\varphi)=(\hat{w}_n,\hat{q})} \|A(\mathbf{z},\varphi)\|_{0,\Omega}$$

Opt test norm $$\|(\mathbf{v},\eta)\|_{\mathrm{opt}}^2 = \|A_h(\mathbf{v},\eta)\|_{0,\Omega}^2 + \left(\sup_{(\hat{w}_n,\hat{q})\in Q} \frac{\left\langle (\hat{w}_n,\hat{p}),(\eta,\mathbf{v}\cdot\mathbf{n}) \right\rangle_{\partial\Omega_h}}{\|(\hat{w}_n,\hat{p})\|_Q} \right)^2 .$$

So we choose the V-norm as: $$\|(\mathbf{v},\eta)\|_V^2 = \|A_h(\mathbf{v},\eta)\|_{0,\Omega}^2 + \|(\mathbf{v},\eta)\|_{0,\Omega}^2.$$

Theorem: There are constants $M > 0$ and $\gamma > 0$, independent of wavenumbers $k > k_0$, s.t.

$$\gamma\|(\mathbf{v},\eta)\|_V \leq \|(\mathbf{v},\eta)\|_{\mathrm{opt}} \leq M\|(\mathbf{v},\eta)\|_V$$

Moreover,

$$\|(\mathbf{u}-\mathbf{u}_h,\phi-\phi_h)\|_{0,\Omega}^2 + \|(\hat{u}-\hat{u}_h,\hat{\phi}-\hat{\phi}_h)\|_Q^2$$
$$\leq \frac{M^2}{\gamma^2}\left(\inf_{(\mathbf{w}_h,q_h)} \|(\mathbf{u}-\mathbf{w}_h,\phi-q_h)\|_{0,\Omega}^2 + \inf_{(\hat{w}_h,\hat{q}_h)} \|(\hat{u}-\hat{w}_h,\hat{\phi}-\hat{q}_h)\|_Q^2 \right)$$

Error estimation

Conforming p-optimal H^1-interpolant (Demkowicz,Gopalakrishnan,Schöberl)

$$\|\psi - \Pi_{hp}\psi\|_{0,\Omega} + h\|\nabla(\psi - \Pi_{hp}\psi)\|_{0,\Omega} \leq C\frac{\ln(\tilde{p})^2}{\tilde{p}^s} h^{s+1}|\psi|_{H^{s+1}(\Omega)}, \qquad s+1 \in (\frac{3}{2}, p+1]$$

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Error estimation (for traces & fluxes of globally continuous polynomials of degree $p+1$)

$$\inf_{(\hat{w}_h,\hat{q}_h)} \|(\hat{u}_n - \hat{w}_h, \hat{\phi} - \hat{q}_h)\|_Q$$

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$$\inf_{(\hat{w}_h, \hat{q}_h)} \|(\hat{u}_n - \hat{w}_h, \hat{\phi} - \hat{q}_h)\|_Q \leq \quad \|A(\mathbf{u} - \Pi_{hp}\mathbf{u}, \phi - \Pi_{hp}\phi)\|_{0,\Omega}$$

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$$\begin{aligned} \inf_{(\hat{w}_h,\hat{q}_h)} \|(\hat{u}_n - \hat{w}_h, \hat{\phi} - \hat{q}_h)\|_Q \leq & \quad \|A(\mathbf{u} - \Pi_{hp}\mathbf{u}, \phi - \Pi_{hp}\phi)\|_{0,\Omega} \\ \lesssim & \quad \|ik(\mathbf{u} - \Pi_{hp}\mathbf{u}) + \nabla(\phi - \Pi_{hp}\phi)\|_{0,\Omega} \end{aligned}$$

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\end{aligned}$$

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\end{aligned}$$

where

$$\|\psi\|^2_{s+1,k,\Omega}=\sum_{j=0}^{s+1}k^{2(s+1-j)}|\psi|^2_{H^j(\Omega)}, \qquad \forall s=1,\dots,p$$

The ε-scaling approach

$$\|(\mathbf{v},\eta)\|_{V,\varepsilon} := \|A_h(\mathbf{v},\eta)\|_{0,\Omega}^2 + \varepsilon^2\|(\mathbf{v},\eta)\|_{0,\Omega}^2$$

What happens in the *eyeball norm* ?

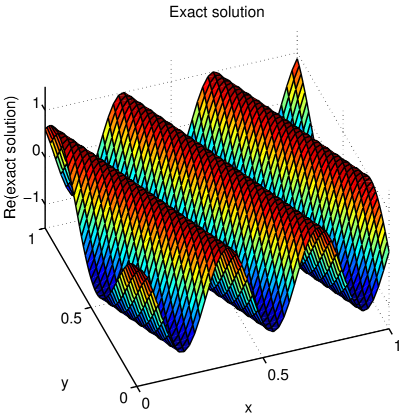

Figure : Numerical traces of a plane wave propagating at angle $\pi/8$

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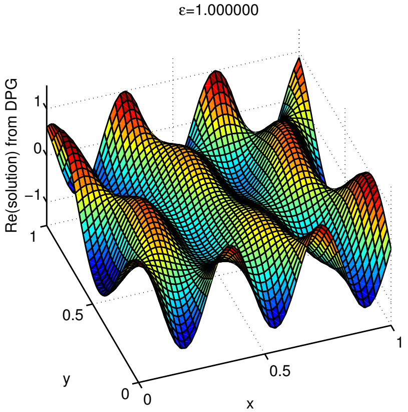

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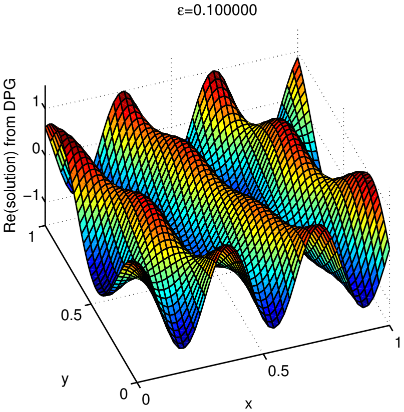

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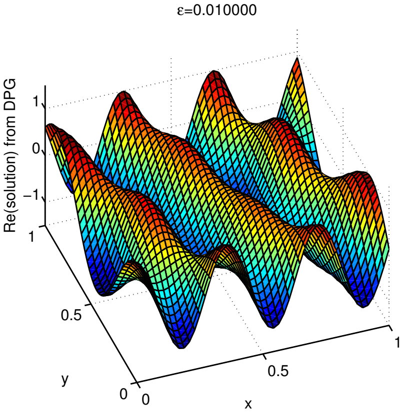

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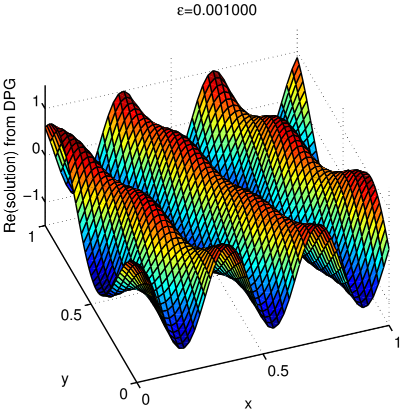

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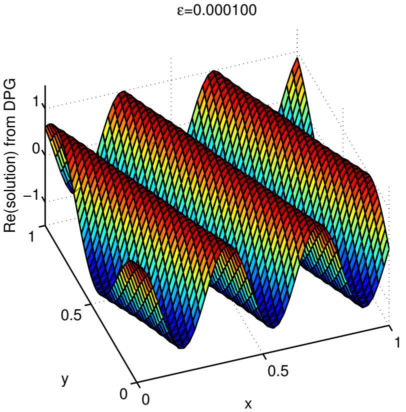

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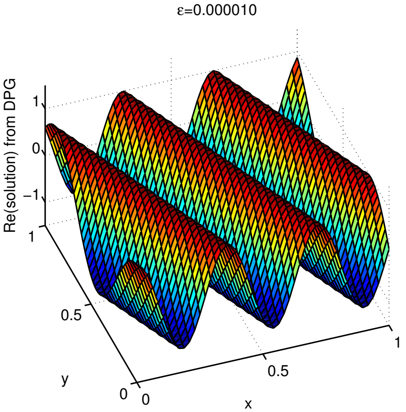

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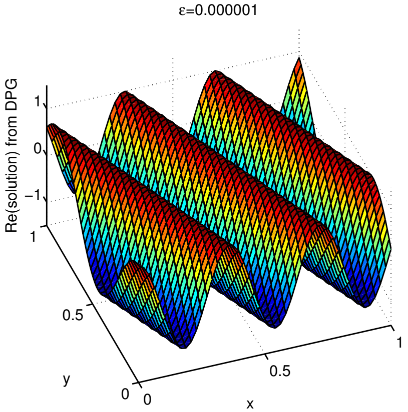

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The ε-scaling approach

Theorem
Let $(\hat{u}_h^\varepsilon, \hat{\phi}_h^\varepsilon)$ be the discrete DPG solution of fluxes and traces using the ε-scaling approach. If $\varepsilon \to 0^+$, then

$$\|(\hat{u}, \hat{\phi}) - (\hat{u}_h^\varepsilon, \phi_h^\varepsilon)\|_Q \longrightarrow \inf_{(\hat{w}_h, \hat{q}_h)} \|(\hat{u}, \hat{\phi}) - (\hat{w}_h, \hat{q}_h)\|_Q$$

Dispersion of the lowest order method

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Recall that

$$U := L^2(\Omega)^N \times L^2(\Omega) \times \underbrace{\mathrm{Tr}_{\partial\Omega_h}\Big(H(\,\mathrm{div}\,,\Omega) \times H^1(\Omega) + \ \mathrm{B.C.}\Big)}_{=:Q}$$

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Hence, the lowest order choice is:

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Hence, the lowest order choice is:

- Piecewise constants for field variable $\mathbf{u}$ (on each element K).

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- Piecewise constants for fluxes $\hat{u}$ (on each edge of ∂K).

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- Piecewise linear (on each edge of ∂K) and globally continuous for traces $\hat{\phi}$.

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- Piecewise constants for fluxes $\hat{u}$ (on each edge of ∂K).
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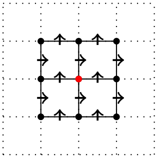
For the numerical results that will be shown later, the enriched space approaching $V = H(\,\mathrm{div}\,,\Omega_h) \times H^1(\Omega_h)$ for the computation of optimal test functions is

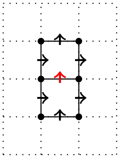
$$V^r = \Big\{(\mathbf{v},\eta) : (\mathbf{v},\eta)\big|_K \in (\mathcal{Q}_{r,r-1} \times \mathcal{Q}_{r-1,r}) \times \mathcal{Q}_{r,r}\Big\}, \qquad \text{where } r \geq 2.$$

Dispersion Analysis

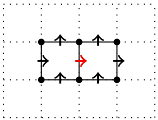
(Discontinuous field variables are condensed out)



(a) 21-point stencil



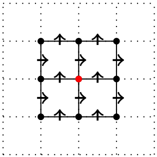
(b) 13-point stencil



(c) 13-point stencil

Dispersion Analysis

(Discontinuous field variables are condensed out)



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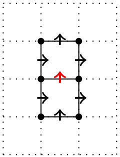

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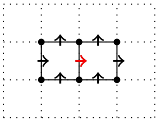
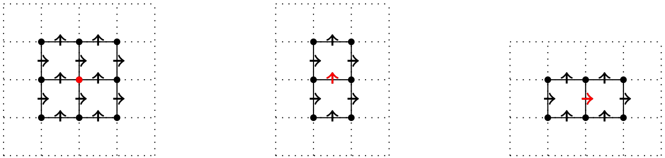


(c) 13-point stencil

- Plane waves $Ae^{k(x_1 \cos\theta + x_2 \sin\theta)}$ are exact solutions with zero sources.

Dispersion Analysis

(Discontinuous field variables are condensed out)



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- Plane waves $Ae^{k(x_1 \cos\theta + x_2 \sin\theta)}$ are exact solutions with zero sources.
- We work with the assumption that the discrete solution is interpolating a plane wave of the type

$$\hat{p}(\vec{x}) = \alpha e^{i\vec{k}_h\cdot\vec{x}}, \quad \widehat{u}_{nh}(\vec{x}) = \beta e^{i\vec{k}_h\cdot\vec{x}}, \quad \widehat{u}_{nv}(\vec{x}) = \gamma e^{i\vec{k}_h\cdot\vec{x}}.$$

Dispersion Analysis

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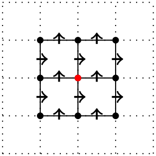

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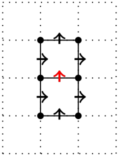

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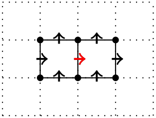

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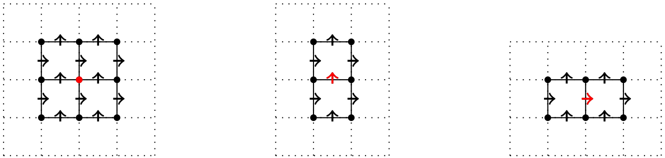
where $\vec{k}_h = k_h(\cos(\theta), \sin(\theta))$ for some $0 \le \theta < 2\pi$ representing the direction of propagation and α, β, γ are unknown amplitudes.

Dispersion Analysis

(Discontinuous field variables are condensed out)



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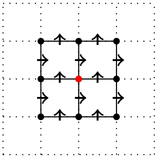
Dispersion Analysis

(Discontinuous field variables are condensed out)



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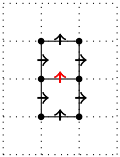

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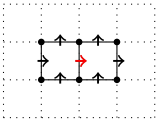

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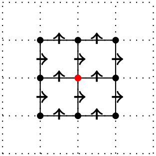

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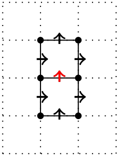

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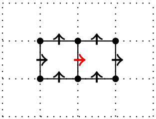

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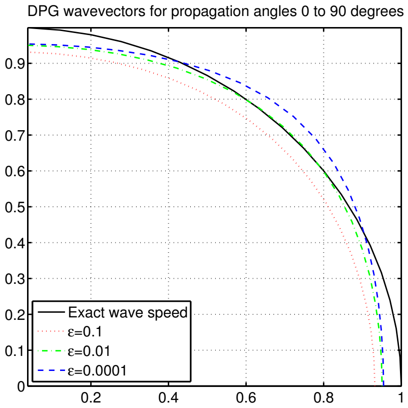
- Plane waves $Ae^{k(x_1 \cos\theta + x_2 \sin\theta)}$ are exact solutions with zero sources.
- We work with the assumption that the discrete solution is interpolating a plane wave of the type

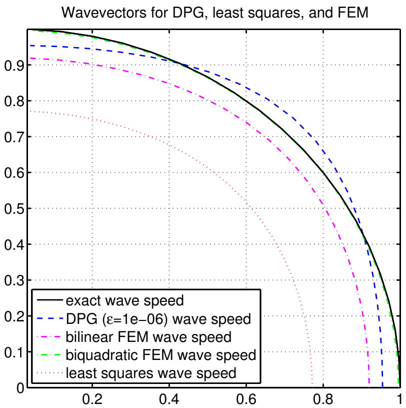
$$\hat{p}(\vec{x}) = \alpha e^{i\vec{k}_h\cdot\vec{x}}, \quad \widehat{u}_{nh}(\vec{x}) = \beta e^{i\vec{k}_h\cdot\vec{x}}, \quad \widehat{u}_{nv}(\vec{x}) = \gamma e^{i\vec{k}_h\cdot\vec{x}}.$$

  where $\vec{k}_h = k_h(\cos(\theta), \sin(\theta))$ for some $0 \leq \theta < 2\pi$ representing the direction of propagation and α, β, γ are unknown amplitudes.
- We want to compute k_h as a function of the exact wavenumber k, the direction of propagation θ and some of the discretization and stabilization parameters (kh, r and ε).

Numerical results: dependence on θ



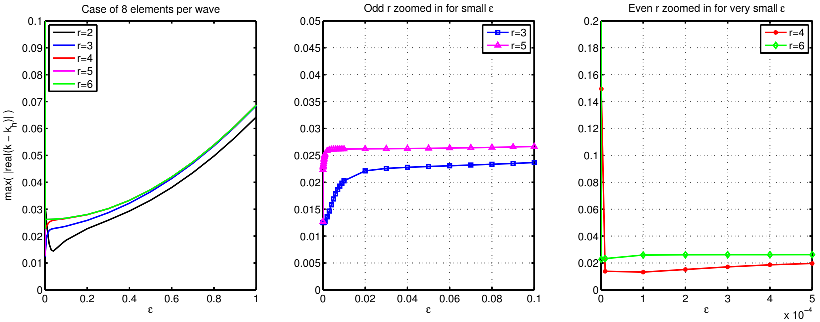
(a) Dependence of $\Re(\vec{k}_h)$ on ε



(b) Comparison of methods

Figure : The curves traced out by the discrete wavevectors $\vec{k}_h$ as θ goes from 0 to $\pi/2$. These plots were obtained using $k = 1$ and $h = 2\pi/4$.

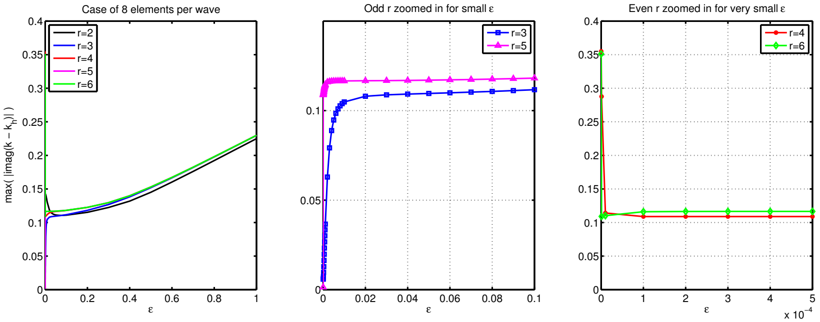
Numerical results: dispersive errors $\rho = \max_{\theta} |\mathfrak{Re}(k_h) - k|$



(a) Dispersive errors: Plots of ρ vs. ε

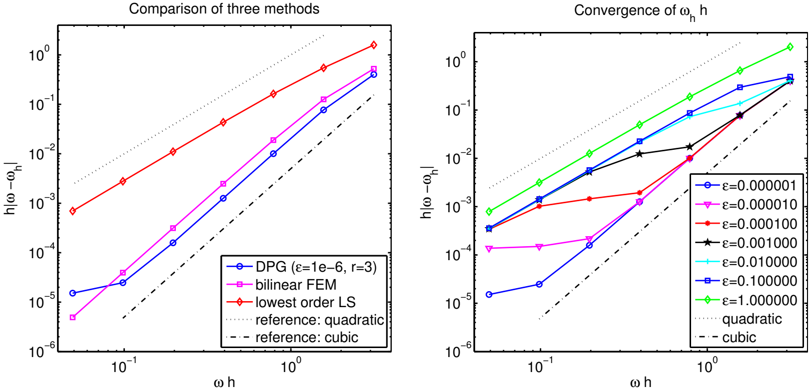
Figure : The discrepancies between exact and discrete wavenumbers as a function of ε, when $k = 1$ and $h = 2\pi/8$.

Numerical results: dissipative errors $\eta = \max_{\theta} |\Im m(k_h)|$



(a) Dissipative errors: Plots of η vs. ε

Figure : The discrepancies between exact and discrete wavenumbers as a function of ε, when $k = 1$ and $h = 2\pi/8$.

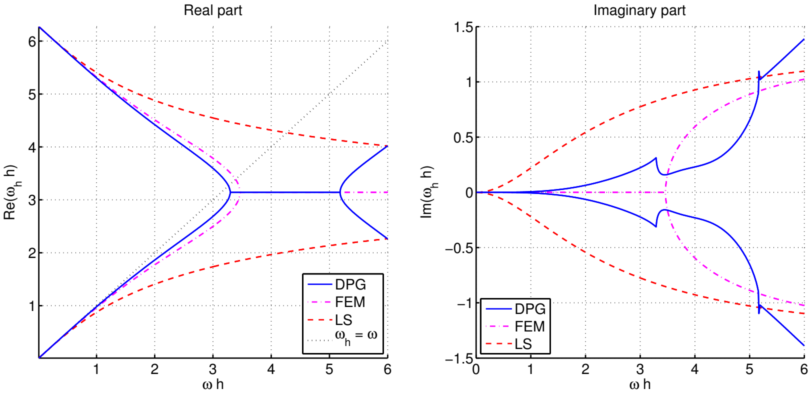


(a) Plot of $|k_h h - kh|$ for three methods

(b) Case of DPG with $r = 3$ and various ε

Figure : Rates of convergence of $|k_h h - kh|$ to zero for small kh, in the case of propagation angle $\theta = 0$.

Observe that $|k_h h - kh| = O(kh)^{\alpha+1}$ means $|k_h - k| = kO(kh)^{\alpha}$.



(a) $\Re e(k_h h)$ as a function of kh

(b) $\Im m(k_h h)$ as a function of kh

Figure : A comparison of discrete wavenumbers obtained by three lowest order methods in the case of propagation angle $\theta = 0$.

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- DPG is a Least-Squares method, so it has a Hermitian Positive Definite stiffness matrix.
- In order to be competitive the future approaches must explore *hp* adaptivity, solvers and/or plane waves.