Pyramidal finite elements

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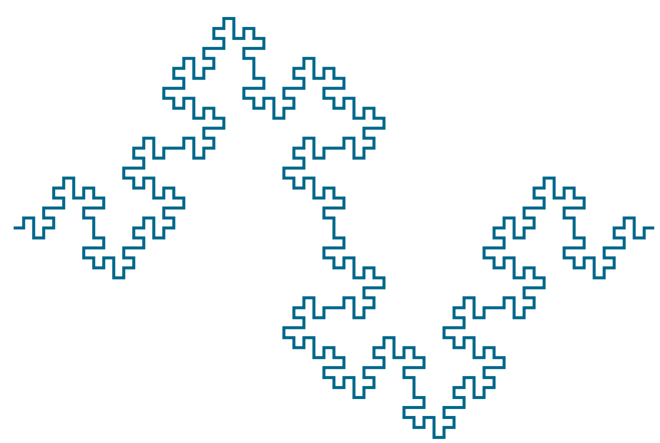


Outline

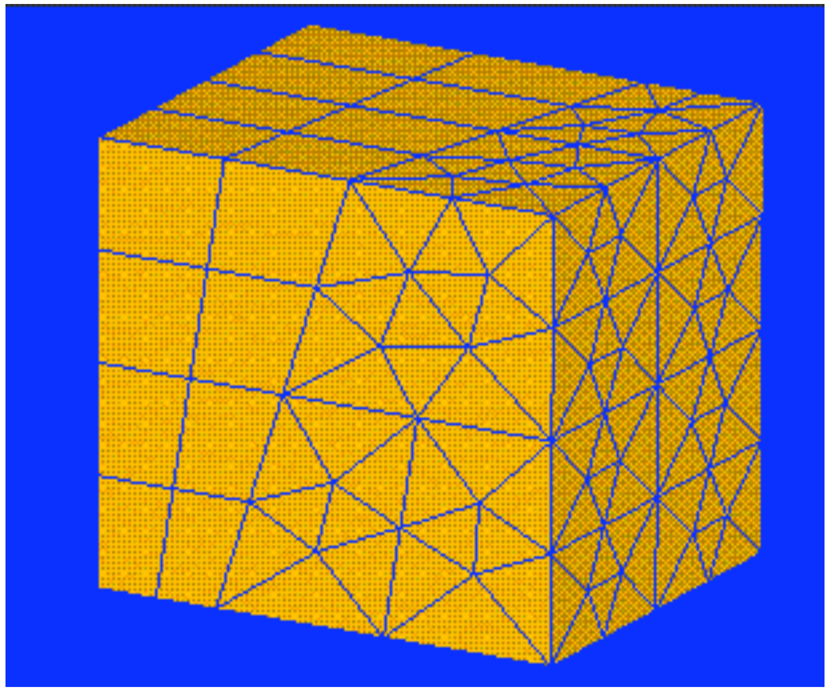
High order conforming elements on pyramids

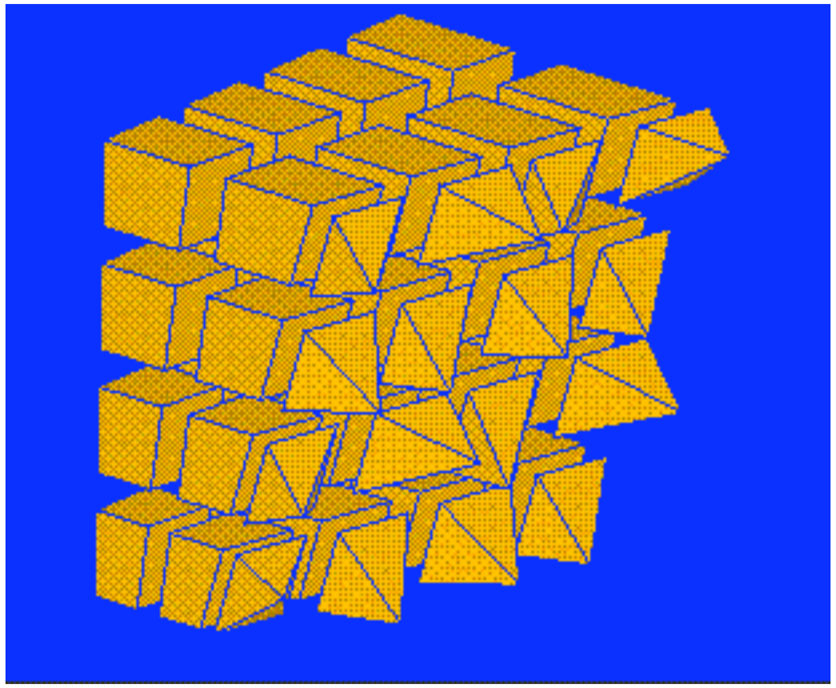
Quadrature

Numerical experiments

Why?

- Arise naturally as 'glueing' elements in hybrid meshes
- Because the design and proofs presented interesting challenges.





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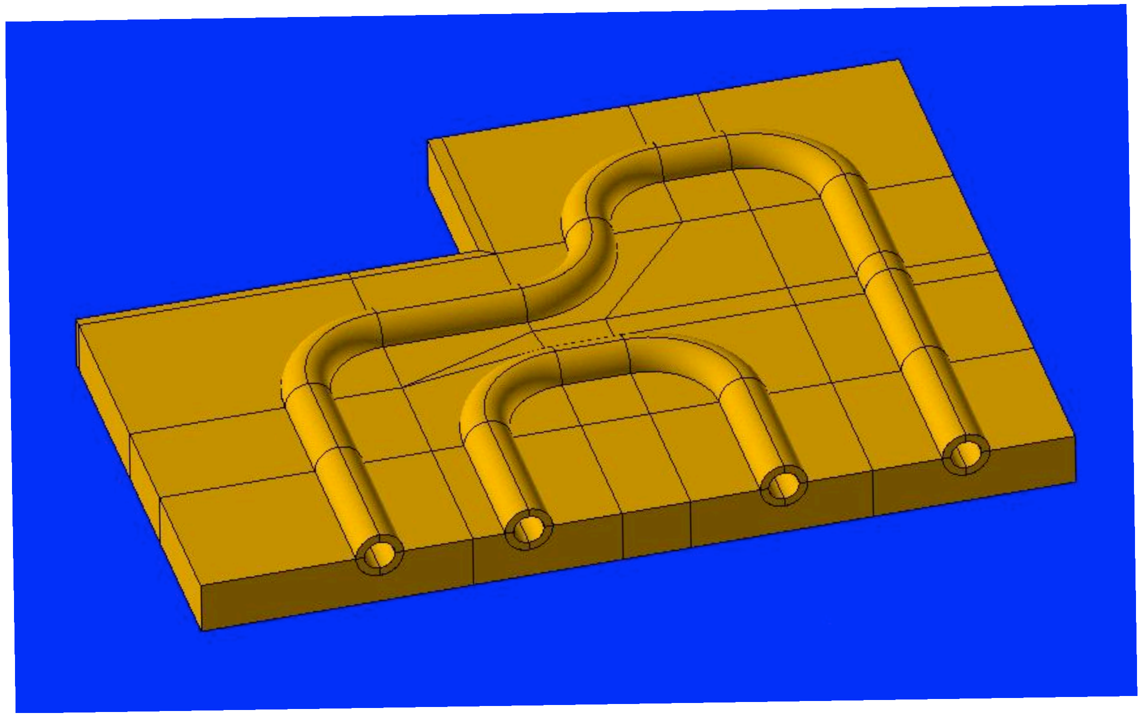
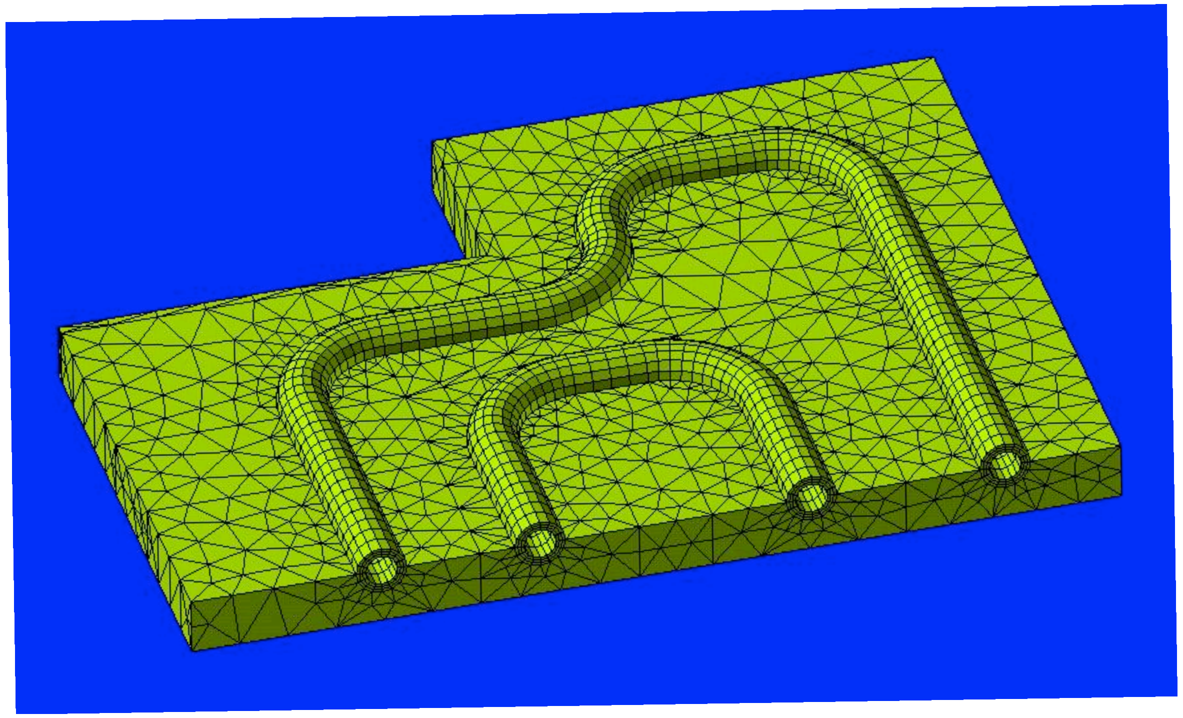
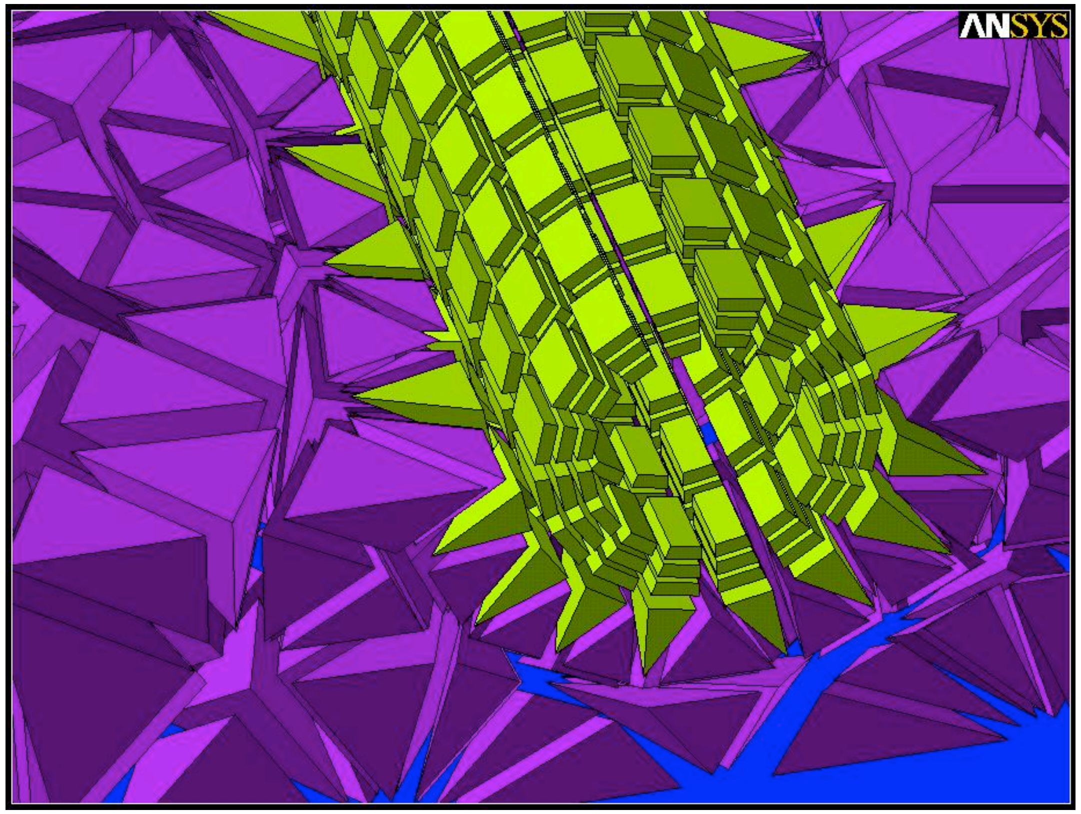
Department of Civil and Environmental Engineering, Carnegie Mellon University
http://www.andrew.cmu.edu/user/sowen/hextet/hextotet2.htm

Pyramidal elements as 'glueing' elements







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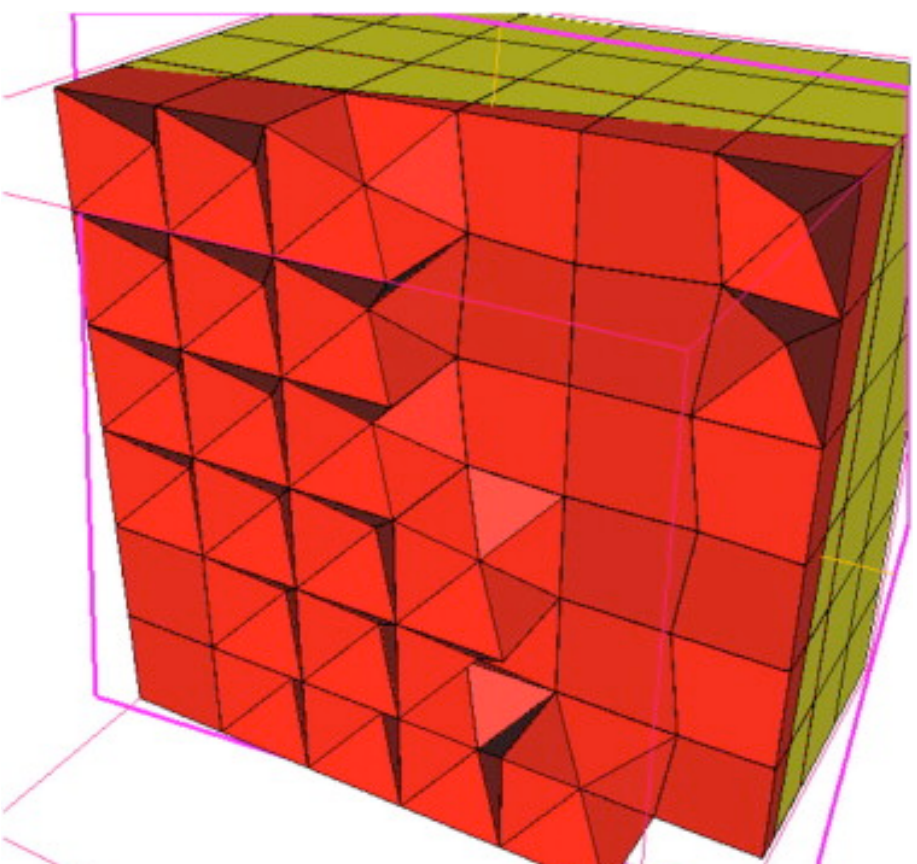


M. Bergot, M. Duruflé / Journal of Computational Physics 232 (2013) 189–213 211

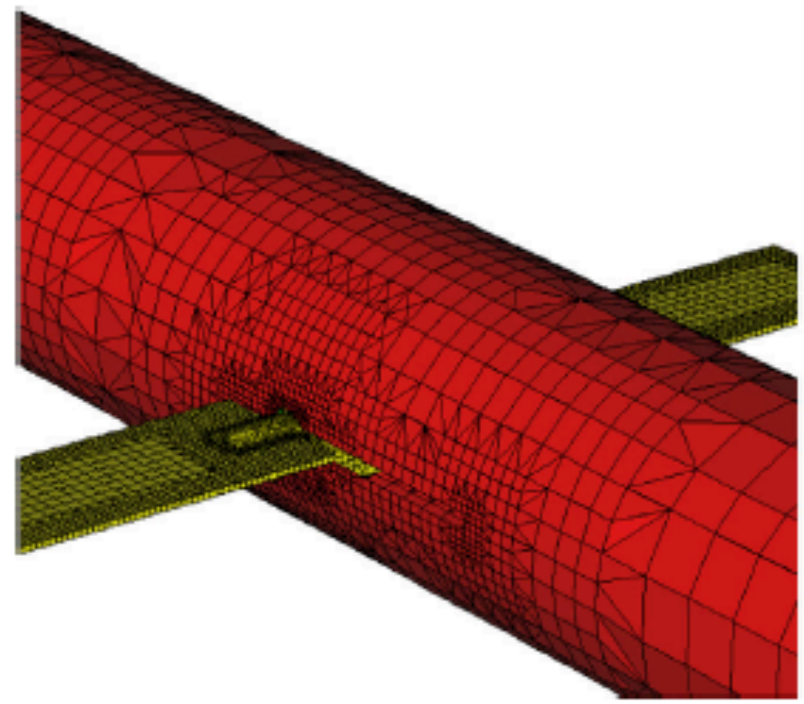
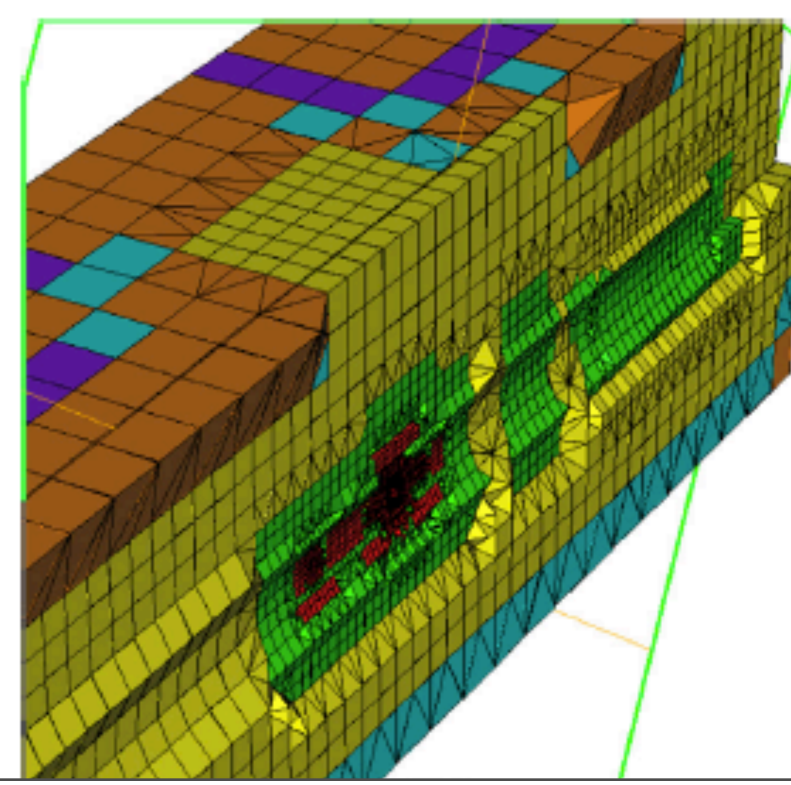


Fig. 13. Surface of the hybrid mesh used for the scattering by a satellite.



Bergot and Durufle, JCP 2013

WHY?

> *A successful 3-D finite element code for Maxwell's equations must include all four kinds of geometrical shapes: tets, hexes, prisms and pyramids. The theory of exact sequences and higher order elements for the pyramid element remains one of the most urgent research issues.*

- L. Demkowicz.
Mixed Finite Elements, Compatibility Conditions, and Applications, C.I.M.E. Summer School held in Cetraro 2006.

HOW?

> *Finite element exterior calculus is an approach to the design and understanding of finite element discretizations for a wide variety of systems of partial differential equations.*

- Arnold, Falk, Winther, Acta Numerica, 2006.

Ground rules

1) *Compatibility.*
2) *Approximation.* The discrete spaces $\mathcal{U}^{(s),k}(K)$ should allow for high-order approximation.
3) *Stability:* The elements satisfy a commuting diagram property:

$$\begin{array}{ccccccc}
H^r(K) & \xrightarrow{\nabla} & \mathbf{H}^{r-1}(\text{curl}, K) & \xrightarrow{\nabla\times} & \mathbf{H}^{r-1}(\text{div}, K) & \xrightarrow{\nabla\cdot} & H^{r-1}(K) \\
\Pi^{(0)}\downarrow & & \Pi^{(1)}\downarrow & & \Pi^{(2)}\downarrow & & \Pi^{(3)}\downarrow \\
\mathcal{R}^{(0),k}(K) & \xrightarrow{\nabla} & \mathcal{R}^{(1),k}(K) & \xrightarrow{\nabla\times} & \mathcal{R}^{(2),k}(K) & \xrightarrow{\nabla\cdot} & \mathcal{R}^{(3),k}(K)
\end{array}$$

Here $\Pi^{(s)}$, s=0,1,2,3, denote interpolation operators induced by the degrees of freedom, and r is chosen so that the interpolation operators are well defined.

We shall focus on affine families

Our work is for affine-mapped pyramidal elements and shape-regular meshes.

Bilinear transformations can lead to loss of approximation (Arnold, Boffi, Falk, 2002, Boffi 2006.)

Recently, Bergot and Duruflé (JCP 2013) provide elements which allow for bilinear transformations of pyramids.

Continuous elements on pyramids

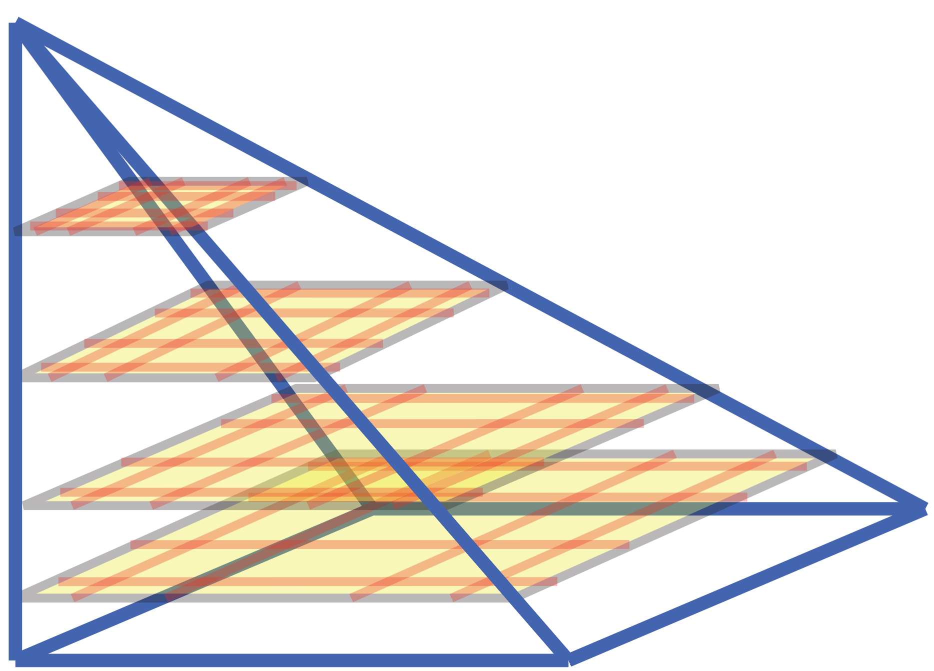
- Wachspress 1975. General polyhedral elements with 3-vertices. Generalises to pyramids.
- Bedrosian 1992. First and second order pyramidal elements.
- Macro-element based approaches. Wieners 1997...
- Sherwin 1997; Chatzi 2000. Attempts at high order.
- Bergot, Cohen, Durufle, 2010, 2013, 2014. Includes survey. "Optimal" high order elements.

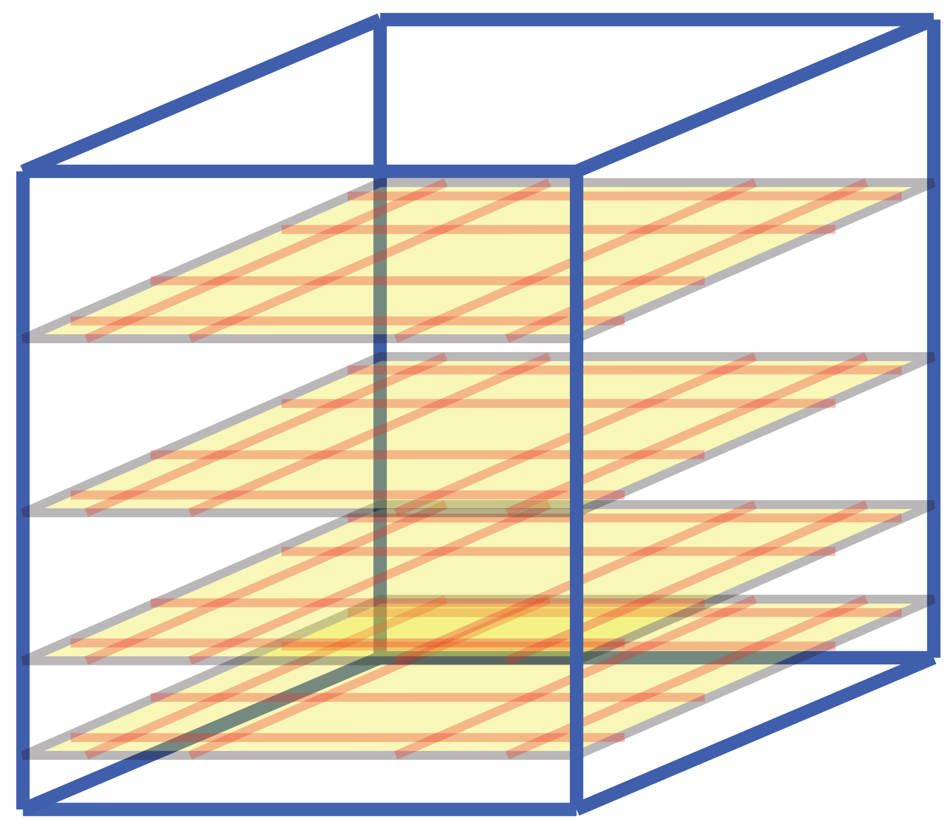
Conforming pyramidal elements for the de Rham complex

- Graglia, 1999. First and second order edge and face elements.
- Gradinaru and Hiptmair, 1999. First order elements. Proof of commutativity.
- Zaglmayr. High order: local exact sequences.
- Bossavit, 2008. Canonical construction of first order.
- Bergot et. al: 2010, 2013, 2014.
- Nigam and Phillips, (arXiV) 2007, 2011, 2012. High order: infinite pyramid.

Attempt #1

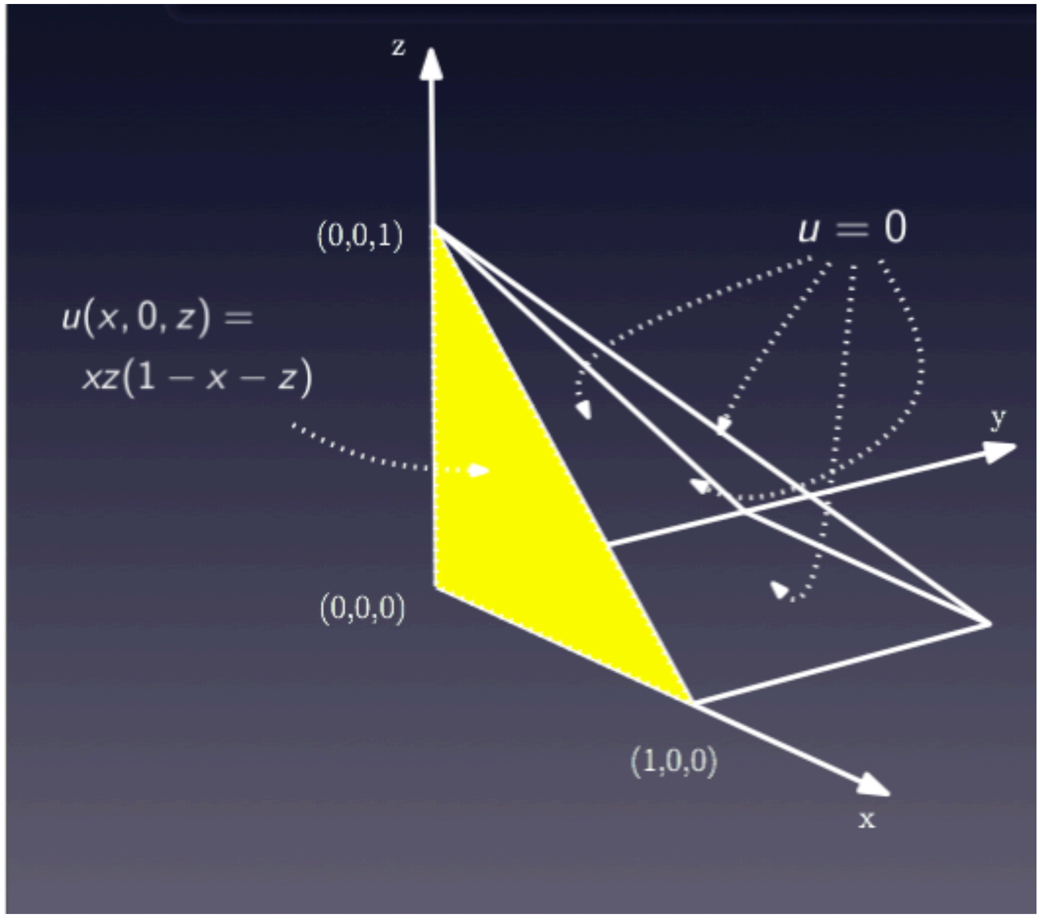
- High-order conforming FEM on hexahedra are well-known...
- ... so can we use Duffy transform from cube to pyramid?





Theorem

There is no (high order) conforming pyramidal continuous finite element whose approximation space consists purely of polynomials.
(Bedrossian 92, Wieners 97, Warren 02, NP 07.)



$$
\begin{aligned}
u(x,y,z) &= \frac{xz(1-x-z)(1-y-z)}{1-z} \\
p(x,y,z) &= xz(1-x-z)(1-y-z) \\
&\qquad (r(x,z) + ys(x,y,z))
\end{aligned}
$$

p is a polynomial, conforming, interpolant of u (?)

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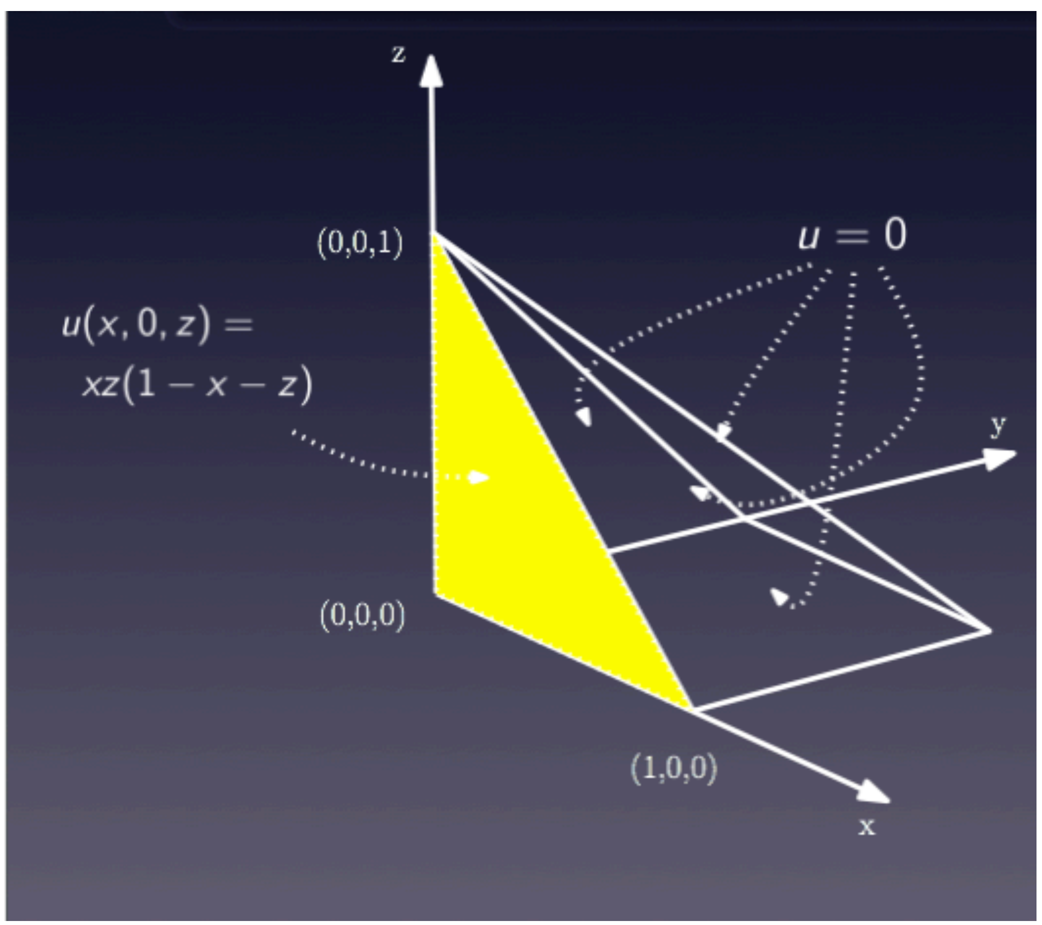

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p(x,y,z) &= xz(1-x-z)(1-y-z) \\
&\qquad (r(x,z) + ys(x,y,z)) \\
p(x,0,z) &= xz(1-x-z)(1-z)r(x,z) \\
u(x,0,z) &= xz(1-x-z) \\
\Rightarrow \boxed{r(x,z) = \frac{1}{1-z}}
\end{aligned}
$$

Negative result: many versions

It is *impossible* to construct pyramidal conforming, compatible high-order finite elements using only polynomials.

There is no conforming global interpolant onto element-wise polynomials for pyramidal elements.

If $\pi_r : H^{r+1}(K) \to P_r$ is a projector onto element-wise polynomials, $\|u - \pi_r u\|_{1,\Omega}$ cannot be bounded.

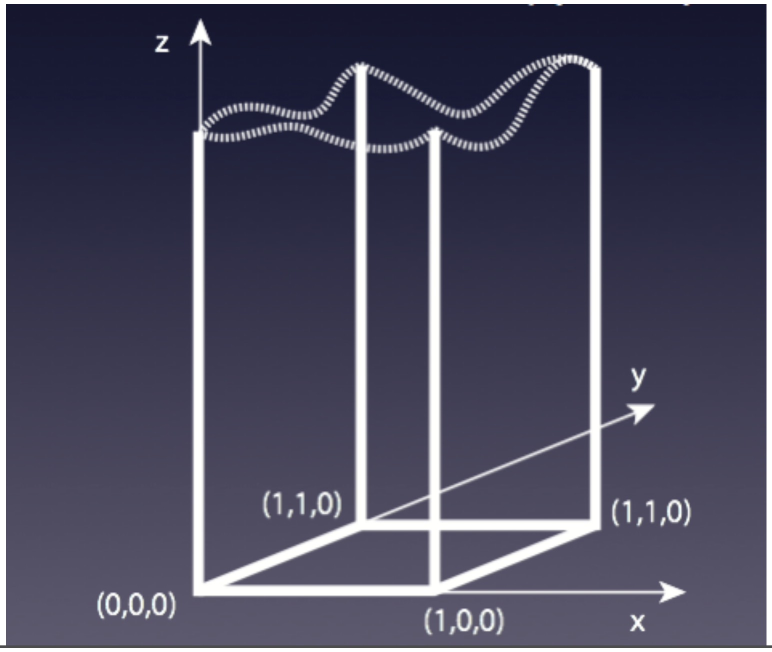
This has consequences for construction and for analysis.

A strange 'reference' element

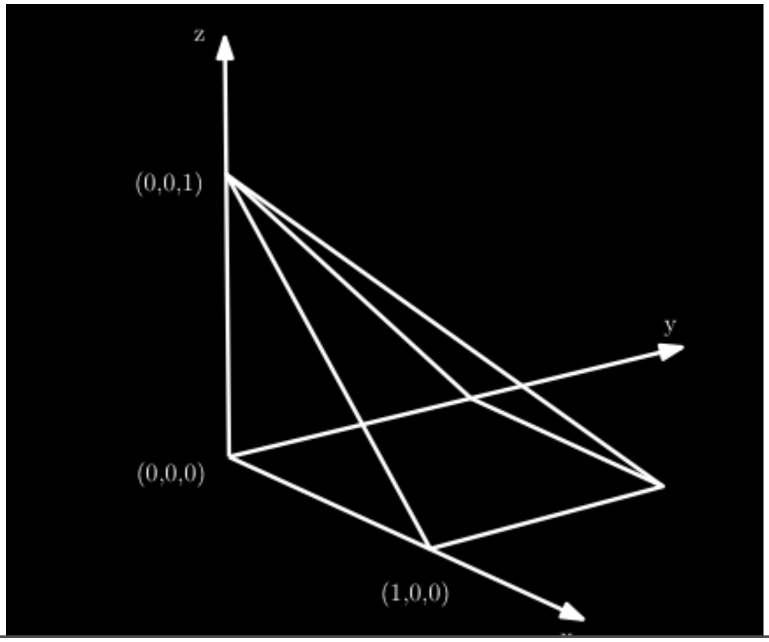
- Let K denote reference pyramidal element with square base

$$K = \{(\xi, \eta, \zeta) \,|\, 0 \leq \zeta \leq 1, 0 \leq \xi, \eta \leq \zeta\}.$$

- Use 'infinite reference pyramidal element' K_∞
- Use pullback mapping induced by the bijection



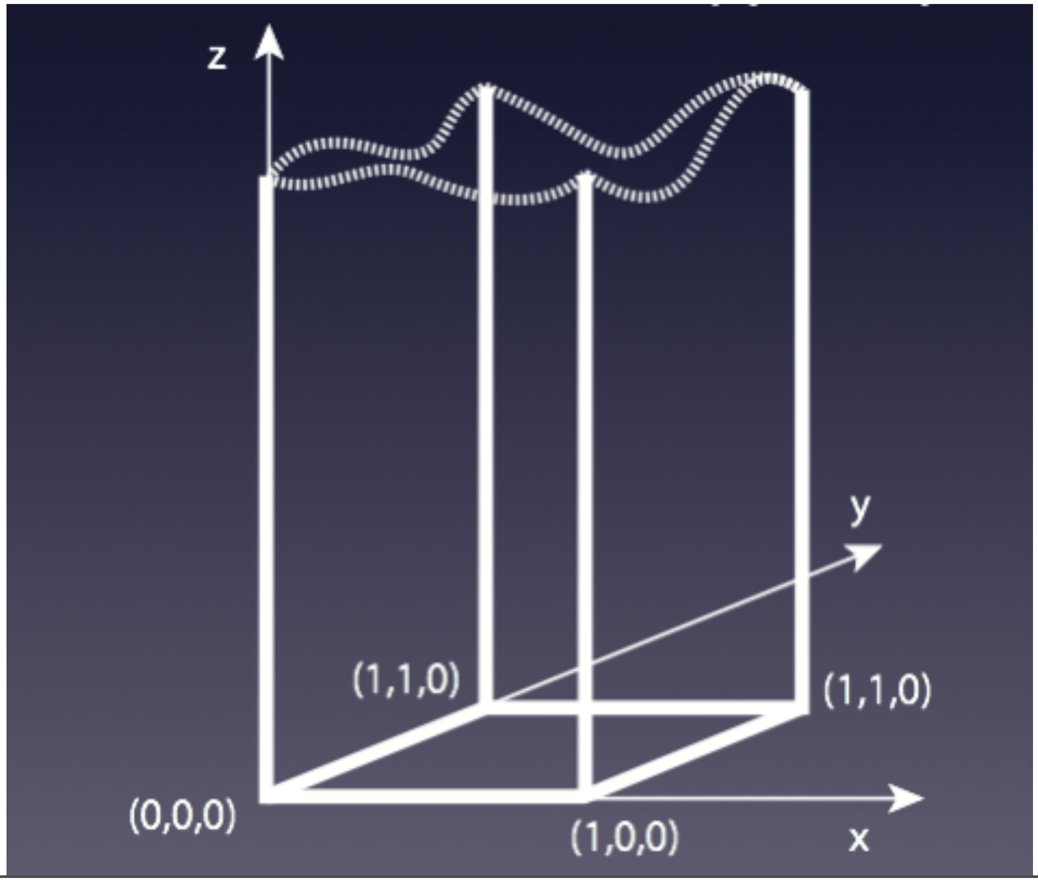
$$\phi : K_\infty \to K$$



What happens on K_∞?

- Start on K_∞
- Use ‘k-weighted’ tensorial polynomials

$$Q_k^{l,m,n}(x,y,z) = \left\{ \frac{u}{(1+z)^k} \; : \; u \in Q^{l,m,n}(x,y,z) \right\}.$$



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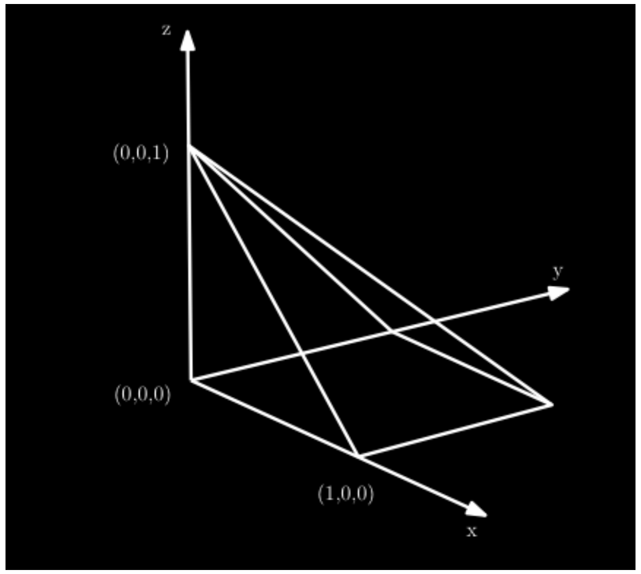

$$Q_k^{l,m,n}(x,y,z) = \left\{ \frac{u}{(1+z)^k} \ : \ u \in Q^{l,m,n}(x,y,z) \right\}.$$

- Consider $p_1(x,y,z) = x$ on K_∞ $\longrightarrow (\phi^{-1})^* p = \frac{\xi}{1-\zeta}$ on K.

What happens on K_∞?

- Start on K_∞
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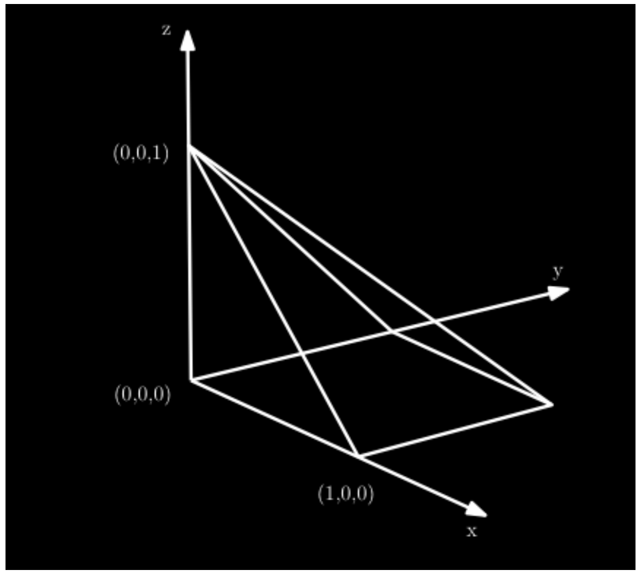

$$\mathcal{Q}_k^{l,m,n}(x,y,z) = \left\{ \frac{u}{(1+z)^k} \;:\; u \in \mathcal{Q}^{l,m,n}(x,y,z) \right\}.$$

- Consider $p_1(x,y,z) = x$ on $K_\infty \longrightarrow (\phi^{-1})^* p = \frac{\xi}{1-\zeta}$ on K.
  If $\alpha_\lambda(t) = (\lambda(1-t), 0, t)$ then $\lim_{t\to 1}(\phi^{-1})^* p(\alpha_\lambda(t)) = \lambda$
  $\Rightarrow$ Must not have p_1 in H^1_w- approximation space.

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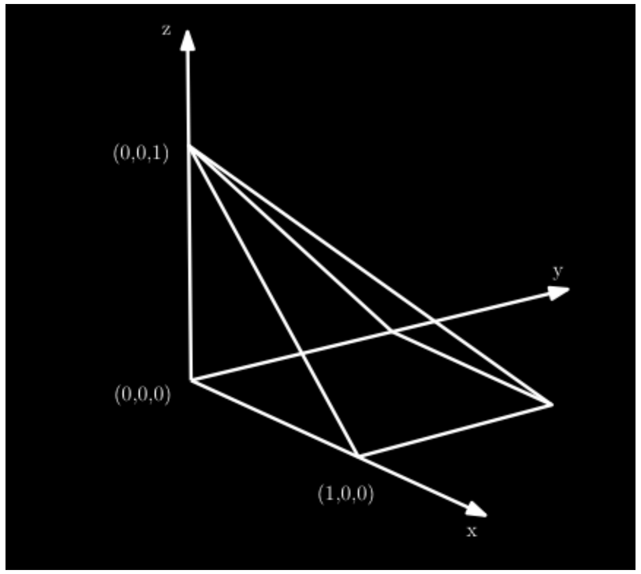

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  $\Rightarrow$ Must not have p_1 in H^1_w- approximation space.
- Consider the function

$$p_2(x,y,z) = \frac{z^k}{(1+z)^k}, \ \text{on} K_\infty \longrightarrow (\phi^{-1})^* p_2 = \zeta^k.$$

$\Rightarrow$ Must retain p_2 in the H^1_w- approximation space.

High order elements on K_∞

We have introduced 3 families of approximation spaces on the infinite pyramid for $s = 0, 1, 2, 3$, approximation order k:

1. 'First family' $\mathcal{U}^{(s),k}$ (NP 2007, 2011)
2. 'Second family' $\mathcal{R}^{(s),k}$ (NP 2012)
3. 'Serendipity' $\mathcal{S}^{(s),k}$ (NPP, 2014)

Family 1

Definition: Underlying spaces via 'k-weighted' tensorial polynomials

Define spaces $\overline{\mathcal{U}^{(s),k}}$ on K_∞, for $s = \{0,1,2,3\}$ and $k \geq 0$

$$\overline{\mathcal{U}^{(0),k}} = \{u \in Q_k^{k,k,k} : \nabla u \in Q_{k+1}^{k-1,k,k-1} \times Q_{k+1}^{k,k-1,k-1} \times Q_{k+1}^{k,k,k-1}\}$$

$$\overline{\mathcal{U}^{(1),k}} = \{u \in Q_{k+1}^{k-1,k,k} \times Q_{k+1}^{k,k-1,k} \times Q_{k+1}^{k,k,k-1} : \nabla \times u \in Q_{k+2}^{k,k-1,k-1} \times Q_{k+2}^{k-1,k,k-1} \times Q_{k+2}^{k-1,k-1,k}\}$$

$$\overline{\mathcal{U}^{(2),k}} = \{u \in Q_{k+2}^{k,k-1,k-1} \times Q_{k+2}^{k-1,k,k-1} \times Q_{k+2}^{k-1,k-1,k} : \nabla \cdot u \in Q_{k+3}^{k-1,k-1,k-1}\}$$

$$\overline{\mathcal{U}^{(3),k}} = \{u \in Q_{k+3}^{k-1,k-1,k-1}\}$$

First Finite element Family $\mathcal{U}^{(s),k}$: functions $u \in \overline{\mathcal{U}^{(s),k}}$ such that on K,$(\phi^{-1}) * u$ has appropriate polynomial traces onto faces and edges.

Family 2: High order elements on K_∞

$$Q_k^{l,m,n}(x,y,z) = \left\{ \frac{u}{(1+z)^k} \; : \; u \in Q^{l,m,n}(x,y,z) \right\}.$$

$$Q_k^{[l,m]} = \{ \frac{x^a y^b (1+z)^{k-c}}{(1+z)^k} \; : \quad c \leq k, \; a \leq c + l - k, \; b \leq c + m - k \}.$$

These spaces can be characterised via a decomposition into spaces of exactly r-weighted tensor product polynomials,

$$Q_k^{[l,m]} = \bigoplus_{r=0}^{k} Q_r^{r+l-k, r+m-k, 0}.$$

Family 2: High order elements on K_∞

Definition

Define spaces $\mathcal{R}^{(s),k}$ on K_∞, for $s = \{0, 1, 2, 3\}$ and $k \geq 0$

$$
\begin{aligned}
\mathcal{R}^{(0),k}(K_\infty) &= Q_k^{[k,k]}, \\
\mathcal{R}^{(1),k}(K_\infty) &= \left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right) \oplus \{\nabla u : u \in Q_k^{[k,k]}\}, \\
\mathcal{R}^{(2),k}(K_\infty) &= \left(\{0\} \times \{0\} \times Q_{k+2}^{[k-1,k-1]}\right) \\
&\qquad \oplus \left\{\nabla \times u : u \in \left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right)\right\}, \\
\mathcal{R}^{(3),k}(K_\infty) &= Q_{k+3}^{[k-1,k-1]}.
\end{aligned}
$$

$$
\mathcal{R}^{(s),k}(K) := \left\{(\phi^{-1})^* u : u \in \mathcal{R}^{(s),k}(K_\infty)\right\}, s = 0, 1, 2, 3.
$$

Properties

$$
\begin{aligned}
\mathcal{R}^{(0),k} &= Q_k^{[k,k]}, \\
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\mathcal{R}^{(3),k} &= Q_{k+3}^{[k-1,k-1]}.
\end{aligned}
$$

The exterior derivatives, $d : \mathcal{R}^{(s),k} \to \mathcal{R}^{(s+1),k}$ are well defined. ∇ is injective on $Q^{[k,k]}/\mathbb{R}$;

Properties

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&\quad \oplus \left\{\nabla \times u : u \in \left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right)\right\}, \\
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\end{aligned}
$$

The exterior derivatives, $d : \mathcal{R}^{(s),k} \to \mathcal{R}^{(s+1),k}$ are well defined. ∇ is injective on $Q^{[k,k]}/\mathbb{R}$; curl is injective on $\left(Q_{k+1}^{[k-1,k]} \times Q_{k+1}^{[k,k-1]} \times \{0\}\right)$ and div is a bijection from $\left(\{0\} \times \{0\} \times Q_{k+2}^{[k-1,k-1]}\right)$ to $Q_{k+3}^{[k-1,k-1]}$.

Properties

Theorem (NP 2012): Properties of second family

- Discrete sequence

$$\mathcal{R}^{(0),k}(K) \xrightarrow{\nabla} \mathcal{R}^{(1),k}(K) \xrightarrow{\nabla\times} \mathcal{R}^{(2),k}(K) \xrightarrow{\nabla\cdot} \mathcal{R}^{(3),k}(K)$$

  is exact.

- Discrete spaces are conforming

$$\mathcal{R}^{(0),k}(K) \subset H^1(K), \quad \mathcal{R}^{(1),k}(K) \subset \mathbf{H}(\mathrm{curl}, K),$$

$$\mathcal{R}^{(2),k}(K) \subset \mathbf{H}(\mathrm{div}, K), \quad \mathcal{R}^{(3),k}(K) \subset L^2(K)$$

Same result holds for first family $\mathcal{U}^{(s),k}(K)$, (NP 2007, 2011).

Properties

Theorem (NP 2012): Properties of second family

- Polynomial approximation properties are:

$$P^k(K) \subset \mathcal{R}^{(0),k}(K) \quad \text{and} \quad P^{k-1}(K) \subset \mathcal{R}^{(s),k}(K), s = 1, 2, 3.$$

- Compatibility through traces of $\mathcal{R}^{(s),k}$, $s = 0, 1, 2$ with relevant traces of Lagrange, curl or div-conforming elements on neighbouring tets and boxes.

- Helmholtz decompositions hold for the bubbles.

Same result holds for first family $\mathcal{U}^{(s),k}(K)$, (NP 2007, 2011).

Unisolvency, conformance, exactness, commuting diagram

$$\begin{array}{ccccccc}
H^r(K) & \xrightarrow{\nabla} & \mathbf{H}^{r-1}(\mathrm{curl}, K) & \xrightarrow{\nabla\times} & \mathbf{H}^{r-1}(\mathrm{div}, K) & \xrightarrow{\nabla\cdot} & H^{r-1}(K) \\
\Pi^{(0)}\downarrow & & \Pi^{(1)}\downarrow & & \Pi^{(2)}\downarrow & & \Pi^{(3)}\downarrow \\
\mathcal{R}^{(0),k}(K) & \xrightarrow{\nabla} & \mathcal{R}^{(1),k}(K) & \xrightarrow{\nabla\times} & \mathcal{R}^{(2),k}(K) & \xrightarrow{\nabla\cdot} & \mathcal{R}^{(3),k}(K)
\end{array}$$

Exterior degrees of freedom analogous to those from neighbouring tets or hexes. Volume degrees of freedom based on projection-based interpolation

Reminder: high-order serendipity elements

- Serendipity elements on hexahedra are a subset of $Q^{k,k,k}$
- Arnold and Awanou (2011) define high-order H^1-conforming serendipity spaces as

$$\begin{aligned}\mathcal{S}_k(box) &:= P^k(box) \oplus span\{xy^a z^b, x^a y^b z, x^a y^b z, a+b=k+1\}\\ &\oplus span\{xyz^k, xy^k z, x^k yz\}\end{aligned}$$

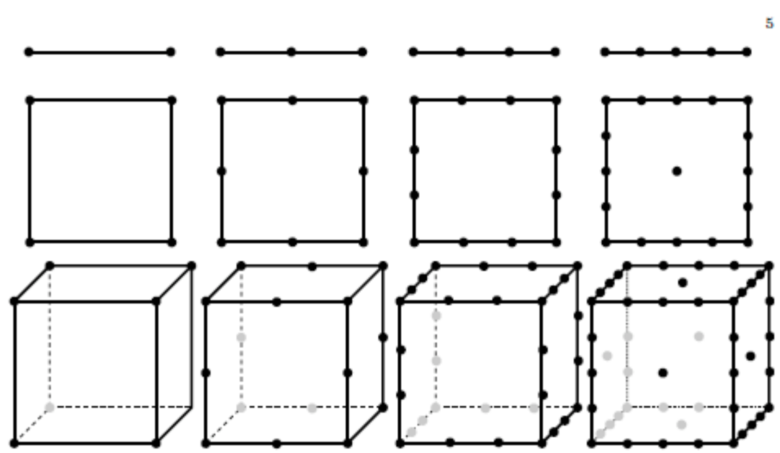


Fig. 1 Degrees of freedom for $\mathcal{S}_r(I^n)$ for $r = 1, 2, 3, 4$, $n = 1, 2, 3$. Dots indicate the number of degrees of freedom associated to each face.

High-order 'serendipity' elements on a pyramid?

- Must be compatible with neighbouring tetrahedral and (serendipity) hexahedra through boundary traces
- Must be subspace of $\mathcal{R}^{(0),k}(K)$
- Must allow for arbitrary order approximation by polynomials.

High-order 'serendipity' elements on infinite pyramid

Definition (NP2013)

$$
\begin{aligned}
\mathcal{S}^{(0),k}(K_\infty) \quad &:= span\{\frac{x^\alpha y^\beta z^\gamma}{(1+z)^{k-1}} / \alpha+\beta+\gamma \leq k-1\} \\
&\oplus span\{\frac{x^\alpha y^\beta z}{(1+z)^{k-1}} / \alpha+\beta = k-1 \ \} \\
&\oplus span\{\frac{x y^\alpha z^\beta}{(1+z)^{k-1}} / \alpha+\beta = k-1, \ \beta < k-1, \ \beta \neq 1\} \\
&\oplus span\{\frac{x^\alpha y z^\beta}{(1+z)^{k-1}} / \alpha+\beta = k-1, \ 1 \neq \alpha, \ \beta < k-1, \ \beta \neq 1\} \\
&\oplus span\{\frac{x^{k-1} y z}{(1+z)^{k-1}}, \frac{x y^{k-1} z}{(1+z)^{k-1}}\}
\end{aligned}
$$

Quadrature

Error due to quadrature

- Elliptic bilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$,

$$a(u, v) := \int_\Omega A(\nabla u, \nabla v)\, dx$$

  A positive definite covariant tensor, entries in $W^{k,\infty}(\Omega)$.
- Let $V_h \subset H_0^1(\Omega)$be a polynomial approximation space (degree k).
- $S_{K,k}(\cdot)$ be a quadrature rule, which satisfies

$$S_{K,k}(\partial_i u \partial_j v) = \int_K (\partial_i u \partial_j v), \qquad \forall i, j, \quad \forall u, v \in V_h$$

  over element K.
- Discrete bilinear form $a_h(u, v) := \sum_{K \in \mathcal{T}_h} S_{K,k}(A(\nabla u, \nabla v))$.
- Approximate

$$a_h(u, v) \approx a(u, v), \forall v \in V_h.$$

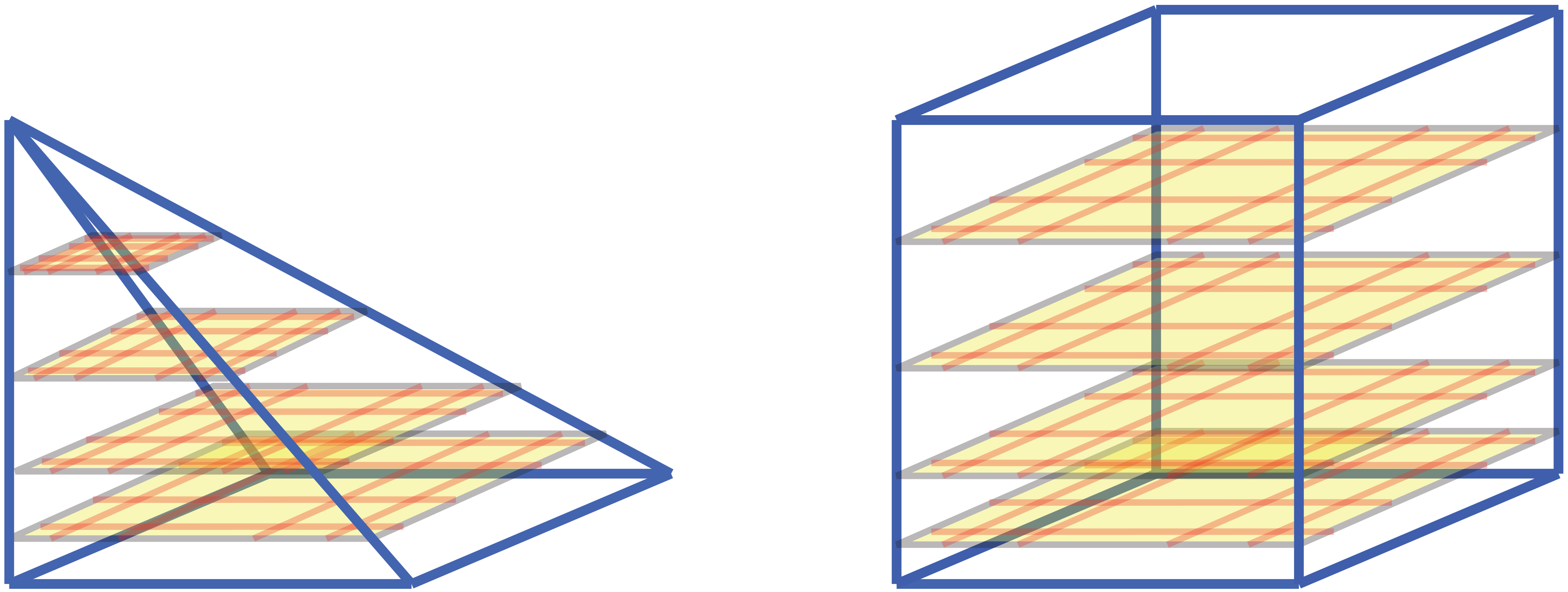
Refresher: quadrature

- *"Our basic objective is to give sufficient conditions on the quadrature scheme which ensure that the effect of the numerical integration does not decrease [the] order of convergence"*,
  - Ciarlet 1978.

- If approximation space $\subset H_0^1(\Omega)$, and true solution is $u \in H^{k+1}(\Omega)$, want h^k convergence in the H^1 norm.

- Rule of thumb: If approximation space has polynomials of degree k, quadrature rule should be exact at $2k-2$.

Quadrature on pyramids



- Conical product formulae (Stroud, 72).
- Duffy transform + Gauss Legendre / Jacobi.
- k^3 evaluations

Conical product formulae

- kth degree formula is exact for polynomials of degree $2k$ on the pyramid.
- Exact for products of any pair of kth order pyramidal shape functions from each family of elements, including the rational functions.
- Numerical evidence that they perform well for continuous elements (Bergot et al, 2010).

A nitpicky question

We *know* the quadrature rule is exact for pairs of basis functions. Does this suffice for the analysis of errors?

What we would like

Let $\forall v \in V_h$,

$$\begin{aligned} a(u,v) &= \int_\Omega A(\nabla u, \nabla v) &= f(v), \\ a_h(u_h,v) &= \sum_{K\in\mathcal{T}_h} S_{K,k}(A(\nabla u_h, \nabla v)) &= f(v). \end{aligned}$$

We'd like to conclude:

$$\|u-u_h\|_1 \le Ch^k\left(|u|_{k+1} + \|A\|_{k,\infty}\|u\|_{k+1}\right)$$

Analyze variational crime via First Strang Lemma:

$$\begin{aligned} \|u-u_h\|_1 &\le C \inf_{v_h\in V_h}\left(\|u-v_h\|_1 + \sup_{w_h\in V_h}\frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_1}\right) \\ &\le C\left(h^k|u|_{k+1} + h^k\|A\|_{k,\infty}\|u\|_{k+1}\right) \end{aligned}$$

What is needed?

- Need estimate of the global consistency error.

$$\begin{aligned}\sup_{\substack{w_h \in V_h \\ \|w_h\|_1 = 1}} \left| \int_\Omega A(\Pi_h u, w_h) - \sum_{K \in \mathcal{T}_h} S_{K,k}(\Pi_h u, w_h) \right| &\leq Ch^k \|A\|_{k,\infty} \|\Pi u\|_{k+1} \\ &\leq Ch^k \|A\|_{k,\infty} \|u\|_{k+1}\end{aligned}$$

where $\Pi_h : H_0^1(\Omega) \to V_h$ is a *bounded interpolation operator*. The constant $C = C(\Omega, k)$ is independent of h (and changed from line to line).

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- Need a local estimate to control the consistency error: let

$$\text{quadrature error} = E_{k,K}(\phi) = 0 \quad \forall \phi \in P_{2k-2}\ K.$$

- *Usually* use Bramble-Hilbert lemma, and get: There $\exists C$ s.t. $\forall A \in W^{k,\infty}(K), \quad \forall p, q \in P_k(K)$

$$|E_{k,K}(A\partial_i p \partial_j q))| \leq Ch^k\|A\|_{k,\infty,K}\|\partial_i p\|_{k-1,K}\|\partial_j q\|_{0,K}$$

- Can we do the same?

Attempt #2

- Our quadrature formula is exact for basis functions....
- Our approximation spaces contain polynomials....

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ergo, Optimistic Conjecture: Let $K \in \mathcal{T}_h$ be a pyramid.
Since the quadrature rule (with error functional $E_{k,K}$) integrates products of shape functions in $\mathcal{R}_k^{(1),k}(K)$ exactly, desired estimate $\forall v, w \in \mathcal{R}_k^{(1),k}(K)$,

$$|E_{k,K}(Avw)| \leq Ch^k \|A\|_{k,\infty,K} \|w\|_{0,K} \|v\|_{k-1,K}$$

holds.

Rational functions can't be ignored

- $\mathcal{R}^{(0),k}$ contains $P_k(K)$.
- $\mathcal{R}^{(0),k}$ also contains rational polynomials!

Take the $\mathcal{R}^{(0),k}(\Omega)$ shape function associated with the base vertex, $(1,1,0)$:

$$v(\xi,\eta,\zeta) = \frac{\xi\eta}{1-\zeta}.$$

The third partial ζ-derivative $\frac{\partial^3 v}{\partial\zeta^3} \notin L^2(\Omega)$:

$$\int_\Omega \left(\frac{\partial^3 v}{\partial\zeta^3}\right)^2 d\hat{x} = \int_0^1 \int_0^{1-\zeta} \int_0^{1-\zeta} \left(\frac{-6\xi\eta}{(1-\zeta)^4}\right)^2 d\xi\, d\eta\, d\zeta = \int_0^1 \frac{9}{(1-\zeta)^2} d\zeta.$$

Hence $v \notin H^3(\Omega)$.

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holds. This conjecture cannot be used.

Negative result

It is *impossible* to analyze quadrature errors for pyramidal high-order finite elements by considering only polynomials. Direct application of classical arguments fail when we attempt to use the Bramble-Hilbert lemma to obtain the estimate

$$|\Pi_{k,K}^{(s)} u|_{k,K} \leq C|u|_{k,K}$$

where $\Pi_{k,K}^{(s)}$ is any bounded interpolant to $\mathcal{R}^{(s),k}(K)$.

The fix

Go back to the Bramble-Hilbert lemma...

Bramble-Hilbert Lemma

Let $\Omega \subset \mathbb{R}^n$ be open. For some integer $k \geq 0$ and $p \in [0, \infty]$ let the linear functional, $f : W^{k+1,p}(\Omega) \to \mathbb{R}$ have the property that $\forall \psi \in P^k(\Omega)$, $f(\psi) = 0$. Then there exists a constant $C(\Omega)$ such that

$$\forall v \in W^{k+1,p}(\Omega), \quad |f(v)| \leq C(\Omega) \|f\|_{W^{k+1,p}(\Omega)'} |v|_{k+1}$$

... and modify it.

Don't apply it to the whole approximation space at once!

Resolution: the local estimate

- Observation: On pyramid, the components of each basis function in $\mathcal{R}^{(0),k}(K)$ live in spaces spanned by e_{abr}:

$$e_{abr}(x,y,z) := x^a y^b (1-z)^{r-a-b} \quad r \leq k \text{ and } a,b \leq r+1$$

- Regularity increases with r: $e_{abr} \in H^{r+1}(K) \quad \forall a,b$

$$|E_{k,K}[A^{ij}\partial_i e_{abr}\partial_j v]| \leq Ch^r \|A\|_{r,\infty,K}\|e_{abr}\|_{r,K}|w|_{1,K}$$

- Let Aij be element-wise polynomial. Quadrature is still exact:

$$E_{k,K}[A^{ij}\partial_i e_{abr}\partial_j w] = 0 \quad \forall A^{ij} \in P^{k-r}, \quad w \in V_k(K)$$

- Need to modify the Bramble Hilbert Lemma and scaling argument to get "missing" h^{k-r}.

Modification via decomposition

Fix $\alpha \geq 0$ and let $k \geq \alpha$ be an integer. Suppose that:

- Let R^k be finite dimensional, $P^k \subset R^k \subset H^\alpha(K)$;
- $\Pi : H^\alpha(K) \to R^k$ a bounded linear projection;
- $\exists V_r \subset H^r(K)$ for each $r \in \{0, \ldots, k\}$ such that decomposition holds:

$$R^k = V_0 \oplus \cdots \oplus V_k.$$

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Interpolation-on-decomposition estimate (NP 2012)

Let $\forall u \in H^k(K)$, interpolant $\Pi u \in R^k = \; v_0 + \cdots + v_k, \; v_r \in V_r$.

- For each r satisfying $\alpha \leq r \leq k$:

$$|v_r|_r \leq C|u|_r.$$

- If $\subset V_r$ has poly. of homogenous degree r, then

$$|v_r|_r \leq C|u|_{r+1} + |u|_r, \quad \forall r \in [\alpha - 1, -1].$$

Our approximation spaces $\mathcal{R}^{(s),k}$ satisfy the decomposition properties needed. Therefore, combining errors on subspaces,

Theorem (NP 2012)

The consistency error for the elliptic bilinear form

$$a(\cdot,\cdot) := \int_\Omega A(\nabla u, \nabla v)\,dx$$

is

$$\sup_{v\in V_h} \frac{|a(\Phi_h^{(0)}u, v) - a_h(\Phi_h^{(0)}u, v)|}{\|v\|_1} < Ch^k\|A\|_{k,\infty,\Omega}\|u\|_{k+1,\Omega}.$$

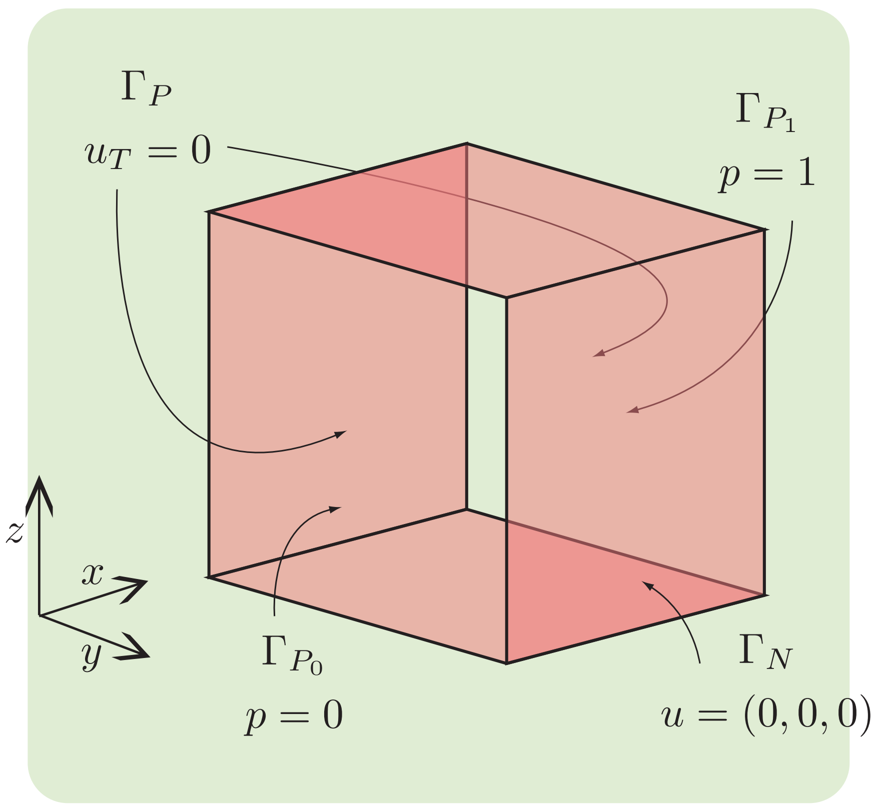
Here, $\Phi_h^{(0)}$ is a bounded projection operator onto $\mathcal{R}^{(0),k}$. Analogous results hold for $\mathcal{R}^{(s),k}$, $s = 0, 1, 2, 3$

Nitpicky question answered.

We have a full accounting of quadrature errors for the second family of (affine) finite elements on the pyramid.
We do *not* have a complete analysis for non-affine pyramids.

Stokes flow in a pipe with square cross-section

$$
\begin{aligned}
-\triangle u + \nabla p &= f && \text{in } \Omega, \\
\nabla \cdot u &= 0 && \text{in } \Omega \\
u_T &= g_T && \text{on } \Gamma \\
u_N &= g_N && \text{on } \Gamma_N \\
p &= \phi && \text{on } \Gamma_P
\end{aligned}
$$



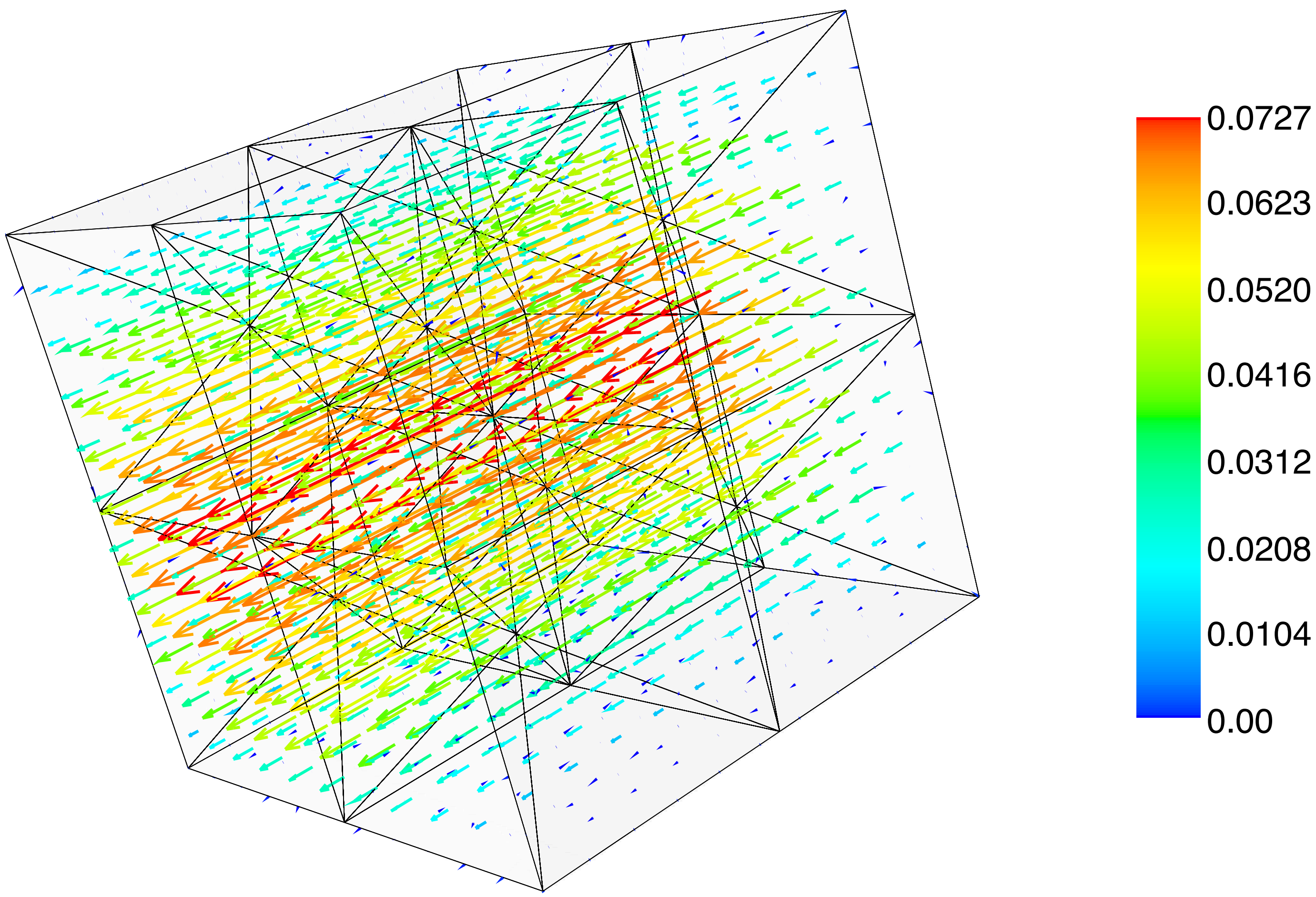
Find $w \in H(\mathrm{curl}, \Omega)$, $u \in H(\mathrm{div}, \Omega; g_N)$ and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
(w, \tau)_\Omega - (u, \nabla \times \tau)_\Omega &= (g_T, n \times \tau)_\Gamma && \forall \tau \in H(\mathrm{curl}, \Omega) \\
-(\nabla \times w, v)_\Omega + (p, \nabla \cdot v)_\Omega &= (\phi, v \cdot n)_{\Gamma_P} - (f, v)_\Omega && \forall v \in H(\mathrm{div}, \Omega; 0) \\
(\nabla \cdot u, q)_\Omega &= 0 && \forall q \in L^2(\Omega)
\end{aligned}
$$

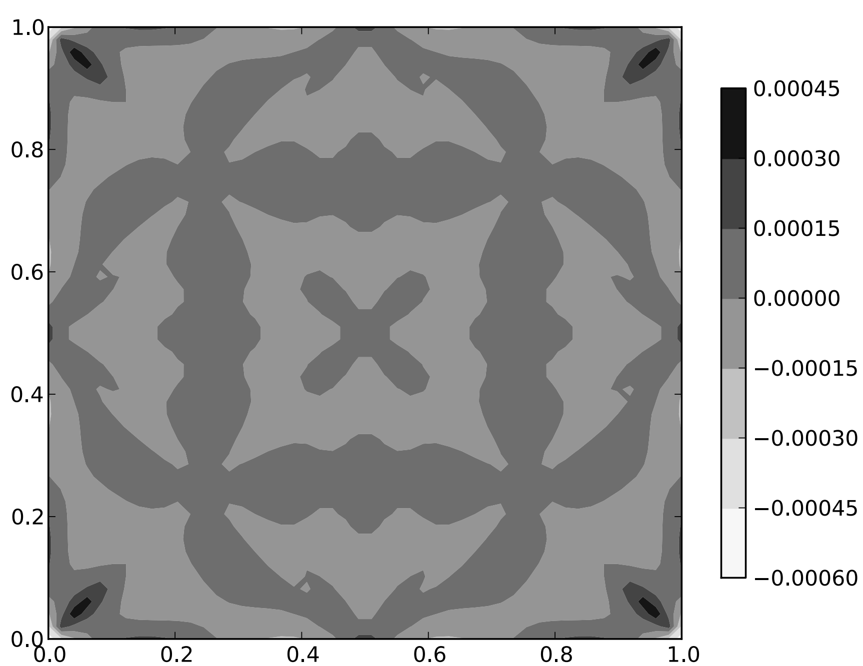
$$
\begin{bmatrix} I & C^t & 0 \\ C & 0 & D^t \\ 0 & D & 0 \end{bmatrix}
\begin{bmatrix} w \\ u \\ p \end{bmatrix}
=
\begin{bmatrix} G \\ \Phi - F \\ 0 \end{bmatrix}
$$

$[0\ D]$ has closed range, and first block is invertible on $ker[0\ D]$.
Exactness of discrete sequence allows us to show these.

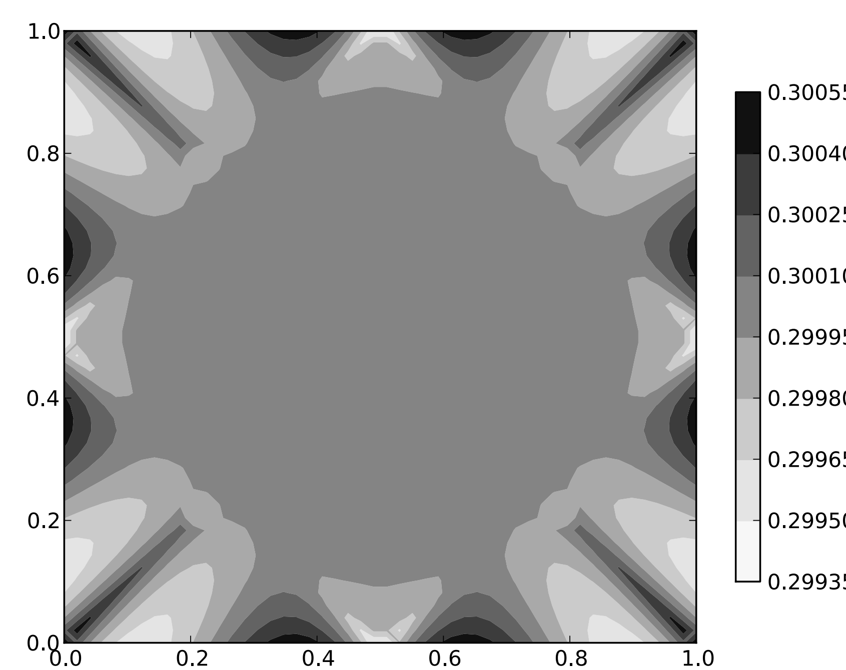
Numerical experiments



2x2x2 grid of cubes, each containing 6 4th order pyramidal elements



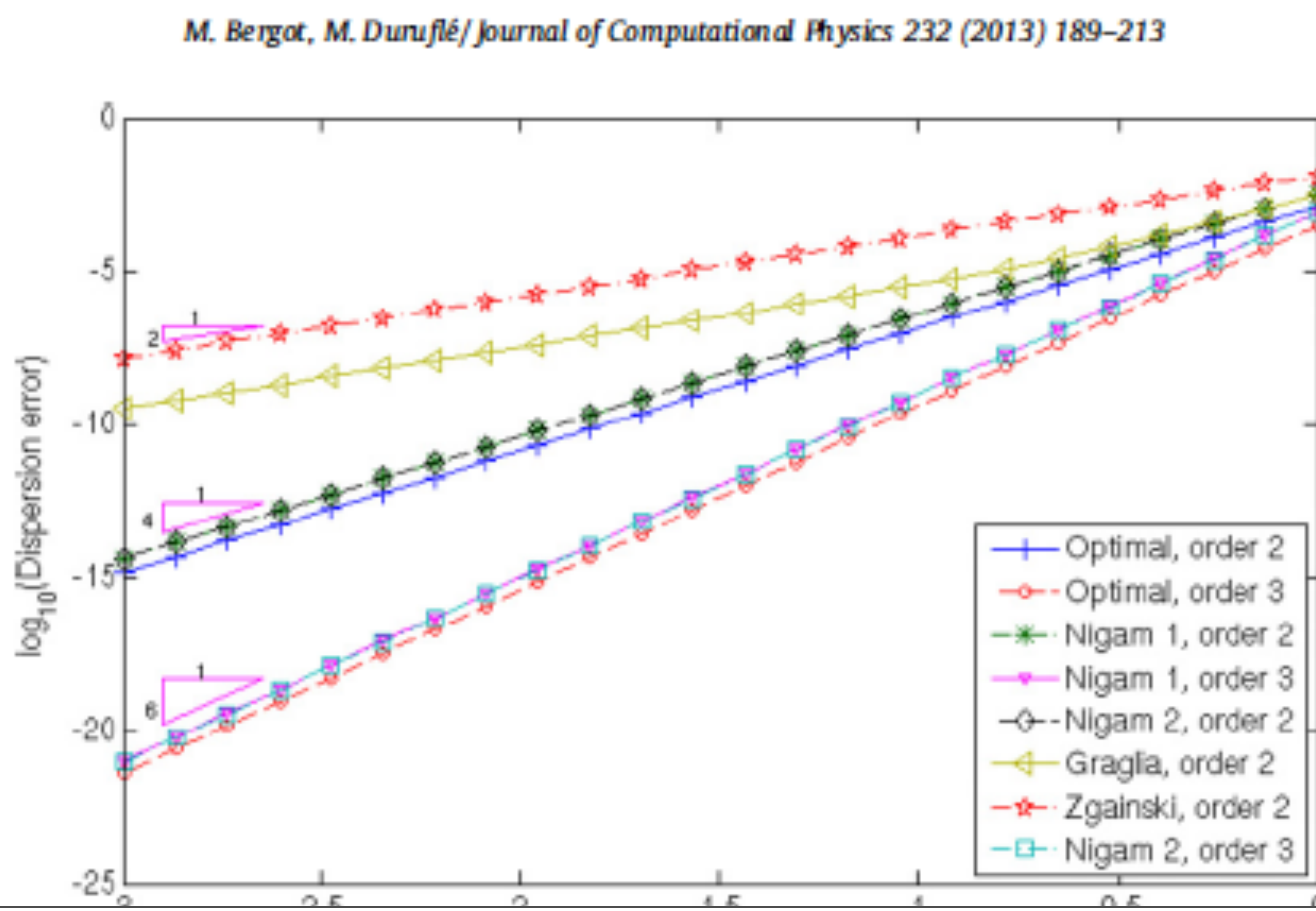
error at *x*=0.3



pressure at *x*=0.3

Dispersion error

From: High-order optimal edge elements for pyramids, prisms, and hexahedra, Bergot and Duruflé, JCP 2013.
Affine mesh.



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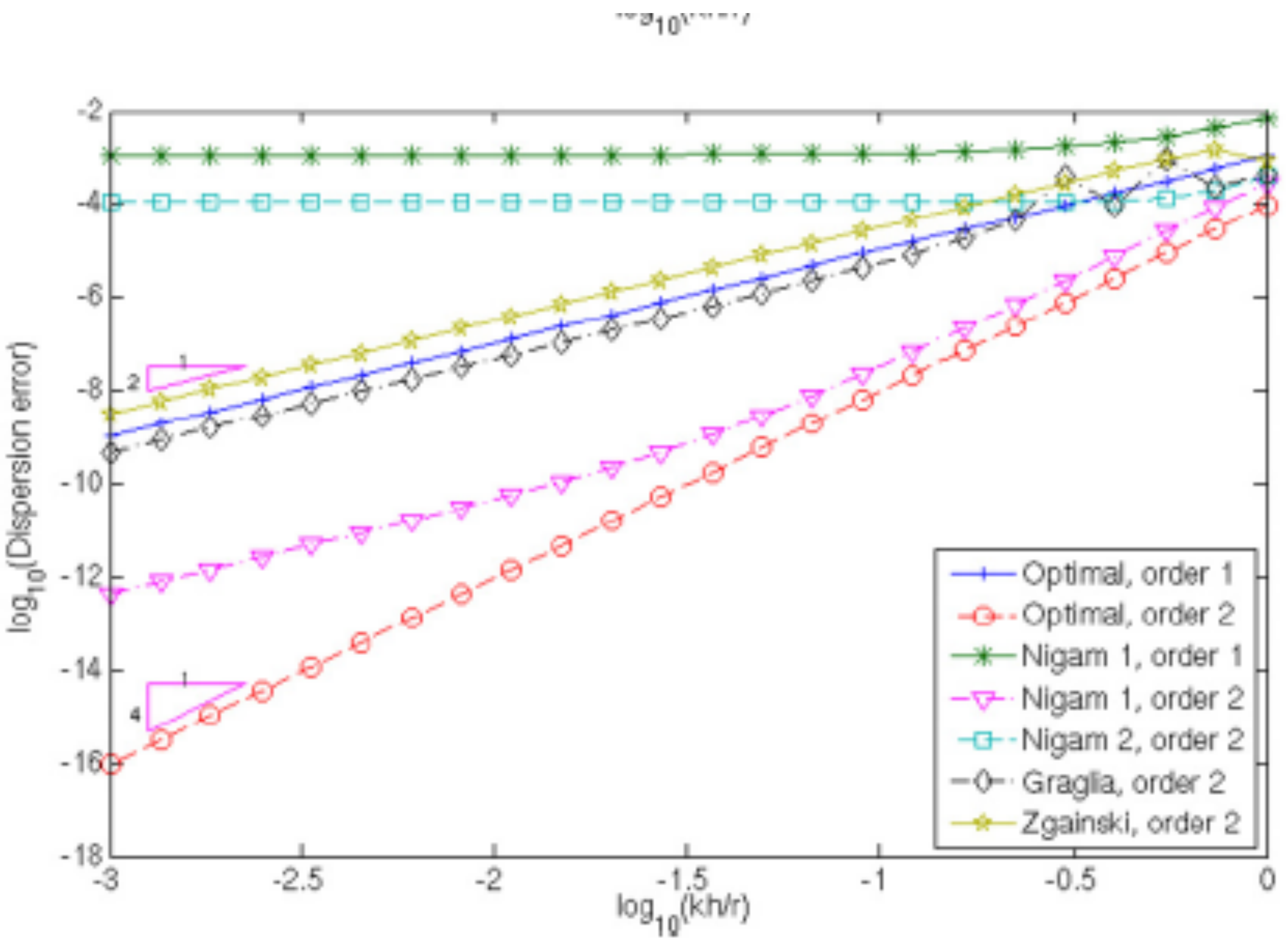
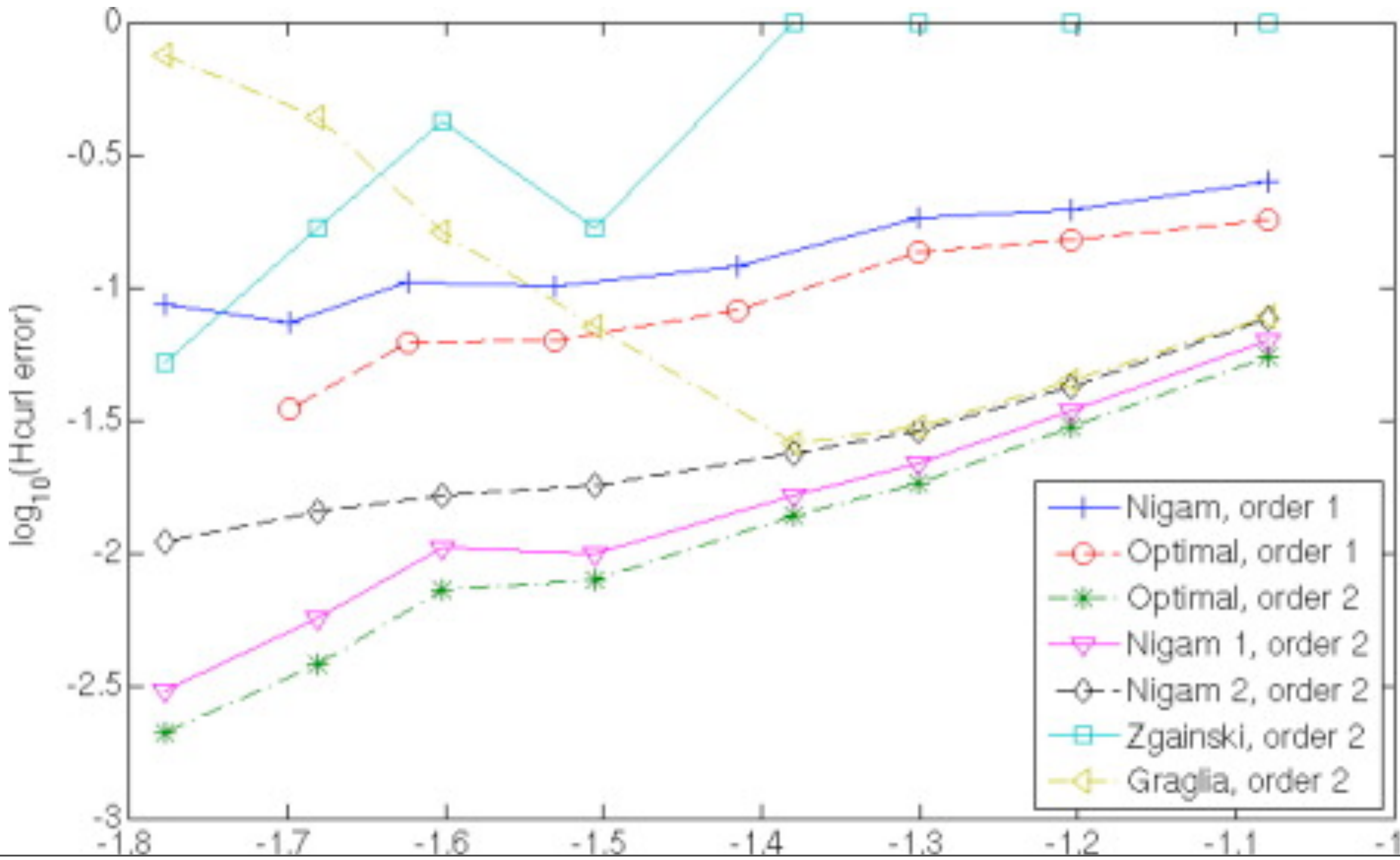


Gaussian source in cavity

From: High-order optimal edge elements for pyramids, prisms, and hexahedra, Bergot and Duruflé, JCP 2013.

Mesh contains affine and non-affine elements.



Eigenvalue calculations

From: High-order optimal edge elements for pyramids, prisms, and hexahedra, Bergot and Duruflé, JCP 2013.

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Table 2
Properties of different finite element spaces.

Property	Zgainski, $r=2$	Graglia, $r=2$	Nigam/Phillips 1	Nigam/Phillips 2	Optimal
Convergence with affine pyramids	$O(h)$	$O(h)$	$O(h^r)$	$O(h^r)$	$O(h^r)$
Convergence with non-affine pyramids	$O(h)$	$O(1)$	$O(h^{r-1})$	$O(1)$	$O(h^r)$
Spurious modes	Yes	Yes	No	No	No
Compatibility	No	Yes	Yes	Yes	Yes

Summary

- Constructed (two families of) high order compatible pyramidal elements for the spaces of the de Rham complex.
- The elements satisfy a commuting diagram property.
- Stroud's conical product rules can be used to construct numerical integration formulae that do not decrease the order of convergence.
- Some supporting numerical results.

Thank you for your attention!

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