Trefftz-Discontinuous Galerkin Methods for Maxwell's Equations

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Outline

- The time-harmonic Maxwell equations
- Trefftz-discontinuous Galerkin methods
- Error analysis

Time-harmonic Maxwell's equations

Electric-field based formulation with impedance boundary conditions

$$\begin{cases} \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \varepsilon \, \mathbf{E} = 0 & \text{in } \Omega \\ (\mu^{-1} \nabla \times \mathbf{E}) \times \mathbf{n} - i\omega\vartheta(\mathbf{n} \times \mathbf{E}) \times \mathbf{n} = \mathbf{g} & \text{on } \partial\Omega \end{cases}$$

- bounded polyhedral domain $\Omega \subset \mathbb{R}^3$
- angular frequency (wave number) $\omega \geq \omega_0 > 0$ (wave length $\lambda = 2\pi/\omega$)
- assume $\varepsilon, \mu, \vartheta \in \mathbb{R}$ to be constant, $\varepsilon, \mu > 0$, $\vartheta \neq 0$
- $\mathbf{g} \in L^2_T(\partial\Omega)$

Time-harmonic Maxwell's equations

Fredholm alternative $\rightarrow$ well-posedness of weak formulation in

$$H_{\mathsf{imp}}(\mathrm{curl};\Omega) = \{\boldsymbol{v} \in H(\mathrm{curl};\Omega) : (\mathbf{n} \times \boldsymbol{v}) \times \mathbf{n} \in L^2_T(\partial\Omega)\}$$

with $\nabla \cdot (\varepsilon \mathbf{E}) = 0$ and stability bound

$$\|\mu^{-1/2}\nabla \times \mathbf{E}\|_{0,\Omega} + \omega\|\varepsilon^{1/2}\mathbf{E}\|_{0,\Omega} \leq C_{\mathsf{stab}}\|\mathbf{g}\|_{0,\partial\Omega}$$

(see P. Monk's book)

Time-harmonic Maxwell's equations

Numerical issues

- Oscillating solutions $\rightarrow$ the number of d.o.f. to obtain a given accuracy increases with the frequency ω

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- Oscillating solutions $\rightarrow$ the number of d.o.f. to obtain a given accuracy increases with the frequency ω
- *h*-version FEM is affected by pollution effect:

$$\underbrace{\|\text{discretisation error}\|}_{\|u-u_{hp}\|} \leq C(\omega) \underbrace{\|\text{best approximation error}\|}_{\inf_{v_{hp}\in V_p(\mathcal{T}_h)} \|u-v_{hp}\|}$$

where $C(\omega)$ is an *increasing* function of ω

[Babuška & Sauter, 2000]

Trefftz-FEM

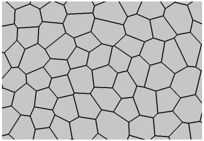
Trefftz FEM: basis functions are, element by element, solutions to the PDE (for time-harmonic wave problems: oscillating functions with the same frequency as the problem)

$\rightarrow$ improved the accuracy vs. number of d.o.f. as compared to standard (polynomial-based) finite element methods

Trefftz-FEM

$$\begin{cases} \mathcal{L}u = 0 \quad \text{in } \Omega \qquad (\mathcal{L} \text{ elliptic operator}) \\ + \text{ b.c.} \end{cases}$$

Trefftz spaces for $\mathcal{L}$



- mesh $\mathcal{T}_h$ of Ω
- local Trefftz spaces $T(E) = \{v : \mathcal{L}v = 0\}$
- Trefftz spaces $T(\mathcal{T}_h)$: discontinuous functions whose restrictions to each $E \in \mathcal{T}_h$ belong to $T(E)$

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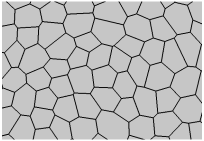

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Trefftz finite element spaces

- let $V_p(E) \subset T(E)$ be *finite dimensional* local spaces
- Trefftz finite element spaces $V_p(\mathcal{T}_h)$: discontinuous functions whose restrictions to each $E \in \mathcal{T}_h$ belong to $V_p(E)$

Trefftz-FEM for Maxwell's equations

- UWVF (Trefftz-DG)
  - [Cessenat, 1996], [Cessenat & Després, 2002], [Huttunen, Malinen & Monk, 2007], [Darrigrand & Monk, 2007, 2012]
  - Error analysis: [Hiptmair, Moiola & Perugia, 2013]
- Other Trefftz approaches
  - [Copeland, 2009], [Copeland, Langer & Pusch, 2009]
  - [Kretzschmar, Schnepp, Tsukerman & Weiland, 2014]

Trefftz-DG methods for time-harmonic Maxwell's equations

Maxwell's equation: $\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \varepsilon \, \mathbf{E} = 0 \quad \text{in } \Omega$

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- Introduce a mesh $\mathcal{T}_h$; multiply by test functions and integrate by parts twice on each element K (ultra weak formulation)

$$\int_K \mathbf{E} \cdot (\nabla \times (\mu^{-1} \nabla \times \overline{\boldsymbol{v}}) - \omega^2 \varepsilon \, \overline{\boldsymbol{v}}) + \int_{\partial K} \mathbf{n} \times \mathbf{E} \cdot \left(\mu^{-1} \nabla \times \overline{\boldsymbol{v}} \right) + \int_{\partial K} \mathbf{n} \times (\mu^{-1} \nabla \times \mathbf{E}) \cdot \overline{\boldsymbol{v}} = 0$$

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- Replace traces by numerical fluxes on ∂K : $\mathbf{E}\to\widehat{\mathbf{E}},\ \mu^{-1}\nabla\times\mathbf{E}\to i\omega\widehat{\mathbf{H}}$

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For every $K \in \mathcal{T}_h$, $\quad \displaystyle\int_{\partial K} \mathbf{n} \times \widehat{\mathbf{E}} \cdot (\mu^{-1}\nabla \times \overline{\boldsymbol{v}}) + \int_{\partial K} \mathbf{n} \times (i\omega\widehat{\mathbf{H}}) \cdot \overline{\boldsymbol{v}} = 0$

Example of Maxwell-Trefftz spaces

Plane wave (PW) Maxwell-Trefftz spaces

scalar PW:	$\boldsymbol{x} \mapsto e^{i\kappa \boldsymbol{d}\cdot \boldsymbol{x}}$		Helmholtz solutions
vector PW:	$\boldsymbol{x} \mapsto \boldsymbol{a}\, e^{i\kappa \boldsymbol{d}\cdot \boldsymbol{x}}$,	$\boldsymbol{a}\cdot\boldsymbol{d} = 0$	Maxwell solutions
		$(\mathrm{div} = 0)$	$\kappa = \omega\sqrt{\varepsilon\mu}$

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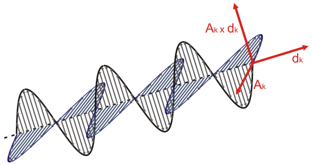

Basis (3D): $p = (q+1)^2$ directions $\{\boldsymbol{d}_\ell\}_{\ell=1}^{p}$

$\{\boldsymbol{a}_\ell\}_{\ell=1}^{p}$ unit vectors s.t. $\boldsymbol{a}_\ell \perp \boldsymbol{d}_\ell$

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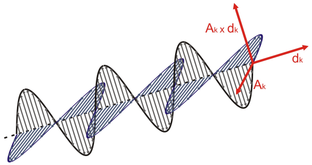

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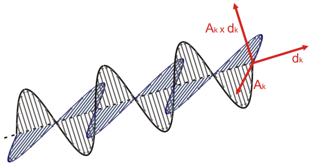

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(dimension $2(q+1)^2$)

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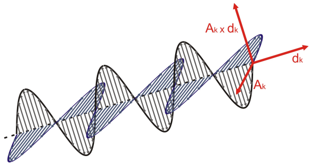

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- Spherical wave Maxwell-Trefftz spaces → [A. Moiola, PhD Thesis]

Error analysis

Theoretical analysis [Hiptmair, Moiola & Perugia, MCOM (2013)]

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	Ingredients	$\rightarrow$	Results
1)	suitable choice of numerical fluxes	(stabilisation)	unconditional well-posedness

Error analysis

Theoretical analysis

[Hiptmair, Moiola & Perugia, MCOM (2013)]

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1)  $=$  2)  $\neq$  3)  $\simeq$  but...

Error analysis: unconditional well-posedness

Trefftz-DG elemental formulation

For every $K \in \mathcal{T}_h$, $$\int_{\partial K} \mathbf{n} \times \widehat{\mathbf{E}} \cdot \left(\mu^{-1} \nabla \times \overline{\boldsymbol{v}}\right) + \int_{\partial K} \mathbf{n} \times (i\omega \widehat{\mathbf{H}}) \cdot \overline{\boldsymbol{v}} = 0$$

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Numerical fluxes on interior faces:

$$\widehat{\mathbf{E}} = \{\!\{\mathbf{E}\}\!\} - \frac{\beta}{i\omega} [\![\mu^{-1} \nabla_h \times \mathbf{E}]\!]_T$$

$$i\omega \widehat{\mathbf{H}} = \{\!\{\mu^{-1} \nabla_h \times \mathbf{E}\}\!\} + \alpha\, i\omega\, [\![\mathbf{E}]\!]_T \qquad \text{with } \alpha, \beta > 0$$

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On boundary faces:

$$\widehat{\mathbf{E}} = \mathbf{E} - \delta \vartheta^{-1} \left(\frac{1}{i\omega} \mathbf{n} \times (\mu^{-1} \nabla_h \times \mathbf{E}) + \vartheta (\mathbf{n} \times \mathbf{E}) \times \mathbf{n} + \frac{1}{i\omega} \mathbf{g} \right)$$

$$i\omega \widehat{\mathbf{H}} = \frac{1}{i\omega\mu} \nabla_h \times \mathbf{E} - (1-\delta) \left(\frac{1}{i\omega\mu} \nabla_h \times \mathbf{E} - \vartheta (\mathbf{n} \times \mathbf{E}) - \frac{1}{i\omega} \mathbf{n} \times \mathbf{g} \right) \qquad \delta \in [0,1]$$

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- The choice $\alpha = \beta = \delta = 1/2$ gives the UWVF by Cessenat & Després [Gabard, 2007], [Buffa & Monk, 2008], [Gittelson, Hiptmair & Perugia, 2009]

Error analysis: unconditional well-posedness

Trefftz-DG method

Find $\mathbf{E}_{hp} \in V_p(\mathcal{T}_h)$ such that $\quad \mathcal{A}_{hp}(\mathbf{E}_{hp}, \boldsymbol{v}_{hp}) = \ell_{hp}(\boldsymbol{v}_{hp}) \qquad \forall \boldsymbol{v}_{hp} \in V_p(\mathcal{T}_h)$

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In order to have coercivity, simplest possible choice of norm:

$$\|\boldsymbol{v}\|_{DG}^2 := |\mathrm{Im}\ [\mathcal{A}_{hp}(\boldsymbol{v}, \boldsymbol{v})]|$$

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Explicitly,

$$\begin{aligned}\|\boldsymbol{v}\|_{DG}^2 = {} & \omega^{-1}\|\beta^{1/2}[\![\mu^{-1}\nabla_h \times \boldsymbol{v}]\!]_T\|_{L^2(\mathcal{F}_h^I)^3}^2 + \omega\,\|\alpha^{1/2}[\![\boldsymbol{v}]\!]_T\|_{L^2(\mathcal{F}_h^I)^3}^2 \\ & + \omega^{-1}\|\delta^{1/2}\vartheta^{-1/2}\mathbf{n} \times (\mu^{-1}\nabla_h \times \boldsymbol{v})\|_{L^2(\mathcal{F}_h^B)^3}^2 + \omega\,\|(1-\delta)^{1/2}\vartheta^{1/2}(\mathbf{n} \times \boldsymbol{v})\|_{L^2(\mathcal{F}_h^B)^3}^2\end{aligned}$$

which is actually a norm on $T(\mathcal{T}_h)$

Error analysis: unconditional well-posedness

Continuity + coercivity of $\mathcal{A}_{hp}(\cdot,\cdot) \Rightarrow$

Well-posedness and quasi-optimality in DG–norm (for any value of k and h)

$$\|\mathbf{E} - \mathbf{E}_{hp}\|_{DG} \leq 3 \inf_{\boldsymbol{v}_{hp} \in V_p(\mathcal{T}_h)} \|\mathbf{E} - \boldsymbol{v}_{hp}\|_{DG^+}$$

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Remark: Local Trefftz property implies local divergence-free property:

$$\nabla\times(\mu^{-1}\nabla\times\mathbf{E}_{hp})-\omega^2\varepsilon\,\mathbf{E}_{hp}=0 \quad\Rightarrow\quad \nabla\cdot(\varepsilon\,\mathbf{E}_{hp})=0 \qquad \text{in all } K\in\mathcal{T}_h$$

On the other hand, $\|\cdot\|_{DG}$ does not provide control on the normal jumps and traces $\to$ no divergence-free property for $\mathbf{E}_{hp}$ can be expected

Error analysis: error estimates in a mesh-independent norm

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 $\neq$ 

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Error bound in DG–norm $\xrightarrow{\text{duality}}$ error bound in mesh-independent norm

$$\mathbf{w} := \mathbf{E} - \mathbf{E}_{hp} = \mathbf{w}_0 + \nabla p \qquad \text{with} \quad \mathbf{w}_0 \in H(\mathrm{div}^0;\Omega), \quad p \in H_0^1(\Omega)$$

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- Estimate of $\mathbf{w}_0$ in $L^2(\Omega)^3$: modified duality argument [Monk & Wang, 1999]

Error analysis: error estimates in a mesh-independent norm

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Error analysis: error estimates in a mesh-independent norm

Error bound in DG–norm $\overset{\text{duality}}{\longrightarrow}$ error bound in mesh-independent norm

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$$\|\nabla p\|_{H(\mathrm{div};\Omega)'} := \sup_{\boldsymbol{v}\in H(\mathrm{div};\Omega)} \frac{(\nabla p, \boldsymbol{v})_\Omega}{\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)}} \le C_1(\Omega,\omega,h)\,\|\mathbf{w}\|_{DG} \quad \forall \mathbf{w} \in T(\mathcal{T}_h)$$

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For the duality argument:

- regularity/stability of solutions to the adjoint problem with div-free rhs
  - star-shaped polyhedral domains
  - constant coefficients

[Hiptmair, Moiola & Perugia, M^3AS (2011)]

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▶ Scattering problems ?

Error analysis: error estimates in a mesh-independent norm

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Regularity/stability results in star-shaped polyhedra

The solution $\mathbf{\Phi}$ to the Maxwell adjoint problem with rhs $\mathbf{w}_0 \in H(\mathrm{div}^0; \Omega)$ satisfies: $\mathbf{\Phi}, \nabla \times \mathbf{\Phi} \in H^{\frac{1}{2}+s}(\Omega)^3$, $0 < s < 1/2$, and

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Error analysis: error estimates in a mesh-independent norm

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- Extension to time-harmonic Maxwell of previous results proved for Helmholtz ([Melenk, 1995], [Cummings & Feng, 2006], [Hetmaniuk, 2007])

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Final estimate

$$\mathbf{w} := \mathbf{E} - \mathbf{E}_{hp} = \mathbf{w}_0 + \nabla p \qquad \text{with} \quad \mathbf{w}_0 \in H(\mathrm{div}^0;\Omega), \quad p \in H_0^1(\Omega)$$

$$\begin{aligned}
\|\mathbf{w}_0\|_{L^2(\Omega)^3} &\leq C_0(\Omega,\omega,h)\,\|\mathbf{w}\|_{DG} = C_0(\Omega,\omega,h)\,\|\mathbf{E}-\mathbf{E}_{hp}\|_{DG} \\
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$+$ quasi-optimal estimate of $\|\mathbf{E} - \mathbf{E}_{hp}\|_{DG}$

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Error estimate in $H(\mathrm{div};\Omega)'$-norm

For solutions $\mathbf{E} \in H^{1/2+\sigma}(\mathrm{curl};\Omega)$, $\sigma > 0$ (datum $\mathbf{g} \in H^{\tau}(\partial\Omega)$, $\tau > 0$),

$$\|\mathbf{E}-\mathbf{E}_{hp}\|_{H(\mathrm{div};\Omega)'} \leq C(\Omega,\omega,h) \underbrace{\inf_{\boldsymbol{v}_{hp} \in V_p(\mathcal{T}_h)} \|\mathbf{E} - \boldsymbol{v}_{hp}\|_{DG^+}}_{\text{best approximation error}}$$

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  $\rightarrow$ sharp best approximation estimates in weighted Sobolev norms
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Vekua's theory for a 2nd order elliptic operator $\mathcal{L}$

$$\begin{tikzcd}
\text{harmonic functions} \arrow[d, "\text{appr}"] \arrow[r, shift right, "V_1"'] & \{v : \mathcal{L}v = 0\} \arrow[l, shift right, "V_2"'] \arrow[d, "\text{appr}"] \\
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▶ Sharp best approximation estimates of Maxwell solutions in $V_p(\mathcal{T}_h)$?

Summary

Extension of the analysis framework for Trefftz-DG methods developed for the Helmholtz problem to the time-harmonic Maxwell's equations

- unconditional well-posedness and quasi-optimality in mesh-dependent norm
- error analysis in a mesh-independent norm
- *hp*-convergence rates derived from best approximation error estimates of Maxwell solutions in Maxwell-Trefftz spaces

Open issues

- sharp best approximation error estimates
- extension to scattering problems: stability/regularity results in non star-shaped domains.