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Nonparametric adaptive opportunity-based age replacement strategies

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We consider opportunity-based age replacement (OAR) using nonparametric predictive inference (NPI) for the time to failure of a future unit. Based on n observed failure times, NPI provides lower and upper bounds for the survival function for the time to failure X_{n+1} of a future unit which lead to upper and lower cost functions, respectively, for OAR based on the renewal reward theorem. Optimal OAR strategies for unit $n+1$ follow by minimizing these cost functions. Following this strategy, unit $n+1$ is correctively replaced upon failure, or preventively replaced upon the first opportunity after the optimal OAR threshold. We study the effect of this replacement information for unit $n+1$ on the optimal OAR strategy for unit $n+2$. We illustrate our method with examples and a simulation study. Our method is fully adaptive to available data, providing an alternative to the classical approach where the probability distribution of a unit's time to failure is assumed to be known. We discuss the possible use of our method and compare it with the classical approach, where we conclude that in most situations our adaptive method performs very well, but that counter-intuitive results can occur.

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Introduction

In age replacement, a unit is replaced upon failure or upon reaching a predetermined age, whichever occurs first. In opportunity-based age replacement (OAR), a unit is replaced upon failure or upon the first opportunity after reaching a predetermined threshold age, whichever occurs first. Within theory of stochastic processes, the optimal preventive replacement age, in the sense of leading to minimal expected costs per unit of time when the strategy is used for a sequence of similar units over a long period of time, is derived by application of the renewal reward theorem, see, for example, Barlow and Proschan.[1] Due to its mathematical simplicity, this procedure is attractive even though one realizes that the resulting optimal strategy may only be used for a few such cycles, for example because the unit would normally undergo some technical updates within reasonable period of time, or one wishes to change the policy in light of new information that may occur during the process.

Age replacement has predominantly been studied from a classical Operational Research perspective, where the probability distribution for the time to failure of the unit is assumed to be known. However, Mazzuchi and Soyer[2] studied age replacement within a Bayesian framework, allowing the assumed parametric failure time distribution to be updated when new data from the process become available. Instead of using the renewal reward criterion, they minimize the expected costs per unit of time over a single cycle to avoid the assumption of long-term use of the same strategy. Sheu *et al*[3] also present a Bayesian approach to preventive maintenance modelling. As an alternative, recognizing that one often has scarce information in the form of observed failure times, one could also base replacement decisions entirely on expert judgments.[4,5]

As an alternative to these approaches, OR-based decision making aspects for age replacement have been combined with nonparametric predictive inference (NPI) for the failure time distribution,[6,7] where the age replacement problem formulation is based on renewal theory, but instead of assuming a known probability distribution for the time to failure of a unit, imprecise predictive survival functions for the time to failure of the next unit are used, based on failure times of n previous units. Such NPI-based methods enable study of the way that resulting optimal replacement strategies adapt to available data.

Although age replacement has been widely studied, OAR has not received much attention, see Dekker and Dijkstra[8] and Jhang and Sheu[9] for overviews. In this paper we consider the same OAR model as Dekker and Dijkstra,[8] where the opportunities are generated independently of the time to failure of the unit considered, but we use imprecise predictive survival functions for the failure time distribution

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of the next unit. As Dekker and Dijkstra,[8] we restrict attention to the case in which the opportunities occur according to a Poisson process.

The outline of this paper is as follows. First, we provide the necessary details of NPI, referring to the literature for justifications and further discussions. Then we describe the OAR model and we derive the lower and upper cost functions and optimal OAR strategies for unit $n+1$. Then we consider unit $n+2$, taking the failure time information resulting from the replacement of unit $n+1$ into account in our NPI-based adaptive replacement strategies. We present the results of a simulation study used to analyse the performance of our method. Finally, we comment on the main conclusions of this work, and discuss the wider relevance of this study. The proofs of some of the analytical results are presented in Appendix A.

Nonparametric predictive inference

We consider a situation where we have observed n failure times, ordered as $x_{(1)} < x_{(2)} < \cdots < x_{(n)}$. Instead of assuming a known probability distribution function for a future failure time X_{n+1}, we specify direct probabilities for X_{n+1} according to Hill's assumption $A_{(n)}$,[10,11] that is,

$$P(X_{n+1} \in (x_{(j)}, x_{(j+1)})) = \frac{1}{n+1}, \quad j = 0, \ldots, n$$

where $x_{(0)} = 0$ and $x_{(n+1)} = \infty$, or $x_{(n+1)} = r$ if we can assume a finite upper bound r for the support of X_{n+1}. $A_{(n)}$ is a post-data assumption related to exchangeability[12], see Hill[13] for a discussion of $A_{(n)}$ and on overview of related work, see Coolen *et al*[14] for an introduction to $A_{(n)}$-based NPI in reliability. The assumption $A_{(n)}$ is not sufficient to derive precise probabilities for all possible events of interest, as it only partially defines a probability distribution for X_{n+1}. Applying De Finetti's 'fundamental theorem of probability'[12] it provides, however, optimal bounds for all probabilities of interest, which are lower and upper probabilities in the theory of interval probability.[11]

In this paper, we study how the optimal OAR strategy for X_{n+2} is affected if information on the failure time X_{n+1} becomes available, once we have applied the optimal OAR strategy for X_{n+1}. The observation of the failure time of unit $n+1$ can either be an actual time of failure, if corrective replacement took place, or a right-censored observation in case of preventive replacement. As Hill's $A_{(n)}$ does not directly allow right-censored observations, we use a generalization of $A_{(n)}$, called right-censored-$A_{(n)}$ and denoted by rc-$A_{(n)}$, developed by Coolen and Yan.[15]

To study the adaptive behaviour of the optimal OAR strategies, we only use the assumption rc-$A_{(n+1)}$ for the situation where the data consist of the n observed failure times, denoted as before, and a right-censored observation for X_{n+1}, denoted by x^c. Note that rc-$A_{(n)}$ also allows the method presented in this paper to be generalized to data sets including multiple right-censored observations, but this would greatly increase the complexity of the expressions and would not add much to the insights we gain later in this paper.

In this paper, we only use lower and upper survival functions for X_{n+1} and X_{n+2} resulting from $A_{(n)}$ and, depending on whether the observation for unit $n+1$ is a failure time or a right-censoring time, $A_{(n+1)}$ or rc-$A_{(n+1)}$. These lower and upper survival functions are the tightest bounds corresponding to these assumptions, using the data as described above. For X_{n+1}, based on $A_{(n)}$, the lower and upper survival functions are denoted by $\underline{S}_{X_{n+1}}(\cdot)$ and $\overline{S}_{X_{n+1}}(\cdot)$, respectively, and are

$$\underline{S}_{X_{n+1}}(x) = S_{X_{n+1}}(x_{(j+1)}) = \frac{n-j}{n+1} \quad \text{for } x \in (x_{(j)}, x_{(j+1)}],\ j = 0, \ldots, n \tag{1}$$

$$\overline{S}_{X_{n+1}}(x) = S_{X_{n+1}}(x_{(j)}) = \frac{n+1-j}{n+1} \quad \text{for } x \in [x_{(j)}, x_{(j+1)}),\ j = 0, \ldots, n \tag{2}$$

For X_{n+2} the lower and upper survival functions depend also on the observation for unit $n+1$. If an actual time of failure is observed for unit $n+1$, then the assumption $A_{(n+1)}$ leads to lower and upper survival functions for X_{n+2} which are directly obtained from (1) and (2) by replacing n by $n+1$. In case of a right-censored observation x^c for unit $n+1$, we distinguish between two cases that are relevant in this paper, both using the assumption rc-$A_{(n+1)}$. The lower and upper survival functions below are easily justified using the theory presented by Coolen and Yan,[15] and can be regarded as predictive alternatives to the well-known product-limit estimator of Kaplan and Meier.[14,15] We first consider the lower survival functions, as these not only decrease at the observed $x_{(j)}, j = 1, \ldots, n$, but also at x^c. The first case is when $x^c = x_{(k)}$ for some $k \in \{1, \ldots, n\}$, a situation which also occurred with NPI for age replacement,[7] giving for $x \in (x_{(j)}, x_{(j+1)}]$,

$$\underline{S}_{X_{n+2}}(x) = \frac{n-j+1}{n+2} \quad \text{for } j = 0, \ldots, k-1 \tag{3}$$

$$\underline{S}_{X_{n+2}}(x) = \frac{(n-j)(n-k+2)}{(n+2)(n-k+1)} \quad \text{for } j = k, \ldots, n \tag{4}$$

The second case which is relevant in this paper, but did not occur in Coolen-Schrijner and Coolen,[7] is $x^c \in (x_{(k)}, x_{(k+1)})$ for some $k \in \{0, \ldots, n\}$. The lower survival function for X_{n+2} in this case is

$$\underline{S}_{X_{n+2}}(x) = \frac{n-j+1}{n+2} \quad \text{for } x \in (x_{(j)}, x_{(j+1)}],\ j = 0, \ldots, k-1 \tag{5}$$

$$\underline{S}_{X_{n+2}}(x) = \frac{n-k+1}{n+2} \quad \text{for } x \in (x_{(k)}, x^c] \tag{6}$$

$$\underline{S}_{X_{n+2}}(x) = \frac{(n-k)(n-k+2)}{(n+2)(n-k+1)} \quad \text{for } x \in (x^c, x_{(k+1)}] \tag{7}$$

$$\underline{S}_{X_{n+2}}(x) = \frac{(n-j)(n-k+2)}{(n+2)(n-k+1)} \quad \text{for } x \in (x_{(j)}, x_{(j+1)}],\ j = k+1, \ldots, n \tag{8}$$

For the upper survival functions we do not need to distinguish between these two cases.[15] Let $k \in \{0, \ldots, n\}$ be such that $x^c \in [x_{(k)}, x_{(k+1)})$, then for $x \in [x_{(j)}, x_{(j+1)})$,

$$\overline{S}_{X_{n+2}}(x) = \frac{n-j+2}{n+2} \text{ for } j = 0, \ldots, k-1 \tag{9}$$

$$\overline{S}_{X_{n+2}}(x) = \frac{(n-j+1)(n-k+2)}{(n+2)(n-k+1)} \text{ for } j = k, \ldots, n \tag{10}$$

Note that in the points $x_{(j)}$ the upper and lower survival functions are identical. These upper and lower predictive survival functions are used to derive the cost functions in the next section.

The opportunity-based age replacement model

In a basic age replacement model (AR),[1] an item is replaced upon failure ('corrective replacement') at a cost c_f or upon reaching the age T ('preventive replacement') at a cost c_p with $0 < c_p < c_f$, whichever occurs first. In the classical setting, a unit's failure time is represented by a random quantity, say X, assumed to belong to a population of independent and identically distributed random quantities. For this case, the survival function for X is denoted by $S_X(x) = P(X > x)$. Let $C(T)$ be the long-run average cost per unit time under this policy and let $R(T)$ be the cost per cycle, which is the period between two consecutive replacements, and $L(T)$ be the length of a cycle, then the renewal reward theorem gives[1,6]

$$C(T) = \frac{E[R(T)]}{E[L(T)]} = \frac{c_f - (c_f - c_p)S_X(T)}{\int_0^T S_X(x)\,\mathrm{d}x} \tag{11}$$

It is not always possible to carry out preventive replacement at any moment in time. In an OAR model,[8] preventive replacements are only possible at opportunities. We assume that these opportunities occur according to a Poisson process with rate $\lambda > 0$, independently of the failure time of the unit. Consequently, both after a corrective and after a preventive replacement, the residual time Y to the next opportunity for replacement after time T is exponentially distributed with mean $1/\lambda$. As we also assume that in case of both kinds of replacement a new unit is installed, both these events may be considered as the end of a renewal cycle.

The OAR rule prescribes replacement of a unit at the first opportunity after threshold age T ('preventive replacement') at a cost $c_p > 0$, or upon failure ('corrective replacement') at a cost $c_f > c_p$ whichever occurs first. Let $C_{op}(T)$ be the long-run average cost per unit time under the OAR rule, $R_{op}(T)$ be the cost per cycle, and $L_{op}(T)$ the length of a cycle under this policy. Then, according to the renewal reward theorem, the long-run average cost per unit time is equal to the expected cost per cycle divided by the expected length of a cycle under the OAR rule, where

$$\begin{aligned} E[R_{op}(T)] &= c_p E[P(X \geqslant T + Y)] \\ &\quad + c_f E[P(X < T + Y)] \\ &= c_f - (c_f - c_p) E[S_X(T + Y)] \end{aligned} \tag{12}$$

and (see Appendix A)

$$\begin{aligned} E[L_{op}(T)] &= E[\min(X, T + Y)] \\ &= \int_0^T S_X(x)\,\mathrm{d}x + E[Y]E[S_X(T + Y)] \end{aligned} \tag{13}$$

Hence, the long-run average cost per unit time under the OAR rule is

$$C_{op}(T) = \frac{c_f - (c_f - c_p)E[S_X(T + Y)]}{\int_0^T S_X(x)\,\mathrm{d}x + E[Y]E[S_X(T + Y)]} \tag{14}$$

Note that the AR can be obtained by taking $Y = 0$ with probability 1. Dekker and Dijkstra[8] consider the case where the lifetimes have a known distribution. Under this assumption, they present a strategy for preventive replacements which minimizes the long-run average cost (14) within the so-called age-based control limit policies. Under an age-based control limit policy, a component is preventively replaced at an opportunity if its age has passed the control limit. In this paper, we do not assume a known distribution function for the lifetimes, indeed we do not even restrict to a parametric family of underlying distributions, but we use the NPI-based lower and upper survival functions, as presented in the previous section, on the basis of the observed failure times for n units, and we study the optimal replacement strategies according to these lower and upper survival functions under an age-based control limit policy.

In Coolen-Schrijner and Coolen[6] we applied the same method to age replacement problems. Using NPI and the renewal reward theorem, optimal age replacement times for unit $n+1$ were obtained by minimizing upper and lower cost functions for X_{n+1}. In Coolen-Schrijner and Coolen[7] we studied the effect on these optimal age replacement times when the observation for X_{n+1} under the optimal age replacement strategy became available. In this paper, we study how our method can be applied to OAR problems, where we consider both the method's performance for unit $n+1$ and the way in which optimal strategies adapt for unit

$n+2$ in the light of an observation for unit $n+1$ from this process. The upper and lower $A_{(n)}$-based survival functions straightforwardly lead to bounds for the cost function (14). In the following sections we derive the lower and upper cost functions for X_{n+1} and the lower and upper cost functions for X_{n+2}, combining NPI with the renewal reward theorem.

Lower cost function $\underline{C}_{X_{n+1},\mathrm{op}}(T)$

In this section, we derive the NPI-based lower cost function, denoted by $\underline{C}_{X_{n+1},\mathrm{op}}(T)$, for unit $n+1$ which will be replaced preventively at the first opportunity after T or upon failure, whichever occurs first. From (14) it follows that $\underline{C}_{X_{n+1},\mathrm{op}}(T)$ is obtained by substituting the upper survival function $\overline{S}_{X_{n+1}}(\cdot)$ for $S_X(\cdot)$, as $C_{\mathrm{op}}(T)$ is decreasing in $S_X(\cdot)$ so

$$\underline{C}_{X_{n+1},\mathrm{op}}(T) = \frac{c_\mathrm{f} - (c_\mathrm{f} - c_\mathrm{p})E[\overline{S}_{X_{n+1}}(T+Y)]}{\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]} \tag{15}$$

As we assume throughout this paper that preventive replacement opportunities occur according to a Poisson process with rate λ, the probability density function of Y is $f_Y(y) = \lambda \mathrm{e}^{-\lambda y}$, for $y \geqslant 0$. For $T \in [x_{(j)}, x_{(j+1)})$, $\overline{S}_{X_{n+1}}(T)$ is given by (2), and

$$\begin{aligned}
E[\overline{S}_{X_{n+1}}(T+Y)] &= \int_0^\infty \overline{S}_{X_{n+1}}(T+y)\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&= \int_0^{x_{(j+1)}-T} \overline{S}_{X_{n+1}}(x_{(j)})\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&\quad + \sum_{l=1}^{n-j} \int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T} \overline{S}_{X_{n+1}}(x_{(j+l)})\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&= \int_0^{x_{(j+1)}-T} \frac{n-j+1}{n+1}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&\quad + \sum_{l=1}^{n-j} \int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T} \frac{n-j-l+1}{n+1}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y \\
&= \frac{n-j+1}{n+1}(1-\mathrm{e}^{-\lambda(x_{(j+1)}-T)}) \\
&\quad + \sum_{l=1}^{n-j}\frac{n-j-l+1}{n+1}(\mathrm{e}^{-\lambda(x_{(j+l)}-T)} - \mathrm{e}^{-\lambda(x_{(j+l+1)}-T)}) \\
&= \frac{1}{n+1}\left((n-j+1) - \sum_{l=j+1}^{n+1}\mathrm{e}^{-\lambda(x_{(l)}-T)}\right)
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\int_0^T \overline{S}_{X_{n+1}}(x)\,\mathrm{d}x &= \sum_{l=0}^{j-1}\int_{x_{(l)}}^{x_{(l+1)}} \overline{S}_{X_{n+1}}(x_{(l)})\,\mathrm{d}x + \int_{x_{(j)}}^{T} \overline{S}_{X_{n+1}}(x_{(j)})\,\mathrm{d}x \\
&= \frac{1}{n+1}\left(\sum_{l=1}^{j} x_{(l)} + (n+1-j)T\right)
\end{aligned} \tag{17}$$

Substituting (16) and (17) into the lower cost function (15) for X_{n+1}, we obtain, for $T \in [x_{(j)}, x_{(j+1)})$ and $j = 0, \ldots, n$,

$$\underline{C}_{X_{n+1},\mathrm{op}}(T) = \frac{jc_\mathrm{f} + (n-j+1)c_\mathrm{p} + (c_\mathrm{f}-c_\mathrm{p})\sum_{l=j+1}^{n+1}\mathrm{e}^{-\lambda(x_{(l)}-T)}}{(n-j+1)(T+E[Y]) + \sum_{l=1}^{j}x_{(l)} - E[Y]\sum_{l=j+1}^{n+1}\mathrm{e}^{-\lambda(x_{(l)}-T)}} \tag{18}$$

As long as we do not assume a known upper bound for the support of X_{n+1} we have that $\underline{C}_{X_{n+1},\mathrm{op}}(T) \to 0$ as $T \to \infty$. This is a minor complication that we avoid, in the first instance, by restricting attention to the interval $(0, x_{(n)}]$ (see Coolen-Schrijner and Coolen[7]), where we use the fact that $\underline{C}_{X_{n+1},\mathrm{op}}(x_{(n)}) = \lim_{T\uparrow x_{(n)}} \underline{C}_{X_{n+1},\mathrm{op}}(T)$ since $\underline{C}_{X_{n+1},\mathrm{op}}(\cdot)$ is a continuous function as will be shown at the end of this section.

Denote by $\underline{T}^j_{n+1,\mathrm{op}}$ the value for the age replacement threshold T for which $\underline{C}_{X_{n+1},\mathrm{op}}(T)$ is minimal over the interval $[x_{(j)}, x_{(j+1)})$. Then the optimal OAR threshold $\underline{T}^*_{n+1,\mathrm{op}}$ over $(0, x_{(n)}]$ is, of course,

$$\underline{T}^*_{n+1,\mathrm{op}} = \arg\min_{0\leqslant j\leqslant n-1} \underline{C}_{X_{n+1},\mathrm{op}}(\underline{T}^j_{n+1,\mathrm{op}}) \tag{19}$$

The proof of Lemma 1 is given in Appendix A.

Lemma 1 *For OAR with unit $n+1$ replaced upon failure or upon the first opportunity after threshold T, we have for $T \in [x_{(j)}, x_{(j+1)})$ and $j = 0, \ldots, n-1$,*

$$\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow$$

$$\frac{(c_\mathrm{f}-c_\mathrm{p})(\overline{S}_{X_{n+1}}(T) - E[\overline{S}_{X_{n+1}}(T+Y)])}{E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]} - \underline{C}_{X_{n+1},\mathrm{op}}(T) = 0 \tag{20}$$

If there does not exist a $T \in [x_{(j)}, x_{(j+1)})$ such that $\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0$, then $\underline{C}'_{X_{n+1},\mathrm{op}}(T)$ is minimal over this interval in one of the end-points.

Substituting (2) and (16) into (20) yields

$$\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow \frac{(c_\mathrm{f} - c_\mathrm{p})\sum_{l=j+1}^{n+1} \mathrm{e}^{-\lambda(x_{(l)}-T)}}{E[Y]\{(n-j+1) - \sum_{l=j+1}^{n+1} \mathrm{e}^{-\lambda(x_{(l)}-T)}\}} - \underline{C}_{X_{n+1},\mathrm{op}}(T) = 0 \tag{21}$$

To prove that (21) minimizes the lower cost function, we must show that $\underline{C}''_{X_{n+1},\mathrm{op}}(T) > 0$ for $T \in (x_{(j)}, x_{(j+1)})$ and $j = 0, \ldots, n-1$. This proof is included in Appendix A, in a manner that immediately also covers this optimality check as required in the similar situations in the following sections.

To obtain the optimal OAR threshold $\underline{T}^*_{n+1,\mathrm{op}}$ for unit $n+1$ we must solve (21) for all intervals $[x_{(j)}, x_{(j+1)})$, $j = 0, \ldots, n-1$, giving us the values $\underline{T}^j_{n+1,\mathrm{op}}$. Then $\underline{T}^*_{n+1,\mathrm{op}}$ is equal to the $\underline{T}^j_{n+1,\mathrm{op}}$ for which the corresponding lower costs are minimal. If we assume a known upper bound for the support of X_{n+1}, say r, we also have to calculate the minimum of $\underline{C}_{X_{n+1},\mathrm{op}}$ over the interval $[x_{(n)}, r)$, and comparing this with $\underline{C}_{X_{n+1},\mathrm{op}}(\underline{T}^*_{n+1,\mathrm{op}})$ yields the overall minimal costs (see Coolen-Schrijner and Coolen[7] where a similar procedure was used and discussed in more detail).

The NPI-based lower cost function for unit $n+1$ in the basic age replacement model is[6]

$$\underline{C}_{X_{n+1}}(T) = \frac{c_\mathrm{f} - (c_\mathrm{f} - c_\mathrm{p})\overline{S}_{X_{n+1}}(T)}{\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x}$$

It follows from the proof of Lemma 1 that

$$\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow \frac{(c_\mathrm{f} - c_\mathrm{p})(\overline{S}_{X_{n+1}}(T) - E[\overline{S}_{X_{n+1}}(T+Y)])}{E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]} - \underline{C}_{X_{n+1}}(T) = 0 \tag{22}$$

and as a consequence,

$$\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow \underline{C}_{X_{n+1},\mathrm{op}}(T) = \underline{C}_{X_{n+1}}(T) \tag{23}$$

This is similar to a result by Dekker and Dijkstra[8] but their result holds for all T, so that they obtain a global minimum, while our result holds for each interval, so that we obtain local minima. However, it is easier to calculate the optimal OAR threshold for unit $n+1$ using (21) than using (22), as $\underline{C}_{X_{n+1}}(T)$ is a discontinuous function at the observed failure times while $\underline{C}_{X_{n+1},\mathrm{op}}(T)$ is a continuous function, as

$$\lim_{T \uparrow x_{(j+1)}} \underline{C}_{X_{n+1},\mathrm{op}}(T) = \underline{C}_{X_{n+1},\mathrm{op}}(x_{(j+1)})$$

Upper cost function $\overline{C}_{X_{n+1},\mathrm{op}}(T)$

In this section, we derive the NPI-based upper cost function, denoted by $\overline{C}_{X_{n+1},\mathrm{op}}(T)$, for unit $n+1$, which will be replaced preventively at the first opportunity after T, or upon failure, whichever occurs first. From (14) it follows that $\overline{C}_{X_{n+1},\mathrm{op}}(T)$ is obtained by substituting the lower survival function $\underline{S}_{X_{n+1}}(\cdot)$ for $S_X(\cdot)$, so

$$\overline{C}_{X_{n+1},\mathrm{op}}(T) = \frac{c_\mathrm{f} - (c_\mathrm{f} - c_\mathrm{p})E[\underline{S}_{X_{n+1}}(T+Y)]}{\int_0^T \underline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y]E[\underline{S}_{X_{n+1}}(T+Y)]} \tag{24}$$

For $T \in (x_{(j)}, x_{(j+1)}]$, $j = 0, \ldots, n$, $\underline{S}_{X_{n+1}}(T)$ is given by (1), and

$$\int_0^T \underline{S}_{X_{n+1}}(x)\mathrm{d}x = \sum_{l=0}^{j-1} \int_{x_{(l)}}^{x_{(l+1)}} \underline{S}_{X_{n+1}}(x_{(l)})\mathrm{d}x + \int_{x_{(j)}}^{T} \underline{S}_{X_{n+1}}(x_{(j)})\mathrm{d}x = \frac{1}{n+1}\left(\sum_{l=1}^{j} x_{(l)} + (n-j)T\right) \tag{25}$$

and

$$\begin{aligned} E[\underline{S}_{X_{n+1}}(T+Y)] &= \int_0^{x_{(j+1)}-T} \underline{S}_{X_{n+1}}(x_{(j+1)})\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y + \sum_{l=1}^{n-j} \int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T} \underline{S}_{X_{n+1}}(x_{(j+1+l)})\lambda\mathrm{e}^{-\lambda y}\mathrm{d}y \\ &= \frac{n-j}{n+1}(1 - \mathrm{e}^{-\lambda(x_{(j+1)}-T)}) + \sum_{l=1}^{n-j} \frac{n-j-l}{n+1}(\mathrm{e}^{-\lambda(x_{(j+l)}-T)} - \mathrm{e}^{-\lambda(x_{(j+l+1)}-T)}) \\ &= \frac{1}{n+1}\left((n-j) - \sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\right) \end{aligned} \tag{26}$$

Substituting (25) and (26) into (24) gives, for $T \in (x_{(j)}, x_{(j+1)}]$ and $j = 0, \ldots, n$,

$$\overline{C}_{X_{n+1},\mathrm{op}}(T) = \frac{(j+1)c_\mathrm{f} + (n-j)c_\mathrm{p} + (c_\mathrm{f} - c_\mathrm{p})\sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}}{(n-j)(T + E[Y]) + \sum_{l=1}^{j} x_{(l)} - E[Y]\sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}} \tag{27}$$

From (27) it follows that $\overline{C}_{X_{n+1},\mathrm{op}}(T) = \overline{C}_{X_{n+1},\mathrm{op}}(x_{(n)})$ for $T \geqslant x_{(n)}$, so we only have to calculate the minimum of the upper cost function over the interval $(0, x_{(n)}]$, and if the minimum is obtained in $x_{(n)}$, then we might as well not replace preventively at all. Denote by $\overline{T}^j_{n+1,\mathrm{op}}$ the value of T for which $\overline{C}_{X_{n+1},\mathrm{op}}(T)$ is minimal over the interval $(x_{(j)}, x_{(j+1)}]$. The optimal OAR threshold $\overline{T}^*_{n+1,\mathrm{op}}$ for X_{n+1} is

$$\overline{T}^*_{n+1,\mathrm{op}} = \arg\min_{0 \leqslant j \leqslant n-1} \overline{C}_{X_{n+1},\mathrm{op}}(\overline{T}^j_{n+1,\mathrm{op}}) \tag{28}$$

The proof of Lemma 2 is omitted as it is similar to the proof of Lemma 1.

Lemma 2 *For OAR with unit $n+1$ replaced upon failure or upon the first opportunity after threshold T, we have, for $T \in (x_{(j)}, x_{(j+1)}]$ and $j=0, \ldots, n-1$,*

$$\overline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow$$
$$\frac{(c_{\mathrm{f}} - c_{\mathrm{p}})(\underline{S}_{X_{n+1}}(T) - E[\underline{S}_{X_{n+1}}(T+Y)])}{E[Y]E[\underline{S}_{X_{n+1}}(T+Y)]} - \overline{C}_{X_{n+1},\mathrm{op}}(T) = 0 \tag{29}$$

If there does not exist a $T \in (x_{(j)}, x_{(j+1)}]$ such that $\overline{C}'_{X_{n+1},\mathrm{op}}(T) = 0$, then $\overline{C}'_{X_{n+1},\mathrm{op}}(T)$ is minimal over this interval in one of the end-points.

With $\overline{C}_{X_{n+1}}(T)$ denoting the NPI-based upper cost function for unit $n+1$ in the basic age replacement model,[7] we also have

$$\overline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow \overline{C}_{X_{n+1},\mathrm{op}}(T) = \overline{C}_{X_{n+1}}(T) \tag{30}$$

Substituting (1) and (26) into Lemma 2, and using the fact that $\overline{C}''_{X_{n+1},\mathrm{op}}(T) > 0$ (see Appendix A), we have that the optimal OAR threshold $\overline{T}^*_{n+1,\mathrm{op}}$ for unit $n+1$, in the sense of minimizing the upper cost function, equals the solution $\overline{T}^j_{n+1,\mathrm{op}}$ for which the upper costs are minimal, where for all intervals $(x_{(j)}, x_{(j+1)}]$, $j=0, \ldots, n-1$, $\overline{T}^j_{n+1,\mathrm{op}}$ is the solution to

$$\overline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow$$
$$\frac{(c_{\mathrm{f}} - c_{\mathrm{p}}) \sum_{l=j+1}^{n} \mathrm{e}^{-\lambda}(x_{(l)} - T)}{E[Y]\{(n-j) - \sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\}} - \overline{C}_{X_{n+1},\mathrm{op}}(T) = 0 \tag{31}$$

Examples

In this section, we illustrate the OAR results for unit $n+1$, and we discuss the effect of λ on the optimal thresholds.

Example 1

Suppose we have observed the following five lifetimes: 4, 6, 10, 11 and 15. Each preventive replacement costs $c_{\mathrm{p}}=1$, while each corrective replacement costs $c_{\mathrm{f}}=10$. This example was studied for the basic age replacement (AR) problem in Coolen-Schrijner and Coolen.[6] It was shown that the optimal age replacement time that minimizes the upper cost function for X_6 was 4 with corresponding upper costs of 0.7500. The optimal age replacement time that minimizes the lower cost function for X_6 was 4^- with corresponding lower costs of 0.2500. Here, the notation 4^- means 'just before 4', which is only a technicality caused by the discrete nature of our NPI-based upper and lower survival functions which caused the age replacement cost functions to have jumps at the observed failure times, for practical purposes this can just be interpreted as being equal to 4.[6,7] This technicality is less relevant in this paper as our OAR cost functions are continuous.

We now assume that preventive replacement is only possible at opportunities occurring according to a Poisson process with rate $\lambda=2$. Solving (21) and (31) together with (19) and (28) yields $\underline{T}^*_{6,\mathrm{op}} = 2.900$ and $\overline{T}^*_{6,\mathrm{op}} = 8.961$ with corresponding lower and upper costs of 0.3449 and 0.8947, respectively. Dekker and Dijkstra[8] proved that for lifetime distributions with an increasing hazard rate, with sufficiently large limiting value, the optimal OAR threshold is smaller than the optimal age replacement time. Here we do not have such a result, due to the fact that these data do not strongly indicate a lifetime distribution with an increasing hazard rate.

Table 1 gives the optimal OAR thresholds and corresponding costs for different values of λ. Figure 1 shows a plot of $\underline{C}_{X_6}(T)$, $\overline{C}_{X_6}(T)$ and $\underline{C}_{X_6,\mathrm{op}}(T)$ and $\overline{C}_{X_6,\mathrm{op}}(T)$ with $\lambda=1$, 2 and 5. The numbers underneath the curve

Table 1 Optimal times and costs for OAR with different opportunity rate

λ	$\underline{T}^*_{6,\mathrm{op}}$	$\overline{T}^*_{6,\mathrm{op}}$	$\underline{C}_{6,\mathrm{op}}(\underline{T}^*_{6,\mathrm{op}})$	$\overline{C}_{6,\mathrm{op}}(\overline{T}^*_{6,\mathrm{op}})$
1	2.50	8.42	0.3994	0.9360
2	2.90	8.96	0.3449	0.8947
3	3.12	9.20	0.3210	0.8778
4	3.25	3.46	0.3072	0.8659
5	3.35	3.52	0.2982	0.8512
10	3.60	3.69	0.2777	0.8131
15	3.70	3.76	0.2699	0.7969

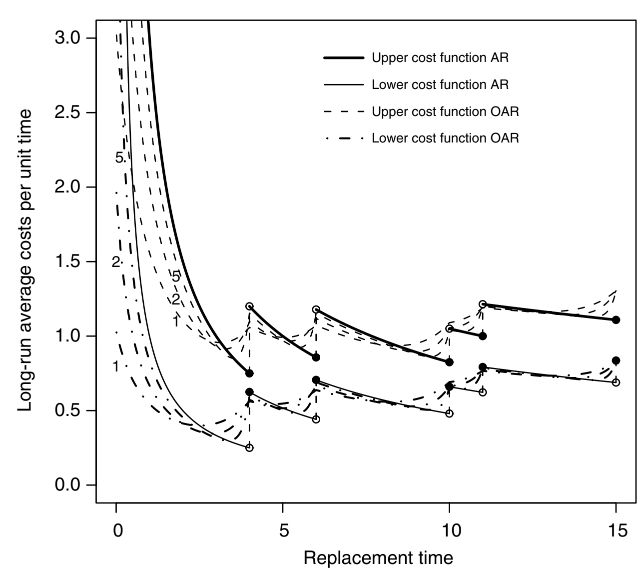

Figure 1 Lower and upper cost functions for AR and OAR ($\lambda=1$, 2, 5).

correspond to the value of λ. If we increase λ, the rate of occurrence of the preventive replacement opportunities, then the cost functions of the OAR model tend to the cost functions of the AR model as one expects, and in this example the optimal OAR threshold for the lower OAR cost function tends to the optimal age replacement time for the lower AR cost function. If λ is not too small (in fact, if $\lambda \geqslant 3.75$), this also holds for the upper cost functions, for smaller λ this OAR optimum threshold is in a later interval. In Example 2 we discuss this convergence further, showing that situations might occur which at first sight may appear to be counter-intuitive.

Intuitively, one may expect that the optimal opportunity-based preventive replacement threshold decreases with decreasing opportunity rate λ. Dekker and Dijkstra[8] report that their numerical observations confirm this intuition, but did not prove or disprove this. In our NPI approach, this intuition is confirmed in nearly all examples and simulated cases we have computed, and we managed to derive a sufficient condition on values for λ and data for this intuition to be correct. However, this also allowed us to construct counter-examples, hence this intuition is not generally correct, one such a counter-example is Example 2.

Example 2

Suppose we have five observed failure times: 0.1, 0.2, 0.3, 0.4 and 0.5, and we compare optimal preventive replacement thresholds for λ equal to 1 and 2. Restricted to the interval $[0, 0.5]$, our lower cost function for $\lambda = 1$ is minimal at $T = 0.5$, whereas for $\lambda = 2$ it is minimal at $T = 0.05$. This is a rather extreme constructed example, with relatively to the observed failure times very few replacement opportunities. The lower cost function for $\lambda = 1$ is very flat, suggesting that for $\lambda = 1$ no preventive replacement at all may well be equally effective as use of threshold $T = 0.5$. The lower cost function for $\lambda = 2$ is slightly less flat, which suggests that in this situation our OAR policy with threshold 0.05 is of benefit. Hence, the larger value of λ, modelling more frequent replacement opportunities, leads to an earlier threshold in this situation. This may well be explained by the fact that, with more frequent preventive replacement opportunities, it may become beneficial to replace preventively if opportunities indeed do occur, and this will also hold for future units which is taken into account in this cost function via the renewal reward theorem. Such a counter-example can also be constructed for the upper cost function. For this constructed example, the optimal replacement thresholds per interval between two consecutive observed failure times occur, for several such intervals, at end-points, which does not often occur but is taken into account in the theory throughout this paper.

Adaptive opportunity-based age replacement for unit $n+2$

Minimization of the lower and upper cost functions with regard to OAR of unit $n+1$, using NPI and the renewal reward theorem as described above, leads to the optimal OAR thresholds $\underline{T}^*_{n+1,\mathrm{op}}$ and $\overline{T}^*_{n+1,\mathrm{op}}$, respectively. We now consider the effect of using such an optimal strategy for unit $n+1$, and the resulting information about the failure time X_{n+1}, on the optimal NPI-based OAR strategy for unit $n+2$ with random failure time X_{n+2}.

Although our cost functions are based on the renewal reward theorem, which assumes that the optimal strategy is used for a long period, it is interesting to study how optimal replacement times would actually adapt to new data from the process under rather minimal assumptions for the failure time distribution. For example, if our study would reveal that the optimal strategy is unlikely to change much on the basis of the new observation, this would suggest that the use of this criterion is not unreasonable even when one wishes to use such an adaptive method.

Suppose we follow an optimal OAR strategy for unit $n+1$, that is unit $n+1$ is replaced upon failure or upon the first opportunity after $\underline{T}^*_{n+1,\mathrm{op}}$ (or $\overline{T}^*_{n+1,\mathrm{op}}$), and suppose that the observation for X_{n+1} from this process becomes available. This observation is either a failure time less than the optimal OAR threshold for X_{n+1} ($\underline{T}^*_{n+1,\mathrm{op}}$ or $\overline{T}^*_{n+1,\mathrm{op}}$) plus the time to the next opportunity (say y_{op}), which occurs if unit $n+1$ is replaced correctively, or it is a right-censored observation at the optimal OAR threshold for X_{n+1} plus y_{op}, which occurs if unit $n+1$ is replaced preventively.

If we observe a failure time for unit $n+1$, then we can directly apply the results of the previous sections with the $n+1$ failure times, with $A_{(n)}$ replaced by $A_{(n+1)}$, to obtain the optimal OAR thresholds $\underline{T}^*_{n+2,\mathrm{op}}$ and $\overline{T}^*_{n+2,\mathrm{op}}$ for unit $n+2$.

The case that unit $n+1$ is preventively replaced at the first opportunity after the optimal OAR threshold, so that the observation for unit $n+1$ is a right-censored observation, leads to a more complicated situation. Here we have to use recently developed theory for NPI with right-censored data,[15] as briefly presented in the section on NPI. In the next two sections, we derive the lower and upper cost functions for X_{n+2} for the situation that the observation for X_{n+1} is a right-censored observation. Although these two sections are again similar in nature, there is an important difference due to the fact that, with one of the $n+1$ observations available being right-censored, the NPI lower survival function for X_{n+2} decreases at this right-censored observation whereas the corresponding upper survival function still only decreases at observed failure times.

$\underline{C}_{X_{n+2},\mathrm{op}}(T)$ with unit $n+1$ preventively replaced

In this section, we determine the lower cost function for unit $n+2$, denoted by $\underline{C}_{X_{n+2},\mathrm{op}}(T)$, where unit $n+2$ will be preventively replaced at the first opportunity after T or upon failure, given that unit $n+1$ is preventively replaced at $x^c \in [x_{(k)}, x_{(k+1)})$ for some $k \in \{0, \ldots, n\}$. From (14) it follows that $\underline{C}_{X_{n+2},\mathrm{op}}(T)$ is obtained by substituting the upper survival function $\overline{S}_{X_{n+2}}(\cdot)$ for $S_X(\cdot)$. Hence, our data consist of n failure times and one right-censored observation at x^c, and the upper survival function for X_{n+2} is given by (9) and (10) in case $x^c \in [x_{(k)}, x_{(k+1)})$.

Lemma 3 provides expressions for $E[\overline{S}_{X_{n+2}}(T+Y)]$ which are used to calculate the lower cost function $\underline{C}_{X_{n+2},\mathrm{op}}(T)$ in examples and simulations later. The proof of Lemma 3 is given in Appendix A.

Lemma 3 *Let $x^c \in [x_{(k)}, x_{(k+1)})$, for some $k \in \{0, \ldots, n\}$, be the right-censored observation for the failure time of unit $n+1$, with further data available consisting of n failure times. Let T be the OAR threshold for unit $n+2$. Using rc-$A_{(n+1)}$, we get the following expressions for $E[\overline{S}_{X_{n+2}}(T+Y)]$, where Y represents the time to the next preventive replacement opportunity and is assumed to be exponentially distributed with rate λ.*

1. *For $T \in [x_{(j)}, x_{(j+1)})$ with $j = 0, \ldots, k-1$,*

$$E[\overline{S}_{X_{n+2}}(T+Y)] = \frac{n-j+2}{n+2} - \frac{1}{n+2}\sum_{l=j+1}^{k} \mathrm{e}^{-\lambda(x_{(l)}-T)} - \frac{(n-k+2)}{(n+2)(n-k+1)}\sum_{l=k+1}^{n+1} \mathrm{e}^{-\lambda(x_{(l)}-T)} \tag{32}$$

2. *For $T \in [x_{(k)}, x_{(k+1)})$,*

$$E[\overline{S}_{X_{n+2}}(T+Y)] = \frac{n-k+2}{n+2} - \frac{(n-k+2)}{(n+2)(n-k+1)}\sum_{l=k+1}^{n+1} \mathrm{e}^{-\lambda(x_{(l)}-T)} \tag{33}$$

3. *For $T \in [x_{(j)}, x_{(j+1)})$ with $j = k+1, \ldots, n$,*

$$E[\overline{S}_{X_{n+2}}(T+Y)] = \frac{(n-j+1)(n-k+2)}{(n+2)(n-k+1)} - \frac{(n-k+2)}{(n+2)(n-k+1)}\sum_{l=j+1}^{n+1} \mathrm{e}^{-\lambda(x_{(l)}-T)} \tag{34}$$

Lemma 4 gives expressions for $\int_0^T \overline{S}_{X_{n+2}}(x)\,\mathrm{d}x$ for the same three cases as in Lemma 3, these will also be used to calculate the lower cost function $\underline{C}_{X_{n+2},\mathrm{op}}(T)$ later. We omit the proof of Lemma 4 as the results follow immediately by substituting the corresponding NPI-based upper survival function, as presented earlier in the paper, into

$$\int_0^T \overline{S}_{X_{n+2}}(x)\mathrm{d}x = \sum_{l=0}^{j-1} \int_{x_{(l)}}^{x_{(l+1)}} \overline{S}_{X_{n+2}}(x)\mathrm{d}x + \int_{x_{(j)}}^{T} \overline{S}_{X_{n+2}}(x)\mathrm{d}x$$

Lemma 4 *Assume the same setting as in Lemma 3. Using rc-$A_{(n+1)}$, we get the following expressions for $\int_0^T \overline{S}_{X_{n+2}}(x)\,\mathrm{d}x$.*

1. *For $T \in [x_{(j)}, x_{(j+1)})$ with $j = 0, \ldots, k-1$,*

$$\int_0^T \overline{S}_{X_{n+2}}(x)\mathrm{d}x = \frac{1}{n+2}\left\{\sum_{l=1}^{j} x_{(l)} + (n-j+2)T\right\} \tag{35}$$

2. *For $T \in [x_{(k)}, x_{(k+1)})$,*

$$\int_0^T \overline{S}_{X_{n+2}}(x)\mathrm{d}x = \frac{1}{n+2}\left\{\sum_{l=1}^{k} x_{(l)} + (n-k+2)T\right\} \tag{36}$$

3. *For $T \in [x_{(j)}, x_{(j+1)})$ with $j = k+1, \ldots, n$,*

$$\int_0^T \overline{S}_{X_{n+2}}(x)\mathrm{d}x = \frac{1}{n+2}\sum_{l=1}^{k} x_{(l)} + \frac{(n-k+2)}{(n+2)(n-k+1)}\left\{\sum_{l=k+1}^{j} x_{(l)} + (n-j+1)T\right\} \tag{37}$$

Substituting (32) and (35), (33) and (36), and (34) and (37), respectively, into

$$\underline{C}_{X_{n+2},\mathrm{op}}(T) = \frac{c_\mathrm{f} - (c_\mathrm{f} - c_\mathrm{p})E[\overline{S}_{X_{n+2}}(T+Y)]}{\int_0^T \overline{S}_{X_{n+2}}(x)\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+2}}(T+Y)]} \tag{38}$$

yields our lower cost function for OAR threshold T applied to unit $n+2$ for Cases 1, 2 and 3 from Lemmas 3 and 4, respectively.

As long as we do not assume a known upper bound for the support of X_{n+2}, we have that $\underline{C}_{X_{n+2},\mathrm{op}}(T) \to 0$ as $T \to \infty$.[7] This is the same minor complication as discussed before for X_{n+1}, which we avoid, in the first instance, by restricting attention to the interval $(0, x_{(n)}]$. Denote by $\underline{T}^j_{n+2,\mathrm{op}}$ the value of $T \in [x_{(j)}, x_{(j+1)})$ at which $\underline{C}_{X_{n+2},\mathrm{op}}(T)$ is minimal over that interval. Then the overall optimal OAR threshold for unit $n+2$, denoted by $\underline{T}^*_{n+2,\mathrm{op}}$ is the value $\underline{T}^j_{n+2,\mathrm{op}}$ for which the corresponding lower costs are minimal. Lemma 5 provides a useful result for computation of the optimal thresholds restricted to intervals between consecutive observed failure times. The proof of Lemma 5 is omitted as it follows immediately from the proof of Lemma 1.

Lemma 5 *For OAR with unit $n+2$ replaced upon failure or upon the first opportunity after threshold T, and where unit $n+1$ was preventively replaced at $x^c \in [x_{(k)}, x_{(k+1)})$ we have,* *for* $T \in [x_{(j)}, x_{(j+1)})$ *and* $j = 0, \ldots, n-1$,

$$\underline{C}'_{X_{n+2},\mathrm{op}}(T) = 0 \Leftrightarrow$$
$$\frac{(c_{\mathrm{f}} - c_{\mathrm{p}})(\overline{S}_{X_{n+2}}(T) - E[\overline{S}_{X_{n+2}}(T + Y)])}{E[Y]E[\overline{S}_{X_{n+2}}(T + Y)]} - \underline{C}_{X_{n+2},\mathrm{op}}(T) = 0 \tag{39}$$

where $\overline{S}_{X_{n+2}}(T)$ *is given by* (9) *and* (10), $E[\overline{S}_{X_{n+2}}(T + Y)]$ *is given in Lemma 3, and* $\underline{C}_{X_{n+2},\mathrm{op}}(T)$ *is given by* (38). *If there does not exist a* $T \in [x_{(j)}, x_{(j+1)})$ *such that* $\underline{C}'_{X_{n+2},\mathrm{op}}(T) = 0$, *then* $\underline{C}'_{X_{n+2},\mathrm{op}}(T)$ *is minimal in one of the end-points.*

We should remark here that we consider it logical in this setting that x^c will result from using the corresponding OAR policy for unit $n+1$, so optimality of the threshold value in the sense of minimization of the lower cost function. There is no strong theoretical argument for this, but from the possible interval probabilistic interpretations of the lower and upper survival functions used in our method this is most natural. In our later examples and simulations, we always work with either the lower cost function for both units considered, or the upper cost function for both units. One could consider mixing such strategies, but we do not consider this or great interest.

In Appendix A we show that $\underline{C}''_{X_{n+2},\mathrm{op}}(T) > 0$, for T in the open intervals created by consecutive observed failure times, so the optimal OAR threshold $\underline{T}^*_{n+2,\mathrm{op}}$ is given by the value of $\underline{T}^j_{n+2,\mathrm{op}}$, for $0 \leqslant j \leqslant n-1$, for which this lower cost function is minimal, where the $\underline{T}^j_{n+2,\mathrm{op}}$ can be obtained by solving (39). If we assume a known finite upper bound for the support of X_{n+1}, say r, we also have to calculate the minimum of $\underline{C}_{X_{n+2},\mathrm{op}}$ over the interval $(x_{(n)}, r)$, and comparing this with $\underline{C}_{X_{n+2},\mathrm{op}}(\underline{T}^*_{n+2,\mathrm{op}})$ would then yield the global minimal costs.

$\overline{C}_{X_{n+2},\mathrm{op}}(T)$ with unit $n+1$ preventively replaced

In this section, we determine the upper cost function for unit $n+2$, denoted by $\overline{C}_{X_{n+2},\mathrm{op}}(T)$, where unit $n+2$ will be preventively replaced at the first opportunity after T or upon failure, given that unit $n+1$ is preventively replaced at $x^c \in [x_{(k)}, x_{(k+1)})$ for some $k \in \{0, \ldots, n\}$. This section is very similar to the previous section, the main difference is due to the fact that our NPI-based lower survival function for X_{n+2} decreases not only at the n observed failure times, but also at the observed right-censoring time x^c for unit $n+1$,[15] where we must distinguish between the case of x^c tied with an $x_{(k)}$ and the case where $x^c \in (x_{(k)}, x_{(k+1)})$. From (14) it follows that $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ is obtained by substituting the lower survival function $\underline{S}_{X_{n+2}}(\cdot)$ for $S_X(\cdot)$. Hence, our data consist of n real failure times and one right-censored observation at x^c, and the lower survival function for X_{n+2} is given by (5)–(8) if $x^c \in (x_{(k)}, x_{(k+1)})$ and by (3) and (4) if $x^c = x_{(k)}$. As the lower survival function is continuous from the left in all observations (failures and right-censored data),[15] formula (6) also holds if the right end-point of the interval is included as well.

Lemma 6 provides expressions for $E[\underline{S}_{X_{n+2}}(T+Y)]$ which are used to calculate the upper cost function $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ in examples and simulations later. The proof of Lemma 6 is given in Appendix A.

Lemma 6 *Let* (a) $x^c \in (x_{(k)}, x_{(k+1)})$ *or* (b) $x^c = x_{(k)}$, *for some* $k \in \{0, \ldots, n\}$, *be the right-censored observation for the failure time of unit* $n+1$, *with further data available consisting of* n *failure times. Let* T *be the OAR threshold for unit* $n+2$. *Using* rc-$A_{(n+1)}$, *we get the following expressions for* $E[\underline{S}_{X_{n+2}}(T+Y)]$ *where* Y *represents the time to the next preventive replacement opportunity and is assumed to be exponentially distributed with rate* λ.

(a) *For* $x^c \in (x_{(k)}, x_{(k+1)})$ *we have*

1. *For* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = 0, \ldots, k-1$,

$$E[\underline{S}_{X_{n+2}}(T + Y)] = \frac{1}{n+2}\left[(n-j+1) - \sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)} - \frac{1}{n-k+1}\left\{\mathrm{e}^{-\lambda(x^c-T)} + \sum_{l=k+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\right\}\right] \tag{40}$$

2. *For* $T \in (x_{(k)}, x^c]$,

$$E[\underline{S}_{X_{n+2}}(T + Y)] = \frac{1}{n+2}\left[(n-k+1) - \sum_{l=k+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)} - \frac{1}{n-k+1}\left\{\mathrm{e}^{-\lambda(x^c-T)} + \sum_{l=k+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\right\}\right] \tag{41}$$

3. *For* $T \in (x^c, x_{(k+1)}]$,

$$E[\underline{S}_{X_{n+2}}(T + Y)] = \frac{1}{n+2}\left(1 + \frac{1}{n-k+1}\right) \times \left[(n-k) - \sum_{l=k+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\right] \tag{42}$$

4. *For* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = k+1, \ldots, n$,

$$E[\underline{S}_{X_{n+2}}(T + Y)] = \frac{1}{n+2}\left(1 + \frac{1}{n-k+1}\right) \times \left[(n-j) - \sum_{l=j+1}^{n} \mathrm{e}^{-\lambda(x_{(l)}-T)}\right] \tag{43}$$

(b) *For* $x^c = x_{(k)}$ *we have that for* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = 0, \ldots, k-1$, $E[\underline{S}_{X_{n+2}}(T + Y)]$ *is given by* (40), *while* *for* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = k, \ldots, n$, $E[\underline{S}_{X_{n+2}}(T+Y)]$ *is given by* (43).

Lemma 7 gives expressions for $\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x$ for the same cases as in Lemma 6, these will also be used to calculate the upper cost function $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ later. As for Lemma 4, the proof is again omitted as the results follow immediately by using the corresponding NPI-based lower survival function.

Lemma 7 *Assume the same setting as in Lemma 6. Using rc-*$A_{(n+1)}$ *we get the following expressions for* $\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x$.

(a) *For* $x^c \in (x_{(k)}, x_{(k+1)})$ *we have*

1. *For* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = 0, \ldots, k-1$,

$$\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x = \frac{1}{n+2}\left[(n-j+1)T + \sum_{l=0}^{j} x_{(l)}\right] \tag{44}$$

2. *For* $T \in (x_{(k)}, x^c]$,

$$\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x = \frac{1}{n+2}\left[(n-k+1)T + \sum_{l=0}^{k} x_{(l)}\right] \tag{45}$$

3. *For* $T \in (x^c, x_{(k+1)}]$,

$$\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x = \frac{1}{n+2}\left[(n-k)T + \sum_{l=0}^{k} x_{(l)}. + \frac{1}{n-k+1}\{(n-k)T + x^c\}\right] \tag{46}$$

4. *For* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = k+1, \ldots, n$,

$$\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x = \frac{1}{n+2}\left[(n-j)T + \sum_{l=0}^{j} x_{(l)} + \frac{1}{n-k+1}\left\{(n-j)T + x^c + \sum_{l=k+1}^{j} x_{(l)}\right\}\right] \tag{47}$$

(b) *For* $x^c = x_{(k)}$ *we have that for* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = 0, \ldots, k-1$, $\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x$ *is given by* (44), *while for* $T \in (x_{(j)}, x_{(j+1)}]$ *with* $j = k, \ldots, n$, $\int_0^T \underline{S}_{X_{n+2}}(x)\,\mathrm{d}x$ *is given by* (47).

For $x^c \in (x_{(k)}, x_{(k+1)})$, substituting (40) and (44), (41) and (45), (42) and (46), and (43) and (47), respectively, into

$$\overline{C}_{X_{n+2},\mathrm{op}}(T) = \frac{c_\mathrm{f} - (c_\mathrm{f} - c_\mathrm{p})E[\underline{S}_{X_{n+2}}(T+Y)]}{\int_0^T \underline{S}_{X_{n+2}}(x)\mathrm{d}x + E[Y]E[\underline{S}_{X_{n+2}}(T+Y)]} \tag{48}$$

yields our upper cost function for OAR threshold T applied to unit $n+2$ for Cases 1, 2, 3 and 4 from part (a) of Lemmas 6 and 7, respectively. For $x^c = x_{(k)}$, the upper cost function for unit $n+2$ is obtained, for $T \in (x_{(j)}, x_{(j+1)}]$, $j = 0, \ldots, k-1$, and for $T \in (x_{(j)}, x_{(j+1)}]$, $j = k, \ldots, n$, by substituting (40) and (44), and (43) and (47), respectively, into (48). As discussed before, $\overline{C}_{X_{n+2},\mathrm{op}}(T) = \overline{C}_{X_{n+2},\mathrm{op}}(x_{(n)})$ for $T \geqslant x_{(n)}$, so we only have to calculate the minimum of the upper costs over the interval $(0, x_{(n)}]$.

For $x^c \in (x_{(k)}, x_{(k+1)})$, denote by $\overline{T}^{j}_{n+2,\mathrm{op}}$, $\overline{T}^{k,c}_{n+2,\mathrm{op}}$ and $\overline{T}^{c,k+1}_{n+2,\mathrm{op}}$ the minima of $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ over the intervals $(x_{(j)}, x_{(j+1)}]$, for $j = 0, \ldots, k-1$, $k+1, \ldots, n-1$, $(x_{(k)}, x^c]$ and $(x^{(c)}, x_{(k+1)}]$, respectively. For $x^c = x_{(k)}$ denote by $\overline{T}^{j}_{n+2,\mathrm{op}}$, $j = 0, \ldots, n-1$ the minima of $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ over the intervals $(x_{(j)}, x_{(j+1)}]$, $j = 0, \ldots, n-1$. Then the overall optimal OAR threshold for unit $n+2$ corresponding to this NPI-based upper cost function, denoted by $\overline{T}^{*}_{n+2,\mathrm{op}}$, is the appropriate optimal threshold per interval for which the corresponding upper costs are minimal. Lemma 8 provides a useful result for computation of the optimal thresholds restricted to intervals between consecutive observations, similarly divided into Cases (a) and (b) as Lemmas 6 and 7. The proof of Lemma 8 is omitted as it follows immediately from the proof of Lemma 1.

Lemma 8 *For OAR with unit* $n+2$ *replaced upon failure or upon the first opportunity after threshold T, and where unit* $n+1$ *was preventively replaced at* (a) $x^c \in (x_{(k)}, x_{(k+1)})$ *and* (b) *at* $x^c = x_{(k)}$, *we have,*

(a) *If* $x^c \in (x_{(k)}, x_{(k+1)})$ *we have that, for* $T \in (x_{(j)}, x_{(j+1)}]$, $j = 0, \ldots, k-1$, $k+1, \ldots, n-1$, *for* $T \in (x_{(k)}, x^c]$, *and for* $T \in (x^c, x_{(k+1)}]$,

$$\overline{C}'_{X_{n+2},\mathrm{op}}(T) = 0 \Leftrightarrow \frac{(c_\mathrm{f} - c_\mathrm{p})(\underline{S}_{X_{n+2}}(T) - E[\underline{S}_{X_{n+2}}(T+Y)])}{E[Y]E[\underline{S}_{X_{n+2}}(T+Y)]} - \overline{C}_{X_{n+2},\mathrm{op}}(T) = 0 \tag{49}$$

where $\underline{S}_{X_{n+2}}(T)$ *is given by* (5)–(8), $E[\underline{S}_{X_{n+2}}(T+Y)]$ *is given in Lemma 6, and* $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ *is given by* (48).

(b) *If* $x^c = x_{(k)}$, (49) *holds for* $T \in (x_{(j)}, x_{(j+1)}]$, $j = 0, \ldots, n-1$, *where* $\underline{S}_{X_{n+2}}(T)$ *is given by* (3) *and* (4), $E[\underline{S}_{X_{n+2}}(T+Y)]$ *is given in Lemma 6, and* $\overline{C}_{X_{n+2},\mathrm{op}}(T)$ *is given by* (48).

For both cases we have that if there does not exist a $T \in (x_{(j)}, x_{(j+1)}]$ such that $\overline{C}'_{X_{n+2},\mathrm{op}}(T) = 0$, then $\overline{C}'_{X_{n+2},\mathrm{op}}(T)$ is minimal over this interval in one of the end-points.

In Appendix A we show that $\overline{C}''_{X_{n+2},\mathrm{op}}(T) > 0$, for T in the open intervals created by the consecutive observed failure times and right-censoring time, so the optimal OAR threshold $\overline{T}^{*}_{n+2,\mathrm{op}}$ is given by the value of $\overline{T}^{j}_{n+2,\mathrm{op}}$ for $j = 0, \ldots, k-1, k+1, \ldots, n-1$, or $\overline{T}^{k,c}_{n+2,\mathrm{op}}$ or $\overline{T}^{c,k+1}_{n+2,\mathrm{op}}$, for which this upper cost function is minimal, for the case that

$x^c \in (x_{(k)}, x_{(k+1)})$, while if $x^c = x_{(k)}$, $\overline{T}^*_{n+2,\text{op}}$ is given by the value of $\overline{T}^j_{n+2,\text{op}}$ for $j = 0, \ldots, n-1$, for which this cost function is minimal, where the optima per interval can be obtained using Lemma 8.

Examples

In this section, we illustrate our OAR results for unit $n+2$, depending on the observation for unit $n+1$. All data values in this section are actually simulated from the Weibull distribution with scale parameter 1 and shape parameter 2, which we also used for the simulation study presented later. These examples also serve to explain the steps used in the simulation study presented in the next section, which allows further conclusions on the performance of our method.

Example 3

Suppose we have observed $n = 5$ failure times: $x_{(1)} = 0.05925$, $x_{(2)} = 0.39199$, $x_{(3)} = 0.88939$, $x_{(4)} = 0.93966$ and $x_{(5)} = 1.04197$. Each preventive replacement cost $c_{\text{p}} = 1$, while each corrective replacement costs $c_{\text{f}} = 50$. Preventive replacement is possible only at opportunities occurring according to a Poisson process with rate $\lambda = 25$. For this situation, we have $\underline{T}^*_{6,\text{op}} = 0.78348$ with corresponding lower costs $\underline{C}_{X_6,\text{op}}(\underline{T}^*_{6,\text{op}}) = 29.00838$. Suppose that the uncensored value for X_6 is 0.99503 (again simulated from the same Weibull distribution), and that the residual time until the next opportunity (y_{op}) is 0.00316 (simulated from the Exponential distribution with rate 25). Hence, we actually have a right-censored observation for unit 6 when we aim at minimization of the lower cost function, as this unit would be replaced preventively at time $0.78348 + 0.00316 = 0.78664$, which is less than the simulated failure time 0.99503 for this unit, so here we have $x^c = 0.78664 \in (x_{(2)}, x_{(3)})$. Applying Lemma 5 for unit 7 yields $\underline{T}^*_{7,\text{op}} = 0.77644$, with $\underline{C}_{X_7,\text{op}}(\underline{T}^*_{7,\text{op}}) = 24.23028$.

For the upper cost function, we get for these same simulated values: $\overline{T}^*_{6,\text{op}} = 0.79596$ with $\overline{C}_{X_6,\text{op}}(\overline{T}^*_{6,\text{op}}) = 53.88973$, so that the value for X_6 is also a right-censored observation $x^c = 0.78664$ when we aim at minimization of the upper cost function, and for unit 7 we get $\underline{T}^*_{7,\text{op}} = 0.75416$ with $\overline{C}_{X_7,\text{op}}(\underline{T}^*_{7,\text{op}}) = 44.40754$.

In this example, the minimal values of the lower and upper cost functions both decrease from unit 6 to 7, which is what we might expect when we have a right-censored observation, as effectively a good preventive replacement was performed and this might lead to more optimism about the quality of the units in the sense that our extra information about unit 6 might lead us to expect that such units are more reliable early on. However, we also see that the optimal replacement thresholds, corresponding to the lower and upper cost functions, have both decreased. This may well be considered counter-intuitive at first sight, but such occurrences are explained by our use of the renewal reward theorem, in the sense that if we have a right-censored observation x^c, then the NPI-based probabilities assigned for the next unit lead to less probability of failure in the interval $(0, x^c)$. But, the optimization criterion based on the renewal reward theorem effectively takes into account the quality of a later unit as well, and now such a later unit is considered less likely to fail early on, hence replacing a unit earlier may indeed be worthwile as we expect the next unit to be more reliable early on, overall leading to lower probability of a costly corrective replacement being required. It will be clear from the examples in this paper, and from our simulation study reported in the next section, that intuition often cannot be trusted upon when aiming at such optimal OAR strategies using the renewal reward criterion.

Example 4

Suppose we have 10 failure times: 0.29329, 0.31628, 0.33891, 0.43982, 0.45423, 0.57758, 0.87230, 1.40275, 1.42630 and 1.83003; costs $c_{\text{p}} = 1$ and $c_{\text{f}} = 10$; and preventive replacement opportunities occurring with rate $\lambda = 10$. For unit 11, this leads to $\underline{T}^*_{11,\text{op}} = 0.15527$ and $\underline{C}_{X_{11},\text{op}}(\underline{T}^*_{11,\text{op}}) = 6.44046$. The simulated value for X_{11} is 0.49210 and the simulated residual time to the next opportunity y_{op} is 0.09331. Hence, for unit 11 we actually have a right-censored observation $x^c = 0.24858$ as this is less than the simulated value 0.49210, and this right-censored observation actually lies in the first interval, so $x^c \in (x_{(0)}, x_{(1)})$. In this case, (33) and (34) reduce to (16) while (36) and (37) reduce to (17). So the upper survival functions for X_{11} and X_{12} are now identical, and also the lower cost functions for units 11 and 12 are now identical, hence $\underline{T}^*_{11,\text{op}} = \underline{T}^*_{12,\text{op}}$ and $\underline{C}_{X_{11},\text{op}}(\underline{T}^*_{11,\text{op}}) = \underline{C}_{X_{12},\text{op}}(\underline{T}^*_{12,\text{op}})$.

For the upper costs we do not have such a similar result as the lower survival function is affected by the right-censored observation even when this falls in the first interval. In this case, $\overline{T}^*_{11,\text{op}} = 0.19049$ with $\overline{C}_{X_{11},\text{op}}(\overline{T}^*_{11,\text{op}}) = 10.49942$, so we get again a right-censored observation in the first interval, and correspondingly for unit 12 we get $\overline{T}^*_{12,\text{op}} = 0.18599$ with $\overline{C}_{X_{12},\text{op}}(\overline{T}^*_{12,\text{op}}) = 10.26470$.

Example 4 showed a case where, due to a right-censored observation for unit $n+1$ in the first interval, the lower cost functions for units $n+1$ and $n+2$ are the same. This results from the fact that, with such a right-censored observation for unit $n+1$ in the first interval, the upper survival functions for unit $n+1$ and unit $n+2$ based on $A_{(n)}$ and rc-$A_{(n+1)}$, respectively, are identical.[15] Because the corresponding lower survival functions are not identical, the upper cost functions for units $n+1$ and $n+2$ are always different. Such a right-censored observation for unit $n+1$ in the first interval is the only situation for which our optimal replacement threshold, corresponding to the lower cost functions, is identical for units $n+1$ and $n+2$.

Example 5

Suppose we have five failure times: 0.43424, 0.73462, 0.79511, 0.91624 and 1.44354; costs $c_p = 1$ and $c_f = 50$; and preventive replacement opportunities occurring with rate $\lambda = 25$. For unit 6 we get $\underline{T}^*_{6,\text{op}} = 0.27323$ with $\underline{C}_{X_6,\text{op}}(\underline{T}^*_{6,\text{op}}) = 3.65993$. The simulated value for X_6 is 0.33862 and the simulated residual time to the next opportunity y_{op} is 0.09942. Hence, the observation for X_6 corresponding to the replacement policy according to the lower cost function is a failure time, as $0.33862 < 0.27323 + 0.09942$. Intuitively, one might expect that the costs now increase, when considering unit 7 compared to unit 6, due to this failure of unit 6, and one might expect that one would wish to replace unit 7 earlier. Indeed, we get $\underline{T}^*_{7,\text{op}} = 0.19395$ and $\underline{C}_{X_7,\text{op}}(\underline{T}^*_{7,\text{op}}) = 5.15587$. For the optimal replacement threshold and corresponding costs when using the upper cost function we get a similar result, as $\overline{T}^*_{6,\text{op}} = 0.35095$ with $\overline{C}_{X_6,\text{op}}(\overline{T}^*_{6,\text{op}}) = 31.34369$, so that the observed value for X_6 is again a failure time, and $\overline{T}^*_{7,\text{op}} = 0.26385$ with $\overline{C}_{X_7,\text{op}}(\overline{T}^*_{7,\text{op}}) = 35.37360$.

We should, however, point out that this intuitively logical decrease of the optimal replacement threshold for unit $n+2$ and increasing costs, in case of an observed failure time for unit $n+1$, does not always occur. For example, suppose we have five failure times: 0.09385, 0.12496, 0.29252, 0.86637 and 1.58141, and further parameters as earlier in this example. The simulated value for X_6 is 1.04793 and the simulated residual time to the next opportunity y_{op} is 0.04607. Then we have $\underline{T}^*_{6,\text{op}} = 1.47690$ with $\underline{C}_{X_6,\text{op}}(\underline{T}^*_{6,\text{op}}) = 46.63504$. So, according to the optimal replacement policy based on this lower cost function, the observation for unit 6 is a failure time, and for unit 7 we get $\underline{T}^*_{7,\text{op}} = 0.78528$ with $\underline{C}_{X_7,\text{op}}(\underline{T}^*_{7,\text{op}}) = 42.16362$. For the upper cost functions we get $\overline{T}^*_{6,\text{op}} = 1.47331$ with $\overline{C}_{X_6,\text{op}}(\overline{T}^*_{6,\text{op}}) = 88.03892$, so that the observation for unit 6 is again a failure time, and $\overline{T}^*_{7,\text{op}} = 0.79329$ with $\overline{C}_{X_7,\text{op}}(\overline{T}^*_{7,\text{op}}) = 70.21287$. Hence, according to both our cost functions our optimal replacement threshold adapts in the sense that the costs decrease with smaller replacement threshold. This happens when the observation $x_{(1)}$ is relatively small and $x_{(n)}$ large, so that there is a large variation in the data. For unit $n+1$, OAR might then not be very effective, as is shown by the large value of the threshold implying that there is a large probability that a cycle would end with corrective replacement. If, in such a case, the failure time for unit $n+1$ is also large, as is the case in this example, one becomes relatively more optimistic about the quality of such units earlier on, so earlier replacement with a new unit might become more cost effective, and such further costs are implicitly taken into account in our optimization criterion based on the renewal reward theorem. This is a further illustration that, when using an optimality criterion based on this theorem in such replacement situations, one might get results which at first may appear to be counter-intuitive.

Simulations

In this section, we present results from a simulation study to illustrate our method and discuss several of its features. All simulations are performed with the statistical package *R*.[16] For the simulations reported here, all failure times are simulated from the Weibull distribution with scale parameter 1 and shape parameter 2, which has increasing hazard rate so preventive replacement may be sensible from theoretical perspective.[1,8] We have also performed simulations from the Weibull distribution with shape parameter 3, the conclusions of which were fully in agreement with those for shape parameter 2, hence we do not present these results explicitly. In this study, we compare our simulation results for the optimal replacement thresholds for the lower and upper cost functions with the theoretical optimal replacement thresholds T^*.[8] As only the ratio c_f/c_p is relevant for the optimum thresholds according to our cost functions, we have set $c_p = 1$ without loss of generality. We have run simulations with c_f equal to 10 or 50. The rate λ with which the preventive replacement opportunities occur is set at 10 or 25. The number of initially observed failure times, n, equals 10 or 50, and in each case we have simulated 1000 times. In the simulations, we do not assume a known upper bound for the support of X_{n+1}. Hence we restrict attention to the interval $(0, x_{(n)}]$, see our earlier discussion. Throughout we use $A_{(n)}$ for our inference leading to $\underline{T}^*_{n+1,\text{op}}$ and $\overline{T}^*_{n+1,\text{op}}$, and $A_{(n+1)}$ or rc-$A_{(n+1)}$ leading to $\underline{T}^*_{n+2,\text{op}}$ and $\overline{T}^*_{n+2,\text{op}}$, depending on whether the observation for unit $n+1$ is a failure time or a right-censored observation.

Table 2 gives the theoretical optimal replacement thresholds T^* and the corresponding minimal costs $C_{\text{op}}(T^*)$ for the Weibull failure time distribution used in our simulations.[8] We have also included the limiting values of these cost functions for $T \to \infty$, denoted by $C_{\text{op}}(\infty)$, which relate to no preventive replacement being carried out, so the value of λ is irrelevant and these values are the same as for the AR model in our approach.[7] To compare our method with the theoretical results, we have calculated $\Lambda_{\text{op}}(T^*) = (C_{\text{op}}(\infty) - C_{\text{op}}(T^*))/C_{\text{op}}(T^*)$, which indicates the loss, relative to the optimal costs, if no preventive replacements were carried out. Note that Dekker and Dijkstra[8] analyse performance of OAR with a similar quantity, but instead they take $C_{\text{op}}(\infty)$ as denominator, which of course has a slightly different interpretation of this performance measure

Table 2 Theoretical results

	$c_f = 10$		$c_f = 50$	
	$\lambda = 10$	$\lambda = 25$	$\lambda = 10$	$\lambda = 25$
T^*	0.256	0.300	0.077	0.109
$C_{\text{op}}(T^*)$	6.269	6.094	16.822	14.542
$C_{\text{op}}(\infty)$	11.284	11.284	56.419	56.419
$\Lambda_{\text{op}}(T^*)$	0.800	0.852	2.354	2.880

but gives effectively the same conclusions. The values of $\Lambda_{\rm op}(T^*)$ for the simulations with Weibull shape parameter 3 (not reported) were larger than for shape parameter 2, which shows that the effectiveness of opportunity-based preventive replacement increases with increasing Weibull shape parameter, which is logical as increasing Weibull shape parameter gives decreasing variance of the failure time distribution.

To present the simulation results we introduce the notation: $\underline{C}^*_{X_{n+i},{\rm op}} = \underline{C}_{X_{n+i},{\rm op}}(\underline{T}^*_{n+i,{\rm op}})$ and $\overline{C}^*_{X_{n+i},{\rm op}} = \overline{C}_{X_{n+i},{\rm op}}(\overline{T}^*_{n+i,{\rm op}})$ and we define $\underline{\Lambda}_{n+i,{\rm op}} = (C_{\rm op}(\underline{T}^*_{n+i,{\rm op}}) - C_{\rm op}(T^*))/C_{\rm op}(T^*)$ and $\overline{\Lambda}_{n+i,{\rm op}} = (C_{\rm op}(\overline{T}^*_{n+i,{\rm op}}) - C_{\rm op}(T^*))/C_{\rm op}(T^*)$ for $i=1, 2$. These Λ's indicate how good our optimal replacement strategies are compared to the corresponding theoretical optimal strategies, judged by comparing the loss in the theoretical long-run average costs per unit of time that would be incurred by using our optimal thresholds instead of the theoretical optimum, as fraction of the long-run average costs per unit of time in the theoretical optimum. Tables 3 and 4 present summaries of the simulation results for the lower and upper cost functions. We discuss the conclusions from the study in the next section.

In this study, we have recorded the number of times that our optimal OAR thresholds, and the corresponding optimal costs, are decreasing or increasing when comparing unit $n+1$ with unit $n+2$, after the information on unit $n+1$, that is, either an observed failure time or a right-censoring time, becomes available, see Tables 5 and 6. The number between brackets is the number of times that the increase or decrease was due to a right-censored observation.

Simulation conclusions

In this section, we discuss the main conclusions and observations from the simulation study.

1. The means and medians of $\underline{T}^*_{n+1,{\rm op}}$, $\underline{T}^*_{n+2,{\rm op}}$, $\overline{T}^*_{n+1,{\rm op}}$ and $\overline{T}^*_{n+2,{\rm op}}$ are all greater than the corresponding theoretical T^*'s. The medians of these optimal OAR thresholds are all less than the corresponding means, so the medians are closer to the theoretical T^*'s than the means. As the distributions of these simulated values are all skewed to the right, the medians may be better indications of the performance of our method than the means.
2. The means and medians of $\underline{C}^*_{X_{n+1},{\rm op}}$ and $\underline{C}^*_{X_{n+2},{\rm op}}$ are less, and the means and medians of $\overline{C}^*_{X_{n+1},{\rm op}}$ and $\overline{C}^*_{X_{n+2},{\rm op}}$ are greater than the corresponding long-run average costs per unit time in the theoretical optima, which is what we expect, although this does, of course, not hold for each individual simulated case due to the variation in the simulated data sets.
3. The means and medians of $\underline{T}^*_{n+1,{\rm op}}$ and $\overline{T}^*_{n+1,{\rm op}}$ tend to be greater than the corresponding means and medians of $\underline{T}^*_{n+2,{\rm op}}$ and $\overline{T}^*_{n+2,{\rm op}}$, respectively. For a few situations they were about the same, for example the medians of the

Table 3 Simulation results for the lower cost function

	$\underline{T}^*_{n+1,{\rm op}}$	$\underline{C}^*_{X_{n+1},{\rm op}}$	$\underline{\Lambda}_{n+1,{\rm op}}$	$\underline{T}^*_{n+2,{\rm op}}$	$\underline{C}^*_{X_{n+2},{\rm op}}$	$\underline{\Lambda}_{n+2,{\rm op}}$
Case 1:			$n=10,\ c_{\rm f}=10,\ \lambda=10$			
Mean	0.3712	5.2202	0.0809	0.3589	5.2965	0.0765
Median	0.3061	4.8860	0.0282	0.2997	5.0650	0.0262
SD	0.2364	1.8492	0.1267	0.2205	1.7948	0.1179
Case 2:			$n=50,\ c_{\rm f}=10,\ \lambda=10$			
Mean	0.2922	5.9467	0.0285	0.2903	5.9476	0.0276
Median	0.2698	5.8850	0.0120	0.2694	5.8801	0.0121
SD	0.1028	1.0901	0.0438	0.1008	1.0784	0.0422
Case 3:			$n=50,\ c_{\rm f}=50,\ \lambda=10$			
Mean	0.1048	16.0122	0.0394	0.1045	16.0163	0.0390
Median	0.0862	15.3060	0.0086	0.0860	15.3389	0.0086
SD	0.0634	5.0226	0.0864	0.0631	4.9915	0.0858
Case 4:			$n=50,\ c_{\rm f}=10,\ \lambda=25$			
Mean	0.3320	5.6478	0.0395	0.3294	5.6567	0.0388
Median	0.3124	5.5397	0.0190	0.3106	5.5556	0.0187
SD	0.1118	1.0545	0.0518	0.1105	1.0445	0.0513
Case 5:			$n=50,\ c_{\rm f}=50,\ \lambda=25$			
Mean	0.1448	12.6139	0.0778	0.1148	12.6387	0.0783
Median	0.1281	11.9794	0.0345	0.1282	12.0195	0.0345
SD	0.0726	4.6308	0.1140	0.0731	4.6022	0.1152

Table 4 Simulation results for the upper cost function

	$\overline{T}^*_{n+1,\mathrm{op}}$	$\overline{C}^*_{X_{n+1},\mathrm{op}}$	$\overline{\Lambda}_{n+1,\mathrm{op}}$	$\overline{T}^*_{n+2,\mathrm{op}}$	$\overline{C}^*_{X_{n+2},\mathrm{op}}$	$\overline{\Lambda}_{n+2,\mathrm{op}}$
Case 1:			$n=10,\ c_{\mathrm{f}}=10,\ \lambda=10$			
Mean	0.4807	7.9555	0.1266	0.4483	7.8842	0.1088
Median	0.4161	7.5862	0.0638	0.3879	7.6969	0.0477
SD	0.2548	2.3318	0.1567	0.2215	2.2288	0.1395
Case 2:			$n=50\ c_{\mathrm{f}}=10,\ \lambda=10$			
Mean	0.3222	6.5694	0.0335	0.3210	6.5626	0.0331
Median	0.3017	6.5180	0.0139	0.2985	6.5269	0.0141
SD	0.1067	1.1333	0.0515	0.1055	1.1185	0.0503
Case 3:			$n=50,\ c_{\mathrm{f}}=50,\ \lambda=10$			
Mean	0.1547	21.0004	0.0866	0.1531	20.9663	0.0845
Median	0.1346	20.3679	0.0400	0.1321	20.4406	0.0370
SD	0.0701	5.3749	0.1152	0.0699	5.3474	0.1145
Case 4:			$n=50,\ c_{\mathrm{f}}=10,\ \lambda=25$			
Mean	0.3614	6.2952	0.0440	0.3571	6.2962	0.0422
Median	0.3390	6.1900	0.0207	0.3367	6.2153	0.0198
SD	0.1194	1.1061	0.0596	0.1160	1.0944	0.0564
Case 5:			$n=50,\ c_{\mathrm{f}}=10,\ \lambda=25$			
Mean	0.1977	18.1813	0.1411	0.1947	18.1602	0.1356
Median	0.1832	17.7959	0.0811	0.1807	17.6933	0.0765
SD	0.0827	5.0747	0.1700	0.0814	5.0551	0.1652

Table 5 Comparison of optimal replacement threshold for units $n+1$ and $n+2$

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*
$\underline{T}^*_{n+1,\mathrm{op}} > \underline{T}^*_{n+2,\mathrm{op}}$	336 (222)	438 (380)	280 (246)	522 (466)	316 (299)
$\underline{T}^*_{n+1,\mathrm{op}} < \underline{T}^*_{n+2,\mathrm{op}}$	324 (229)	514 (437)	373 (362)	442 (357)	259 (241)
$\underline{T}^*_{n+1,\mathrm{op}} = \underline{T}^*_{n+2,\mathrm{op}}$	340 (340)	48 (48)	347 (347)	36 (36)	425 (425)
$\overline{T}^*_{n+1,\mathrm{op}} > \overline{T}^*_{n+2,\mathrm{op}}$	767 (622)	624 (559)	812 (772)	693 (637)	856 (822)
$\overline{T}^*_{n+1,\mathrm{op}} < \overline{T}^*_{n+2,\mathrm{op}}$	233 (94)	376 (288)	188 (165)	307 (202)	144 (116)

Table 6 Comparison of optimal costs for units $n+1$ and $n+2$

	Case 1	*Case 2*	*Case 3*	*Case 4*	*Case 5*
$\underline{C}^*_{X_{n+1},\mathrm{op}} > \underline{C}^*_{X_{n+2},\mathrm{op}}$	473 (451)	822 (817)	611 (608)	830 (823)	541 (540)
$\underline{C}^*_{X_{n+1},\mathrm{op}} < \underline{C}^*_{X_{n+2},\mathrm{op}}$	187 (0)	130 (0)	42 (0)	134 (0)	34 (0)
$\underline{C}^*_{X_{n+1},\mathrm{op}} = \underline{C}^*_{X_{n+2},\mathrm{op}}$	340 (340)	48 (48)	347 (347)	36 (36)	425 (425)
$\overline{C}^*_{X_{n+1},\mathrm{op}} > \overline{C}^*_{X_{n+2},\mathrm{op}}$	784 (716)	855 (847)	942 (937)	851 (839)	940 (938)
$\overline{C}^*_{X_{n+1},\mathrm{op}} < \overline{C}^*_{X_{n+2},\mathrm{op}}$	216 (0)	145 (0)	58 (0)	149 (0)	60 (0)

optimal threshold of the lower cost function in Case 5, this is due to the variation in the simulated data. Hence, these means and medians tend to move towards the theoretical values, indicating that our method adapts well to the additional information on unit $n+1$. From Table 5, it follows that for individual cases these optimal replacement thresholds can move in both directions, both if the observation for unit $n+1$ is a failure time or a right-censored observation, which may perhaps be somewhat counter-intuitive.

4. The means and medians of $\underline{\Lambda}_{n+1,\mathrm{op}}$, $\underline{\Lambda}_{n+2,\mathrm{op}}$, $\overline{\Lambda}_{n+1,\mathrm{op}}$ and $\overline{\Lambda}_{n+2,\mathrm{op}}$ are much smaller than the corresponding theoretical values $\Lambda_{\mathrm{op}}(T^*)$, as given in Table 2, which indicates that our thresholds nearly always lead to better performance than if one would not replace preventively at all.
5. There is no clear move of the means and medians of our optimal lower and upper costs towards the theoretical optimal costs in a similar manner as for the optimal thresholds. The effect of the additional information

on unit $n+1$ tends to be quite small on these cost function values. Table 6 shows that these optimal lower and upper cost function values can move in both directions if the observation for unit $n+1$ is a failure time (see Example 5 for discussion). However, in case of a right-censored observation for unit $n+1$, the optimal lower and upper cost function values never increased. This strongly agrees with intuition, but we have not been able to prove or disprove that this holds generally.

6. The means, medians and standard deviations of $\underline{T}^*_{n+1,\mathrm{op}}$, $\underline{T}^*_{n+2,\mathrm{op}}$, $\overline{T}^*_{n+1,\mathrm{op}}$ and $\overline{T}^*_{n+2,\mathrm{op}}$ for $n=10$ observed failure times are all greater than the corresponding values for $n=50$ observed failure times. This implies that our method indeed adapts well to the available data, in the sense that more data tend to bring these optimal thresholds closer to the theoretical values. The reduction in the variation of these thresholds is also due to the fact that more data provide less fluctuating information about the underlying failure time distribution. The means, medians and standard deviations of the $\underline{\Lambda}_{n+1,\mathrm{op}}$, $\underline{\Lambda}_{n+2,\mathrm{op}}$, $\overline{\Lambda}_{n+1,\mathrm{op}}$ and $\overline{\Lambda}_{n+2,\mathrm{op}}$ are all greater for $n=10$ than for $n=50$, which also indicates better performance of our method when there are more observed failure times.

   The means and medians of $\underline{C}^*_{X_{n+1},\mathrm{op}}$ and $\underline{C}^*_{X_{n+2},\mathrm{op}}$ for $n=10$ are less than the corresponding means and medians of $\underline{C}^*_{X_{n+1},\mathrm{op}}$ and $\underline{C}^*_{X_{n+2},\mathrm{op}}$ for $n=50$. The means and medians of $\overline{C}^*_{X_{n+1},\mathrm{op}}$ and $\overline{C}^*_{X_{n+2},\mathrm{op}}$ for $n=10$ are greater than the corresponding means and medians of $\overline{C}^*_{X_{n+1},\mathrm{op}}$ and $\overline{C}^*_{X_{n+2},\mathrm{op}}$ for $n=50$. So larger n, these lower and upper optimal cost function values tend to be closer to the theoretical optimum cost function values. These optimal lower and upper cost function values also vary less for larger n, which is shown by the smaller standard deviations for $n=50$ than for $n=10$.
7. If we increase λ, the occurrence rate of the preventive replacement opportunities, then the means, medians and standard deviations of the optimal replacement thresholds increase, and the means and medians of the corresponding optimal cost function values decrease. This agrees with intuition, and was also observed in a numerical study by Dekker and Dijkstra.[8] However, as we have shown in Example 2, this does not hold generally.
8. If the cost of corrective replacement, c_{f}, increases then the means, medians and standard deviations of our optimal replacement thresholds decrease, and the means, medians and standard deviations of the corresponding optimal cost function values increase. Hence, if c_{f} increases then we tend to earlier preventive replacements, if opportunities occur, but this cannot prevent the higher average costs. This also agrees with numerical observations by Dekker and Dijkstra.[8]

Conclusions

In this paper, we have developed and analysed theory for OAR from nonparametric predictive inferential perspective, providing a method to derive optimal preventive replacement thresholds that is fully adaptive to failure data from the process. This work extends our earlier studies for age replacement from the same perspective,[6,7] and supports the conclusions in those papers. This paper presents the first combination of aspects from classical stochastic processes, in the form of the homogeneous Poisson process describing the occurrence of preventive replacement opportunities, with NPI. The method proves successful in the sense that for relatively few failure data ($n=10$) the thresholds based on simulated data from a known Weibull distribution were mostly close to the theoretical optimum, and for larger numbers of failure data the method performs indeed better. We explicitly studied the method's ability to take information on unit $n+1$ into account, after applying our optimal age replacement strategy based on the first n failure times. This showed that the method can take this extra information into account appropriately, and also revealed that on some occasions counter-intuitive results may appear, which may be explained by the optimality criterion, for which the renewal reward theorem was used. In particular, via examples and simulations we have shown that it is possible that the optimal OAR threshold may exceed the optimal AR time corresponding to the same failure data and costs, and that increasing rate of occurrence of preventive replacement opportunities does not necessarily imply an increase of the replacement threshold.

Our method requires failure data, which may not be available in many practical situations. However, we regard our analysis as an important alternative approach to the classical OAR problem, as studied by Dekker and Dijkstra,[8] where the failure time distribution is assumed to be known, which is perhaps even less realistic. Both these approaches together can provide detailed insights, for example via simulation studies as we have presented and discussed in this paper, leading to better understanding of the optimality criterion used. Our method is restrictive in the sense that a lower and an upper cost function are derived, without any further comments on which of these to use for decision making or the possibility to choose a cost function in between these two extremes. This is because we did not want to make further assumptions that could influence our optimal replacement thresholds in rather vague manners. If we would have to choose between using the upper or lower cost function, we may have a slight preference for the upper cost function, as this relates to the NPI-based lower survival function and as such can be considered to be conservative. We would prefer, however, to study both these functions simultaneously, if the resulting optimal thresholds are close then one can be fairly confident that it may be reasonable to use one of them, if they differ substantially this may indicate that the data do not strongly support a particular threshold, and that it may not be possible to avoid further assumptions, for example via expert judgements,[4,5] to decide on a value for the threshold. It would be interesting to develop methods that combine NPI, on aspects of decision problems where data are sufficiently available, with subjective methods.

Quite a few counter-intuitive results appeared in this study, and to a lesser extend also in our NPI-based age replacement,[6,7] these are mostly due to the use of the renewal reward theorem to derive our optimality criterion. Although this criterion, taking into account many replacement cycles over a long ('infinite') time horizon, is mathematically attractive and well established, in our future research we will compare it with optimal (opportunity-based) age replacement strategies which take only the cost per unit of time over a single cycle into account,[2] which also fits more naturally with the possibility of changing strategies to adapt to all available information.

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Appendix A

Derivation of (13)

We derive an expression for $E[L_{\mathrm{op}}(T)]$ which is similar to Equation (9) of Dekker and Dijkstra,[8] but in a form more suitable for this paper. This expression holds for exponentially distributed Y, with probability density function $f_Y(.)$.

$$\begin{aligned}
E[L_{\mathrm{op}}(T)] &= E[\min(X, T+Y)] \\
&= \int_0^\infty E[\min(X, T+Y) \mid Y=y] f_Y(y)\,\mathrm{d}y \\
&= \int_{y=0}^{\infty} \int_{x=0}^{T+y} (1-F_X(x)) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= E[\min(X,T)] + \int_{y=0}^{\infty} \int_{x=T}^{T+y} S_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^T S_X(x)\,\mathrm{d}x + \int_{x=T}^{\infty} \int_{y=x-T}^{\infty} S_X(x) f_Y(y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^T S_X(x)\,\mathrm{d}x + \int_0^\infty S_X(T+x) S_Y(x)\,\mathrm{d}x \\
&= \int_0^T S_X(x)\,\mathrm{d}x + \int_0^\infty S_X(T+x) E[Y] f_Y(x)\,\mathrm{d}x \\
&= \int_0^T S_X(x)\,\mathrm{d}x + E[Y] E[S_X(T+Y)]
\end{aligned}$$

where the penultimate equality follows from the fact that $S_Y(x) = E[Y]\mathrm{d}F_Y(x)/\mathrm{d}x$, as Y has an Exponential distribution.

Proof of Lemma 1

$$\begin{aligned}
\underline{C}'_{X_{n+1},\mathrm{op}}(T) = \Bigg[& -(c_{\mathrm{f}}-c_{\mathrm{p}}) \frac{\mathrm{d}}{\mathrm{d}T} E[\overline{S}_{X_{n+1}}(T+Y)] \\
& \times \left(\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)] \right) \\
& - (c_{\mathrm{f}} - (c_{\mathrm{f}}-c_{\mathrm{p}}) E[\overline{S}_{X_{n+1}}(T+Y)]) \\
& \times \left(\frac{\mathrm{d}}{\mathrm{d}T} \int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y] \frac{\mathrm{d}}{\mathrm{d}T} E[\overline{S}_{X_{n+1}}(T+Y)] \right) \Bigg] \\
& \times \left(\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)] \right)^{-2}
\end{aligned} \tag{A.1}$$

Here,

$$\frac{\mathrm{d}}{\mathrm{d}T}E[\overline{S}_{X_{n+1}}(T+Y)] = -\frac{1}{E[Y]}(\overline{S}_{X_{n+1}}(T) - E[\overline{S}_{X_{n+1}}(T+Y)]) \tag{A.2}$$

see, for example Dekker and Dijkstra,[8] and

$$\frac{\mathrm{d}}{\mathrm{d}T}\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x = \overline{S}_{X_{n+1}}(T) \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1) yields

$$\begin{aligned}\underline{C}'_{X_{n+1},\mathrm{op}}(T) = &\frac{E[\overline{S}_{X_{n+1}}(T+Y)]}{\int_0^T \overline{S}_{X_{n+1}}(x)\,\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]} \\ &\times\left\{\frac{(c_\mathrm{f}-c_\mathrm{p})(\overline{S}_{X_{n+1}}(T)-E[\overline{S}_{X_{n+1}}(T+Y)])}{E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]}\right. \\ &\left. - \frac{c_\mathrm{f}-(c_\mathrm{f}-c_\mathrm{p})E[\overline{S}_{X_{n+1}}(T+Y)]}{\int_0^T \overline{S}_{X_{n+1}}(x)\,\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]}\right\}\end{aligned}$$

As the last expression is equal to $\underline{C}_{X_{n+1},\mathrm{op}}(T)$, and both $E[\overline{S}_{X_{n+1}}(T+Y)]$ and $\int_0^T \overline{S}_{X_{n+1}}(x)\mathrm{d}x + E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]$ are positive, we have, for $T\in[x_{(j)}, x_{(j+1)})$ and $j=0,\ldots,n$

$$\begin{aligned}&\underline{C}'_{X_{n+1},\mathrm{op}}(T) = 0 \Leftrightarrow \\ &\frac{(c_\mathrm{f}-c_\mathrm{p})(\overline{S}_{X_{n+1}}(T)-E[\overline{S}_{X_{n+1}}(T+Y)])}{E[Y]E[\overline{S}_{X_{n+1}}(T+Y)]} - \underline{C}_{X_{n+1},\mathrm{op}}(T) = 0\end{aligned}$$

Proof of $\underline{C}''_{X_{n+1},\mathrm{op}}(T)>0, \overline{C}''_{X_{n+1},\mathrm{op}}(T)>0, \underline{C}''_{X_{n+2},\mathrm{op}}(T)>0$ *and* $\overline{C}''_{X_{n+2},\mathrm{op}}(T)>0$

Obviously, we only prove this for $T\in(x_{(j)}, x_{(j+1)})$ and $j=0,\ldots,n-1$, as the second derivatives do not exist in the observed values $x_{(j)}$. Differentiating $C_{\mathrm{op}}(T)$ of Equation (14) with respect to T, and using (A.2) and (A.3) with X_{n+1} replaced by X, obtain

$$C'_{\mathrm{op}}(T) = \frac{E[L(T)]}{E^2[L_{\mathrm{op}}(T)]}g(T) \tag{A.4}$$

where

$$\begin{aligned}g(T) = &\frac{(c_\mathrm{f}-c_\mathrm{p})(S_X(T)-E[S_X(T+Y)])}{E[Y]} \\ &- E[S_X(T+Y)]C(T)\end{aligned} \tag{A.5}$$

Hence, as $E[L(T)]$ and $E[L_{\mathrm{op}}(T)]$ are both strictly positive, we have

$$C'_{\mathrm{op}}(T) = 0 \Leftrightarrow g(T) = 0. \tag{A.6}$$

Now,

$$\begin{aligned}C''_{\mathrm{op}}(T) = &\frac{1}{E^2[L_{\mathrm{op}}(T)]}\left[\left(\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)]\right)g(T)\right. \\ &+ E[L(t)]g'(T) \\ &\left.-\frac{2E[L(T)]g(T)(\frac{\mathrm{d}}{\mathrm{d}T}E[L_{\mathrm{op}}(T)])}{E[L_{\mathrm{op}}(T)]}\right]\end{aligned} \tag{A.7}$$

where

$$\begin{aligned}g'(T) = &\frac{(c_\mathrm{f}-c_\mathrm{p})}{E(Y)}\left(\frac{\mathrm{d}}{\mathrm{d}T}(S_X(T)-E[S_X(T+Y)])\right) \\ &-\frac{\mathrm{d}}{\mathrm{d}T}E[S_X(T+Y)]C(T) - E[S_X(T+Y)]C'(T)\end{aligned} \tag{A.8}$$

with $C'(T)$ the first derivative of the age replacement cost function (11). Moreover, we have

$$\frac{\mathrm{d}}{\mathrm{d}T}E[R(T)] = -(c_\mathrm{f}-c_\mathrm{p})\frac{\mathrm{d}}{\mathrm{d}T}S(T)$$

and using (A.2), with X_{n+1} replaced by X,

$$\frac{\mathrm{d}}{\mathrm{d}T}E[R_{\mathrm{op}}(T)] = \frac{(c_\mathrm{f}-c_\mathrm{p})(S_X(T)-E[S_X(T+Y)])}{E[Y]}$$

so that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}T^2}E[R_{\mathrm{op}}(T)] = &\frac{(c_\mathrm{f}-c_\mathrm{p})}{E[Y]}\frac{\mathrm{d}}{\mathrm{d}T}(S_X(T)-E[S_X(T+Y)]) \\ = &\frac{(c_\mathrm{f}-c_\mathrm{p})}{E^2[Y]}(S_X(T)-E[S_X(T+Y)]) \\ &-\frac{1}{E[Y]}\frac{\mathrm{d}}{\mathrm{d}T}E[R(T)]\end{aligned} \tag{A.9}$$

Substituting (A.9) into (A.8) yields

$$\begin{aligned}g'(T) = &\frac{\mathrm{d}^2}{\mathrm{d}T^2}E[R_{\mathrm{op}}(T)] + \frac{(S(T)-E[S(T+Y)])C(T)}{E[Y]} \\ &- E[S(T+Y)]C'(T) \\ = &\frac{g(T)}{E(Y)} - \frac{\frac{\mathrm{d}}{\mathrm{d}T}E[R(T)] - (\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)])C(T)}{E[Y]} \\ &- E[S(T+Y)]C'(T)\end{aligned} \tag{A.10}$$

Differentiating the age replacement cost function $C(T)$ (11) with the respect to T yields

$$C'(T) = \frac{(\frac{\mathrm{d}}{\mathrm{d}T}E[R(T)])E[L(T)] - (\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)])E[R(T)]}{E^2[L(T)]}$$

and since $E[L(T)]>0$ for all $T>0$,

$$\frac{\mathrm{d}}{\mathrm{d}T}E[R(T)] - (\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)])C(T) = E[L(T)]C'(T)$$

so that (A.10) can be written as

$$g'(T) = \frac{g(T)}{E(Y)} - \left(\frac{E[L(T)]}{E[Y]} + E[S_X(T+Y)]\right)C'(T) \tag{A.11}$$

Using (A.11), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}(E[L(T)]g(T)) &= \left(\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)]\right)g(T) + E[L(T)]g'(T)\\ &= \left(\left(\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)]\right) + \frac{E[L(T)]}{E[Y]}\right)g(T)\\ &\quad - \left(\frac{E[L(T)]}{E[Y]} + E[S_X(T+Y)]\right)E[L(T)]C'(T)\end{aligned} \tag{A.12}$$

Substituting (A.12) into (A.7) yields

$$\begin{aligned}C''_{\mathrm{op}}(T) = &\frac{1}{E^2[L_{\mathrm{op}}(T)]}\Bigg[\Bigg\{\frac{\mathrm{d}}{\mathrm{d}T}E[L(T)]\\ &-\frac{2(\frac{\mathrm{d}}{\mathrm{d}T}E[L_{\mathrm{op}}(T)])E[L(T)]}{E[L_{\mathrm{op}}(T)]} + \frac{E[L(T)]}{E[Y]}\Bigg\}g(T)\\ &-\left\{\frac{E[L(T)]}{E[Y]} + E[S_X(T+Y)]\right\}E[L(T)]C'(T)\Bigg]\end{aligned} \tag{A.13}$$

However, at the optimal OAR threshold T^*_{op} we have $g(T^*_{\mathrm{op}})=0$ according to (A.6). From Lemmas 2.2 and 3.2 of Coolen-Schrijner and Coolen[6] we know that the first derivatives of our NPI based age replacement cost functions $\underline{C}'_{X_{n+1}}(T)$, $\overline{C}'_{X_{n+1}}(T)$, $\underline{C}'_{X_{n+2}}(T)$, $\overline{C}'_{X_{n+2}}(T)$ are all negative. Substituting the lower and upper survival functions in $E[L(T)]$ and $E[S_X(T+Y)]$ yields corresponding lower and upper bounds which are all positive. Then, from (A.13), it follows that $\underline{C}''_{X_{n+1},\mathrm{op}}(T)$, $\overline{C}''_{X_{n+1},\mathrm{op}}(T)$, $\underline{C}''_{X_{n+2},\mathrm{op}}(T)$, $\overline{C}''_{X_{n+2},\mathrm{op}}(T)$ are all positive.

Proof of Lemma 3

1. $T\in[x_{(j)}, x_{(j+1)})$ with $j=0,\ldots,k-1$.
   In this case, Equation (32) can be obtained by noting that

$$\begin{aligned}&E[\overline{S}_{X_{n+2}}(T+Y)] =\\ &\int_0^{x_{(j+1)}-T} \overline{S}_{X_{n+2}}(x_{(j)})f_Y(y)\,\mathrm{d}y\\ &+\sum_{l=1}^{k-j-1}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\overline{S}_{X_{n+2}}(x_{(j+l)})f_Y(y)\,\mathrm{d}y\\ &+\sum_{l=k-j}^{n-j}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\overline{S}_{X_{n+2}}(x_{(j+l)})f_Y(y)\,\mathrm{d}y\\ &=\int_0^{x_{(j+1)}-T}\frac{n-j+2}{n+2}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y\\ &+\sum_{l=1}^{k-j-1}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\frac{n-j-l+2}{n+2}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y\\ &+\sum_{l=k-j}^{n-j}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\frac{(n-j-l+1)(n-k+2)}{(n+2)(n-k+1)}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y\end{aligned}$$

2. $T\in[x_{(k)}, x_{(k+1)})$.
   In this case, Equation (33) can be obtained by noting that

$$\begin{aligned}&E[\overline{S}_{X_{n+2}}(T+Y)] =\\ &\int_0^{x_{(k+1)}-T} \overline{S}_{X_{n+2}}(x_{(k)})f_Y(y)\,\mathrm{d}y\\ &+\sum_{l=1}^{n-k}\int_{x_{(k+l)}-T}^{x_{(k+l+1)}-T}\overline{S}_{X_{n+2}}(x_{(k+l)})f_Y(y)\,\mathrm{d}y\\ &=\int_0^{x_{(k+1)}-T}\frac{n-k+2}{n+2}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y\\ &+\sum_{l=1}^{n-k}\int_{x_{(k+l)}-T}^{x_{(k+l+1)}-T}\frac{(n-k+2)(n-k-l+1)}{(n+2)(n-k+1)}\lambda \mathrm{e}^{-\lambda y}\,\mathrm{d}y\end{aligned}$$

3. $T\in[x_{(j)}, x_{(j+1)})$ with $j=k+1,\ldots,n$.
   In this case, Equation (34) can be obtained by noting that

$$\begin{aligned}E[\overline{S}_{X_{n+2}}(T+Y)] &= \int_0^{x_{(j+1)}-T}\overline{S}_{X_{n+2}}(x_{(j)})f_Y(y)\mathrm{d}y\\ &+\sum_{l=1}^{n-j}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\overline{S}_{X_{n+2}}(x_{(j+1)})f_Y(y)\mathrm{d}y\\ &=\int_0^{x_{(j+1)}-T}\frac{(n-j+1)(n-k+2)}{(n+2)(n-k+1)}\lambda \mathrm{e}^{-\lambda y}\mathrm{d}y\\ &+\sum_{l=1}^{n-j}\int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T}\frac{(n-k+2)(n-j-l+1)}{(n+2)(n-k+1)}\lambda \mathrm{e}^{-\lambda y}\mathrm{d}y\end{aligned}$$

□

Proof of Lemma 6

We only prove part (a)-1 of the lemma, the proofs of the other parts are similar. Suppose $T\in(x_{(j)}, x_{(j+1)}]$ with

$j = 0, \ldots, k-1$, then

$$
\begin{aligned}
E[\underline{S}_{X_{n+2}}(T + Y)] = & \int_{0}^{x_{(j+1)}-T} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
& + \sum_{l=1}^{k-j-1} \int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
& + \int_{x_{(k)}-T}^{x^c-T} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
& + \int_{x^c-T}^{x_{(k+1)}-T} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
& + \sum_{l=1}^{n-k-1} \int_{x_{(k+l)}-T}^{x_{(k+l+1)}-T} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
& + \int_{x_{(n)}-T}^{\infty} \underline{S}_{X_{n+2}}(y + T) f_Y(y)\,\mathrm{d}y \\
= & \int_{0}^{x_{(j+1)}-T} \frac{n-j+1}{n+2} f_Y(y)\,\mathrm{d}y \\
& + \sum_{l=1}^{k-j-1} \int_{x_{(j+l)}-T}^{x_{(j+l+1)}-T} \frac{n-j-l+1}{n+2} f_Y(y)\,\mathrm{d}y \\
& + \int_{x_{(k)}-T}^{x^c-T} \frac{n-k+1}{n+2} f_Y(y)\,\mathrm{d}y \\
& + \int_{x^c-T}^{x_{(k+1)}-T} \frac{(n-k)(n-k+2)}{(n+2)(n-k+1)} f_Y(y)\,\mathrm{d}y \\
& + \sum_{l=1}^{n-k-1} \int_{x_{(k+l)}-T}^{x_{(k+l+1)}-T} \frac{(n-k-l)(n-k+2)}{(n+2)(n-k+1)} f_Y(y)\,\mathrm{d}y \\
& + \int_{x_{(n)}-T}^{\infty} 0 f_Y(y)\,\mathrm{d}y
\end{aligned}
$$

As Y has an Exponential distribution with parameter λ, the result follows.

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