Generalizing the signature to systems with multiple types of components*

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Abstract The concept of signature was introduced to simplify quantification of reliability for coherent systems and networks consisting of a single type of components, and for comparison of such systems' reliabilities. The signature describes the structure of the system and can be combined with order statistics of the component failure times to derive inferences on the reliability of a system and to compare multiple systems. However, the restriction to use for systems with a single type of component prevents its application to most practical systems. We discuss the difficulty of generalization of the signature to systems with multiple types of components. We present an alternative, called the survival signature, which has similar characteristics and is closely related to the signature. The survival signature provides a feasible generalization to systems with multiple types of components.

1 Introduction

Theory of signatures was introduced as an attractive tool for quantification of reliability of coherent systems and networks consisting of components with random failure times that are independent and identically distributed (*iid*), which can be regarded informally as components of 'a single type'. Samaniego [14] provides an excellent introduction and overview to the theory[2]. The main idea of the use of signatures is separation of aspects of the components' failure

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[2] Samaniego [14] assumes *iid* failure times of components, which we follow in this paper; the theory of signatures applies also under the weaker assumption of exchangeability [9]

time distribution and the structure of the system, with the latter quantified through the signature. Let us first introduce some notation and concepts.

For a system with m components, let state vector $\underline{x} = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$, with $x_i = 1$ if the ith component functions and $x_i = 0$ if not. The labelling of the components is arbitrary but must be fixed to define $\underline{x}$. The structure function $\phi : \{0,1\}^m \to \{0,1\}$, defined for all possible $\underline{x}$, takes the value 1 if the system functions and 0 if the system does not function for state vector $\underline{x}$. We restrict attention to coherent systems, which means that $\phi(\underline{x})$ is not decreasing in any of the components of x, so system functioning cannot be improved by worse performance of one or more of its components[3]. We further assume that $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$, so the system fails if all its components fail and it functions if all its components function[4].

Let $T_S > 0$ be the random failure time of the system and $T_{j:m}$ the j-th order statistic of the m random component failure times for $j = 1, \ldots, m$, with $T_{1:m} \leq T_{2:m} \leq \ldots \leq T_{m:m}$. We assume that these component failure times are independent and identically distributed. The system's signature is the m-vector q with j-th component

$$q_j = P(T_S = T_{j:m}) \tag{1}$$

so q_j is the probability that the system failure occurs at the moment of the j-th component failure. Assume that $\sum_{j=1}^{m} q_j = 1$; this assumption implies that the system functions if all components function, has failed if all components have failed, and that system failure can only occur at times of component failures. The signature provides a qualitative description of the system structure that can be used in reliability quantification [14]. The survival function of the system failure time can be derived by

$$P(T_S > t) = \sum_{j=1}^{m} q_j P(T_{j:m} > t) \tag{2}$$

If the failure time distribution for the components is known and has cumulative distribution function (CDF) $F(t)$, then

$$P(T_{j:m} > t) = \sum_{r=m-j+1}^{m} \binom{m}{r} [1 - F(t)]^r [F(t)]^{m-r} \tag{3}$$

The expected value of the system failure time can be derived as function of the expected values of the order statistics $T_{j:m}$, by

[3] This assumption could be relaxed but is reasonable for many real-world systems.

[4] This assumption could be relaxed but for coherent systems it would lead to the trivial cases of systems that either always fail or always function.

$$E(T_S) = \sum_{j=1}^{m} q_j E(T_{j:m}) \tag{4}$$

From these expressions it is clear that the system structure is fully taken into account through the signature and is separated from the information about the random failure times of the components. This enables e.g. straightforward comparison of the reliability of two systems with the same single type of components if their signatures are stochastically ordered [14]. Consider two systems, each with m components and all failure times of the $2m$ components assumed to be *iid*. Let the signature of system A be q^a and of system B be q^b, and let their failure times be T^a and T^b, respectively. If

$$\sum_{j=r}^{m} q_j^a \geq \sum_{j=r}^{m} q_j^b \tag{5}$$

for all $r = 1, \ldots, m$, then

$$P(T^a > t) \geq P(T^b > t) \tag{6}$$

for all $t > 0$. Such a comparison is even possible if the two systems do not have the same number of components, as one can increase the length of a system signature in a way that does not affect the corresponding system's failure time distribution [14], so one can always make the two systems' signatures of the same length. Consider a system with m components, signature $q = (q_1, q_2, \ldots, q_m)$ and failure time T_S. For ease of notation, define $q_0 = q_{m+1} = 0$. Now define a signature q^* as the vector with $m+1$ components

$$q_j^* = \frac{j-1}{m+1} q_{j-1} + \frac{m+1-j}{m+1} q_j \tag{7}$$

for $j = 1, \ldots, m+1$, and define a random failure time T_S^* with probability distribution defined by the survival function

$$P(T_S^* > t) = \sum_{j=1}^{m+1} q_j^* P(T_{j:m+1} > t) \tag{8}$$

with

$$P(T_{j:m+1} > t) = \sum_{r=m-j+2}^{m+1} \binom{m+1}{r} [1 - F(t)]^r [F(t)]^{m+1-r} \tag{9}$$

Then the probability distributions of T_S and T_S^* are identical, so

$$P(T_S > t) = P(T_S^* > t) \tag{10}$$

for all $t > 0$. Note that the signatures q and q^* represent systems with m and $m+1$ components with *iid* failure times and CDF $F(t)$, respectively. Note further that q^* may not actually correspond to a physical system structure, but applying this extension (consecutively) enables two systems with different numbers of components to be compared.

Many systems' structures do not have corresponding signatures which are stochastically ordered. For example, the signatures $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and $(0, \frac{2}{3}, \frac{1}{3}, 0)$, which relate to basic system structures, are not stochastically ordered. An attractive way to compare such systems' failure times, say T^a and T^b, is by considering the event $T^a < T^b$. To get more detailed insight into the difference between the two systems' failure times, one can consider the event $T^a < T^b + \delta$ as function of δ [4]. This way to compare two systems' failure times does not require the failure times of the components in one system to be *iid* with the failure times of the components in the other system. Let system A consist of m_a components with *iid* failure times, and system B of m_b components with *iid* failure times, with components of the different systems being of different types and their random failure times assumed to be fully independent, which means that any information about the failure times of components of the type used in system A does not contain any information about the failure times of components of the type used in system B. The ordered random failure times of the components in system A and of those in system B are denoted by $T^a_{1:m_a} \leq T^a_{2:m_a} \leq \ldots \leq T^a_{m_a:m_a}$ and $T^b_{1:m_b} \leq T^b_{2:m_b} \leq \ldots \leq T^b_{m_b:m_b}$, respectively. Let the failure time distribution for components in system A have CDF F_a and for components in system B have CDF F_b. Using the signatures q^a and q^b of these systems, the probability for the event $T^a < T^b$ is

$$P(T^a < T^b) = \sum_{i=1}^{m_a} \sum_{j=1}^{m_b} q_i^a q_j^b P(T^a_{i:m_a} < T^b_{j:m_b}) \tag{11}$$

with

$$P(T^a_{i:m_a} < T^b_{j:m_b}) = \int_0^\infty f_a^i(t) P(T^b_{j:m_b} > t) dt \tag{12}$$

where $f_a^i(t)$ is the probability density function (PDF) for $T^a_{i:m_a}$, which, with PDF $f_a(t)$ for the failure time of components of in system A, is equal to

$$f_a^i(t) = f_a(t) \sum_{r_a = m_a - i + 1}^{m_a} \binom{m_a}{r_a} [1 - F_a(t)]^{r_a - 1} [F_a(t)]^{m_a - r_a - 1} [r_a - m_a(1 - F_a(t))] \tag{13}$$

and

$$P(T^b_{j:m_b} > t) = \sum_{r_b = m_b - j + 1}^{m_b} \binom{m_b}{r_b} [1 - F_b(t)]^{r_b} [F_b(t)]^{m_b - r_b} \tag{14}$$

In Section 2 of this paper an alternative to the signature is proposed and investigated, we call it the system's survival signature. In Section 3 we discuss the difficulties in generalizing the signature to systems with multiple types of components, where the proposed survival signature appears to have a clear advantage which suggests an interesting and important new area of research. We end the paper with a concluding discussion in Section 4, where we briefly comment on computation of the system signature and survival signature, the possibility to use partial information, the generalization to theory of imprecise reliability in case the components' failure time distribution is not precisely known [5, 6], and the possible use of these concepts if only failure times from tested components are available and nonparametric predictive statistical methods are used for inference on the system reliability [2].

2 The survival signature

The signature was introduced to assist reliability analyses for systems with m components with *iid* failure times, with the specific role of modelling the structure of the system and separating this from the random failure times of the components. In this section, we consider an alternative to the signature which can fulfill a similar role, and which actually is closely related to the signature. Let $\Phi(l)$, for $l = 1, \ldots, m$, denote the probability that a system functions given that *precisely* l of its components function. As in Section 1, we restrict attention again to coherent systems, for which $\Phi(l)$ is an increasing function of l, and we assume that $\Phi(0) = 0$ and $\Phi(m) = 1$. There are $\binom{m}{l}$ state vectors $\underline{x}$ with precisely l components $x_i = 1$, so with $\sum_{i=1}^{m} x_i = l$; we denote the set of these state vectors by S_l. Due to the *iid* assumption for the failure times of the m components, all these state vectors are equally likely to occur, hence

$$\Phi(l) = \binom{m}{l}^{-1} \sum_{\underline{x} \in S_l} \phi(\underline{x}) \tag{15}$$

As the function $\Phi(l)$ is by its definition related to survival of the system, and, as we will see later, it is close in nature to the system signature, we call it the system survival signature.

Let $C_t \in \{0, 1, \ldots, m\}$ denote the number of components in the system that function at time $t > 0$. If the probability distribution of the component failure time has CDF $F(t)$, then for $l \in \{0, 1, \ldots, m\}$

$$P(C_t = l) = \binom{m}{l} [F(t)]^{m-l} [1 - F(t)]^l \tag{16}$$

It follows easily that

$$P(T_S > t) = \sum_{l=0}^{m} \Phi(l) P(C_t = l) \tag{17}$$

The terms in the right-hand side of equation (17) explicitly have different roles, with the term $\Phi(l)$ taking the structure of the system into account, that is how the system's functioning depends on the functioning of its components, and the term $P(C_t = l)$ taking the random failure times of the components into account. Taking these two crucial aspects for determining the survival function for the system failure time into account separately in this way is similar in nature to the use of system signatures as discussed in Section 1.

Equations (2) and (17) imply

$$\Phi(l) = \sum_{j=m-l+1}^{m} q_j \tag{18}$$

which is easily verified by

$$\sum_{j=1}^{m} \sum_{r=m-j+1}^{m} q_j \binom{m}{r} [1-F(t)]^r [F(t)]^{m-r} =$$
$$\sum_{r=1}^{m} \sum_{j=m-r+1}^{m} q_j \binom{m}{r} [1-F(t)]^r [F(t)]^{m-r} \tag{19}$$

Equation (18) is logical when considering that the right-hand side is the probability that the system failure time occurs at the moment of the $(m-l+1)$-th ordered component failure time or later. The moment of the $(m-l+1)$-th ordered component failure time is exactly the moment at which the number of functioning components in the system decreases from l to $l-1$, hence the system would have functioned with l components functioning.

In Section 1 a straightforward comparison was given of the failure times of two systems A and B. Let us denote the survival signatures of these systems by $\Phi^a(l)$ and $\Phi^b(l)$, respectively, and assume that both systems consist of m components and that all these $2m$ components are of the same type, so have *iid* random failure times. The comparison in Section 1 was based on the stochastic ordering of the systems' signatures, if indeed these are stochastically ordered. Due to the relation (18) between the survival signature $\Phi(l)$ and the signature q, this comparison can also be formulated as follows: If

$$\Phi^a(l) \geq \Phi^b(l) \tag{20}$$

for all $l = 1, \ldots, m$, then

$$P(T^a > t) \geq P(T^b > t) \tag{21}$$

for all $t > 0$.

As explained in Section 1, the possibility to extend a signature in a way that retains the same system failure time distribution is an advantage for comparison of different system structures. The survival signature $\Phi(l)$ can be similarly extended as is shown next, by defining explicitly the survival signature $\Phi^*(l)$ that relates to a system with $m+1$ components which has the same failure time distribution as a system with m components and survival signature $\Phi(l)$ (throughout the superscript * indicates the system with $m+1$ components, and all components considered are assumed to have *iid* failure times). For $l = 1, \ldots, m+1$, let

$$\Phi^*(l) = \Phi(l-1) + \frac{m-l-1}{m+1} q_{m-l-1} \tag{22}$$

and from (18), (22) and $\Phi(0) = 0$ we have

$$\Phi^*(1) = \frac{m}{m+1} q_m = \frac{m}{m+1} \Phi(1) \tag{23}$$

and

$$\Phi^*(m+1) = \Phi(m) = 1 \tag{24}$$

and

$$\Phi(l+1) = \Phi(l) + q_{m-l} \tag{25}$$

Furthermore, it is easy to prove that

$$P(C_t = l) = \frac{m+1-l}{m+1} P(C_t^* = l) + \frac{l+1}{m+1} P(C_t^* = l+1) \tag{26}$$

The failure time T_S^* of the extended system with $m+1$ components and survival signature $\Phi^*(l)$ has the same survival function, and hence the same probability distribution, as the failure time T_S of the original system with m components and survival signature $\Phi(l)$. This is proven as follows

$$
\begin{aligned}
P(T_S > t) &= \sum_{l=1}^{m} \Phi(l) P(C_t = l) \\
&= \sum_{l=1}^{m} \Phi(l) \left[\frac{m+1-l}{m+1} P(C_t^* = l) + \frac{l+1}{m+1} P(C_t^* = l+1) \right] \\
&= \Phi^*(1) P(C_t^* = 1) + \sum_{l=1}^{m-1} \left[\Phi(l+1) \frac{m-l}{m+1} + \Phi(l) \frac{l+1}{m+1} \right] P(C_t^* = l+1) \\
&\quad + \Phi^*(m+1) P(C_t^* = m+1) \\
&= \Phi^*(1) P(C_t^* = 1) \\
&\quad + \sum_{l=1}^{m-1} \Phi^*(l+1) P(C_t^* = l+1) + \Phi^*(m+1) P(C_t^* = m+1) \\
&= \sum_{l=1}^{m+1} \Phi^*(l) P(C_t^* = l) = P(T_S^* > t)
\end{aligned}
\tag{27}
$$

Comparison of the failure times of two systems A and B, each with a single type of components but these being different for the two systems, with the use of signatures, was given in Equation (11). This comparison is also possible with the use of the survival signatures, which we denote by $\Phi^a(l_a)$ and $\Phi^b(l_b)$ for systems A and B, respectively. The result is as follows

$$
P(T^a < T^b) = \int_0^\infty f_S^a(t) P(T^b > t) dt \tag{28}
$$

with $f_S^a(t)$ the PDF of T^a, given by

$$
f_S^a(t) = f_a(t) \sum_{l_a=0}^{m_a} \Phi^a(l_a) \binom{m_a}{l_a} [1-F_a(t)]^{l_a-1} [F_a(t)]^{m_a-l_a-1} [l_a - m_a(1-F_a(t))] \tag{29}
$$

and

$$
P(T^b > t) = \sum_{l_b=0}^{m_b} \Phi^b(l_b) \binom{m_b}{l_b} [1-F_b(t)]^{l_b} [F_b(t)]^{m_b-l_b} \tag{30}
$$

Using relation (18) and change of order of summation as in (19), it is easy to show that (28) is actually the same formula as (11), so there is no computational difference in the use of either the signature or the survival signature for such comparison of two systems with each a single type of components. We can conclude that the method using the survival signature as presented in this section is very similar in nature to the method using signatures for systems with components with *iid* failure times. In Section 3 we consider the generalization of the signature and the survival signature to the very important case of reliability inferences for systems with multiple types of components.

3 Systems with multiple types of component

Most practical systems and networks for which the reliability is investigated consist of multiple types of components. Therefore, a main challenge is generalization of the theory of signatures to such systems. Although an obvious challenge for research, little if any mention of it has been made in the literature. We will consider whether or not it is feasible to generalize the standard concept of the signature to systems with multiple types of components, and we will also consider this for the survival signature.

We consider a system with $K \geq 2$ types of components, with m_k components of type $k \in \{1, 2, \ldots, K\}$ and $\sum_{k=1}^{K} m_k = m$. We assume that the random failure times of components of the same type are *iid*, while full independence is assumed for the random failure times of components of different types. Due to the arbitrary ordering of the components in the state vector, we can group components of the same type together, so we use state vector $\underline{x} = (\underline{x}^1, \underline{x}^2, \ldots, \underline{x}^K)$ with $\underline{x}^k = (x_1^k, x_2^k, \ldots, x_{m_k}^k)$ the sub-vector representing the states of the components of type k. We denote the ordered random failure times of the m_k components of type k by $T_{j_k:m_k}^k$, for ease of presentation we assume that ties between failure times have probability zero.

System signatures were introduced explicitly for systems with a single type of components, which are assumed to have *iid* failure times. To generalize the signature approach to multiple types of components, it will be required to take into account, at the moment of system failure T_S which coincides with the failure of a specific component, how many of the components of each other type have failed. The generalized signature can again be defined in quite a straightforward manner, namely by defining

$$q_k(j_k) = P(T_S = T_{j_k:m_k}^k) \tag{31}$$

for $k = 1, \ldots, K$ and $j_k = 1, \ldots, m_k$, so the total signature can be defined as

$$q = (q_1(1), \ldots, q_1(m_1), q_2(1), \ldots, q_2(m_2), \ldots, q_K(1), \ldots, q_K(m_K)) \tag{32}$$

With this definition, the survival function of the system's failure time is

$$P(T_S > t) = \sum_{k=1}^{K} \sum_{j_k=1}^{m_k} q_k(j_k) P(T_{j_k:m_k}^k > t) \tag{33}$$

However, deriving this generalized signature is complex and actually depends on the failure time probability distributions of the different types of components, hence this method does not any longer achieve the separation of the system structure and the failure time distributions as was the case for a single type of components in Section 1. To illustrate this, we consider the calculation of q for the case with $K = 2$ types of components in the system. For ease of notation, let $T_{0:m_2}^2 = 0$ and $T_{m_2+1:m_2}^2 = \infty$. Calculation of $q_1(j_1)$ is possible by

$$q_1(j_1) = P(T_S = T^1_{j_1:m_1}) = \sum_{j_2=0}^{m_2} \left[P(T_S = T^1_{j_1:m_1} \mid T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2}) \times P(T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2}) \right] \tag{34}$$

Derivation of the terms $P(T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2})$ involves the failure time distributions of both component types. It is possible to define the generalized signature instead by the first term in this sum, so the conditional probability of $T_S = T^1_{j_1:m_1}$ given the number of components of type 2 that are functioning at time $T^1_{j_1:m_1}$, but this does not simplify things as the probabilities $P(T^2_{j_2:m_2} < T^1_{j_1:m_1} < T^2_{j_2+1:m_2})$ will still be required for all $j_1 \in \{1,\ldots,m_1\}$ and $j_2 \in \{1,\ldots,m_2\}$, and as these probabilities involve order statistics from different probability distributions this is far from straightforward. Of course, for the general case with a system consisting of $K \geq 2$ types of components, the arguments are the same but the complexity increases as function of K. Calculating the system reliability via this generalized signature involves the calculation of m terms $q_k(j_k)$, each of which requires

$$\prod_{\substack{l=1\\ l\neq j}}^{K} (m_l + 1) \tag{35}$$

probabilities for orderings of order statistics from different probability distributions to be derived. This quickly becomes infeasible, which is probably the reason why such a generalization of the signature has not been addressed in detail in the literature. It may also explain why the signature, although a popular topic in the reliability research literature, does not appear to have made a substantial impact on practical reliability analyses.

We will now investigate if the survival signature, as presented in Section 2, may be better suited for the generalization to systems with multiple types of components. Let $\Phi(l_1, l_2, \ldots, l_K)$, for $l_k = 0, 1, \ldots, m_k$, denote the probability that a system functions given that *precisely* l_k of its components of type k function, for each $k \in \{1, 2, \ldots, K\}$; again we call this function $\Phi(l_1, l_2, \ldots, l_K)$ the system's survival signature, it will be clear from the context whether or not there are multiple types of components. There are $\binom{m_k}{l_k}$ state vectors $\underline{x}^k$ with precisely l_k of its m_k components $x^k_i = 1$, so with $\sum_{i=1}^{m_k} x^k_i = l_k$; we denote the set of these state vectors for components of type k by S^k_l. Furthermore, let $S_{l_1,\ldots,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x^k_i = l_k$, $k = 1, 2, \ldots, K$. Due to the *iid* assumption for the failure times of the m_k components of type k, all the state vectors $\underline{x}^k \in S^k_l$ are equally likely to occur, hence

$$\Phi(l_1,\ldots,l_K) = \left[\prod_{k=1}^{K}\binom{m_k}{l_k}^{-1}\right] \times \sum_{\underline{x}\in S_{l_1,\ldots,l_K}} \phi(\underline{x}) \tag{36}$$

Let $C_t^k \in \{0,1,\ldots,m_k\}$ denote the number of components of type k in the system that function at time $t > 0$. If the probability distribution for the failure time of components of type k is known and has CDF $F_k(t)$, then for $l_k \in \{0,1,\ldots,m_k\}$, $k = 1,2,\ldots,K$,

$$\begin{aligned} P(\bigcap_{k=1,\ldots,K}\{C_t^k = l_k\}) &= \prod_{k=1}^{K} P(C_t^k = l_k) \\ &= \prod_{k=1}^{K}\left(\binom{m_k}{l_k}[F_k(t)]^{m_k-l_k}[1-F_k(t)]^{l_k}\right) \end{aligned} \tag{37}$$

The probability that the system functions at time $t > 0$ is

$$\begin{aligned} &P(T_S > t) = \sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K} \Phi(l_1,\ldots,l_K) P(\bigcap_{k=1}^{K}\{C_t^k = l_k\}) = \\ &\sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}\left[\Phi(l_1,\ldots,l_K)\prod_{k=1}^{K}P(C_t^k = l_k)\right] = \\ &\sum_{l_1=0}^{m_1}\cdots\sum_{l_K=0}^{m_K}\left[\Phi(l_1,\ldots,l_K)\prod_{k=1}^{K}\left(\binom{m_k}{l_k}[F_k(t)]^{m_k-l_k}[1-F_k(t)]^{l_k}\right)\right] \end{aligned} \tag{38}$$

Calculation of (38) is not straightforward but it is far easier than calculation using the generalized signature in (33). Of course, the survival signature $\Phi(l_1,\ldots,l_K)$ needs to be calculated for all $\prod_{k=1}^{K}(m_k+1)$ different $(l_1,\ldots,l_K)$, but this information must be distracted from the system anyhow and is only required to be calculated once for any system, similar to the (survival) signature for systems with a single type of components. The main advantage of (38) is that again the information about system structure is fully separated from the information about the components' failure times, and the inclusion of the failure time distributions is straightforward due to the assumed independence of failure times of components of different types. The difficulty in (33) of having to find probabilities of rankings of order statistics from different probability distributions is now avoided, which leads to a very substantial reduction and indeed simplification of the computational effort.

We can conclude that the survival signature, which is proposed in this paper and which is very closely related to the classical signature in case of systems with components with *iid* failure times, seems to provide an attractive way to generalize the concept of signature to systems with multiple types of components. It should be emphasized that the survival signature provides all that is needed to calculate the survival function for the system's failure time, and as this fully determines the failure time's probability distribution all further inferences of interest can be addressed using the survival signature. While we have assumed independence of the failure times of components of different types, the proposed approach in this paper can also be used if these failure times are dependent, in which case the joint probability distribution for these failure times must of course be used in Equation (37). This would still maintain the main feature of the use of the proposed survival signature in Equation (38), namely the explicit separation of the factors taking into account the information about the system structure and the information about the component failure times. Also with the less attractive generalization of the classical signature it is possible to deal with dependent failure times for components of different types, but it is likely to substantially complicate the computation of probabilities on the orderings of order statistics for failure times of components of different types.

Theoretical properties of the survival signature for systems with multiple types of components are an important topic for future research. This should include analysis of possibilities to extend such a signature while keeping the corresponding systems' failure times distributions the same, which could possibly be used with some adapted form of stochastic monotonicity (on the subvectors relating to components of the same type) for comparison of failure times of systems that share the same multiple types of components. It seems possible to compare the failure times of different systems with multiple types of components using the survival signature along the lines as presented in Section 2 for systems with components with *iid* failure times, but this should also be developed in detail. The first results for this survival signature, as presented in this paper, are very promising, particularly due to the possibility to use the survival signature for systems with multiple types of components, so such further research will be of interest.

We briefly illustrate the use of the survival signature for a system with $K = 2$ types of components, types 1 and 2, as presented in Figure 1 (where the types of the components, 1 or 2, are as indicated).

With $m_1 = m_2 = 3$ components of each type, the survival signature $\Phi(l_1, l_2)$ must be specified for all $l_1, l_2 \in \{0, 1, 2, 3\}$; this is given in Table 1. To illustrate its derivation, let us consider $\Phi(1, 2)$ and $\Phi(2, 2)$ in detail. The state vector is $\underline{x} = (x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2)$, where we order the three components of type 1 from left to right in Figure 1, and similar for the three components of type 2. To calculate $\Phi(1, 2)$, we consider all such vectors $\underline{x}$ with $x_1^1 + x_2^1 + x_3^1 = 1$ and $x_1^2 + x_2^2 + x_3^2 = 2$, so precisely 1 component of type 1 and 2 components of type 2 function. There are 9 such vectors, for only one of these, namely $(1, 0, 0, 1, 0, 1)$, the system functions. Due to the *iid* assumption for the failure times of components of the same type, and independence between components of different types, all these 9 vectors have equal probability to occur, hence $\Phi(1, 2) = 1/9$. To calculate $\Phi(2, 2)$ we need to check all 9 vectors $\underline{x}$ with $x_1^1 + x_2^1 + x_3^1 = 2$ and $x_1^2 + x_2^2 + x_3^2 = 2$. For 4 of these vectors the system functions, namely $(1,1,0,1,0,1)$, $(1,1,0,0,1,1)$, $(1,0,1,1,1,0)$ and $(1,0,1,1,0,1)$, so $\Phi(2,2) = 4/9$.

We consider two cases with regard to the failure time distributions for the components. In Case A, we assume that the failure times of components of type 1 have an Exponential distribution with expected value 1, so with

$$f_1(t) = e^{-t} \quad \text{and} \quad F_1(t) = 1 - e^{-t} \tag{39}$$

and that the failure times of components of type 2 have a Weibull distribution with shape parameter 2 and scale parameter 1, so with

$$f_2(t) = 2te^{-t^2} \quad \text{and} \quad F_2(t) = 1 - e^{-t^2} \tag{40}$$

In Case B, these same probability distributions are used but for the other components type than in Case A, so the failure times of components of type 1 have the above Weibull distribution while the failure times of components of type 2 have the above Exponential distribution.

The survival functions for the failure time of this system, for both Cases A and B, are calculated using Equation (38) and are presented in Figure 2. Type 1 components are a bit more critical in this system, due to the left-most component in Figure 1. The Exponential distribution makes early failures more likely than the Weibull distribution used in this example, which leads initially to lower survival function for Case A than for Case B. It is intesting that these two survival functions cross, it would have been hard to predict this without the detailed computations.

Whilst presenting these survival functions is in itself not of major interest without an explicit practical problem being considered, the fact that the computations based on Equation (38) are straightforward indicates that the survival signature can also be used for larger and more complicated systems. Of course, deriving the survival signature itself is not easy for larger systems, this is briefly addressed in the following section.

l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_2	$\Phi(l_1,l_2)$
0	0	0	2	0	0
0	1	0	2	1	0
0	2	0	2	2	4/9
0	3	0	2	3	6/9
1	0	0	3	0	1
1	1	0	3	1	1
1	2	1/9	3	2	1
1	3	3/9	3	3	1

Table 1: Survival signature of system in Figure 1

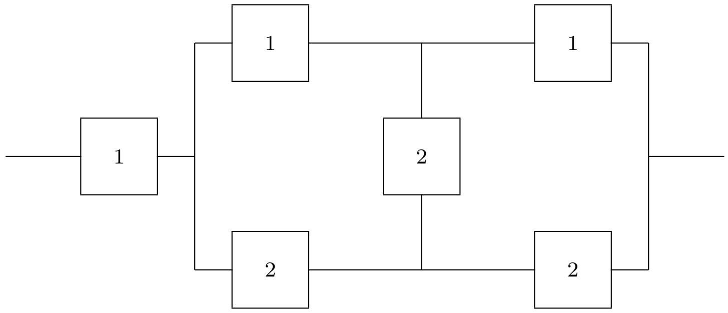

Fig. 1: System with 2 types of components

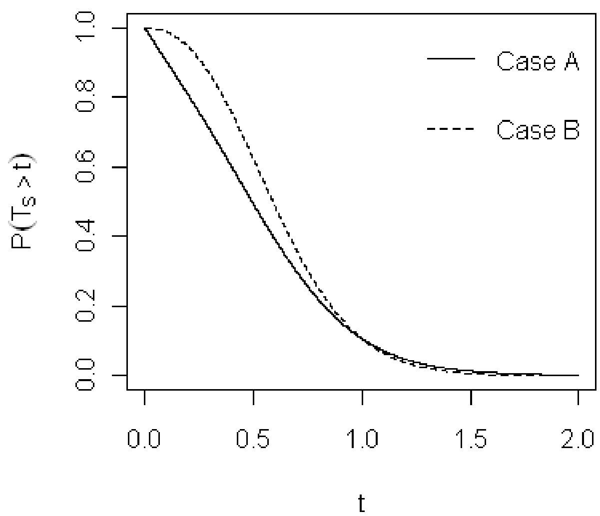

Fig. 2: System survival functions Cases A and B

4 Discussion

Computation of the signature of a system with components with *iid* failure times is difficult unless the number of components is small or the system structure is quite trivial [10, 14], and the same holds for the survival signature introduced in this paper. However, if derivation of the exact survival function for the system's failure time is required then it is unavoidable that all details of the system's structure must be taken into account, hence computation of the signature or the survival signature is necessary. For specific inferences of interest, e.g. if one wants to assess whether or not the system's reliability at a specific time t exceeds a specific value, computation of the exact (survival) signature may not be required. If one has partial information about the signature, then optimal bounds for $P(T_S > t)$ can be derived easily using the most 'pessimistic' and 'optimistic' signatures that are possible given the partial information [1]. Partial information on the survival signature is also quite straightforward to use, due to the fact that $\Phi(l)$ is increasing in l so corresponding bounds for $P(T_S > t)$ are easy to derive. This also holds for the case with multiple types of components, as $\Phi(l_1, \ldots, l_K)$ is increasing in each l_k for $k \in \{1, \ldots, K\}$. Due to the far more complex nature of the generalization of the classical signature to systems with multiple types of components, it is likely that they would be less suited for dealing with partial information, it seems of little interest to investigate this further. If such bounds for $P(T_S > t)$ are already conclusive for a specific inference, then there is no need for further computations which might reduce the effort considerably. In many applications one may not know the precise probability distribution for the components of a system. One way to deal with lack of such exact knowledge is the use of a set of probability distributions, in line with the theory of imprecise reliability [6] which uses lower and upper probabilities [5] for uncertainty quantification. Generalizing the use of (survival) signatures in order to use sets of probability distributions for the components' failure times is difficult as derivation of the optimal corresponding bounds for the survival function for the system's failure time involves multiple related constrained optimisation problems. One may also wish to use a statistical framework for inference on the survival function for the components' failure times. A particularly attractive way to do this is by using nonparametric predictive inference (NPI) [2], as it actually leads to relatively straightforward calculation of the optimal bounds for the survival function for the system's failure time, because the multiple optimisation problems involved can all be solved individually (which is rather trivial in that setting) and their optima can be attained simultaneously [4]. Currently, the generalization to systems with multiple types of components, as presented in this paper using the survival signature, is being investigated within the NPI framework. We expect to report exciting results from this research in the near future, which will generalize recent results on NPI for system reliability [3, 8, 11] to more general systems.

With a suitable generalization of the signature to systems with multiple types of components, as we believe the survival signature to be, there are many challenges for future research. For example, one may wish to decide on optimal testing in order to demonstrate a required level of system reliability, possibly taking costs and time required for testing, and corresponding constraints, into account [13]. It will also be of interest to consider possible system failure due to competing risks, where the NPI approach provides interesting new opportunities to consider unobserved or even unknown failure modes [7, 12]. Of course, the main challenges will result from the application of the new theory to large-scale real-world systems, which we expect to be more feasible with the new results presented in this paper.

References

1. Al-nefaiee AH, Coolen FPA (2012) Nonparametric prediction of system failure time using partially known signatures. Proceedings Statistical Models and Methods for Reliability and Survival Analysis and their Validation, Bordeaux (July 2012), to appear.
2. Coolen FPA (2011) Nonparametric predictive inference. In: International Encyclopedia of Statistical Science; Lovric (Ed.). Springer, Berlin, 968–970.
3. Coolen FPA, Aboalkhair AM, MacPhee IM (2010) Diversity in system reliability following component testing. The Journal of the Safety and Reliability Society 30(4):78–93.
4. Coolen FPA, Al-nefaiee AH (2012) Nonparametric predictive inference for failure times of systems with exchangeable components. Journal of Risk and Reliability, to appear.
5. Coolen FPA, Troffaes MCM, Augustin T (2011) Imprecise probability. In: International Encyclopedia of Statistical Science; Lovric (Ed.). Springer, Berlin, 645–648.
6. Coolen FPA, Utkin LV (2011) Imprecise reliability. In: International Encyclopedia of Statistical Science; Lovric (Ed.). Springer, Berlin, 649–650.
7. Coolen-Maturi T, Coolen FPA (2011) Unobserved, re-defined, unknown or removed failure modes in competing risks. Journal of Risk and Reliability 225:461–474.
8. Coolen-Schrijner P, Coolen FPA, MacPhee IM (2008) Nonparametric predictive inference for systems reliability with redundancy allocation. Journal of Risk and Reliability 222:463–476.
9. De Finetti B (1974) Theory of Probability. Wiley, New York.
10. Eryilmaz S (2010) Review of recent advances in reliability of consecutive k-out-of-n and related systems. Journal of Risk and Reliability 224:225–237.
11. MacPhee IM, Coolen FPA, Aboalkhair AM (2009) Nonparametric predictive system reliability with redundancy allocation following component testing. Journal of Risk and Reliability 223:181–188.
12. Maturi TA, Coolen-Schrijner P, Coolen FPA (2010) Nonparametric predictive inference for competing risks. Journal of Risk and Reliability 224:11–26.
13. Rahrouh M, Coolen FPA, Coolen-Schrijner P (2006) Bayesian reliability demonstration for systems with redundancy. Journal of Risk and Reliability 220:137–145.
14. Samaniego FJ (2007) System Signatures and their Applications in Engineering Reliability. Springer, New York.