# Reliability assessment of phased mission systems subjected to epistemic uncertainty and optimisation of the phase ordering

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**Abstract**

Reliability theory constructs mathematical models for assessment of probability of mission success. Sometimes, a system may be required to perform multiple tasks in a sequence, like an airplane, which has to (at least) take-off, cruise, and land again. Such situations are denoted as Phased Missions and are considered successful if the system successfully completes all the phases. Generally, in each of the phases, different conditions and failure modes may apply. Assessing the mission reliability requires us to merge individual specifications of all the phases into a specification of the whole mission.
The method for computation of phased mission reliability via Survival Signatures was introduced in [5]. Objective of this paper is to introduce its extension to include epistemic uncertainty on component failure rates and phase duration. Furthermore, the paper also introduces a novel perspective, showing how the ordering of the phases influence mission reliability.

## 1 Introduction

Phased mission (PM) system refer to situations where a system has to perform multiple tasks in a sequence [6]. In this paper, we will review how the PM reliability can be assessed with the help of survival signatures (introduced in [4] and extended to PM systems in [5]). An alternative method of modelling PMs via constructing a fault tree of phased mission is presented in [2].
In this paper, we will present novel results which allow us to assess reliability of a PM system with epistemic uncertainty about component failure rates and phase durations (Sect. 3), and our preliminary work on influence of phase ordering on the mission reliability (Sect. 4).

## 2 Phased Mission reliability

For an arbitrary system, its reliability is defined as the probability that it will complete its mission. If we require a system to stay functioning for its whole mission time $T_M$, the reliability can be calculated by assessing the probability of the event:

$$X_M := \left\{ \inf_{t \geq 0} \{t : X_S(t) = 0\} > T_M \right\},$$

where $X_S$ corresponds to a random variable (RV) representing the system state with $X_S = 0$ meaning that the system is not functioning.

In this section, we will introduce mathematical models for reliability assessment of a class of general and subsequently Phased Mission (PM) systems based on Survival Signatures.

### 2.1 System Reliability

A system is a device composed of distinctive subsystems. The reliability of each of the system components may be assessed individually. To assess the reliability of a system, it is necessary to model dependencies between states of the components and the states of the system.
For a binary system $S$ (i.e. we only distinguish two states - functioning and failed) composed of $N$ binary components, this dependency may be described by a structure function $\varphi_S : \{0,1\}^N \to \{0,1\}$. With dependency modelled by a structure function and

states of components being random, the probability of the system functioning may be obtained by calculating the expected value of its structure function:

$$Pr(X_S = 1) = \mathbb{E}\left\{\varphi(X_1, \ldots, X_N)\right\}, \qquad (1)$$

where $X_S \in \{0, 1\}$ represents the state of the system and $X_i \in \{0, 1\}$ the state of component $i$.

The component state can evolve in time and in that case we need to provide a stochastic model for its state at any time point $t$ considered, i.e. describe a stochastic process of component state $\{X_i(t)\}|_t$. Having this description for each of the system components, we may describe the stochastic evolution of the system state. For each time $t$, the probability of the system functioning can be obtained by calculating the expectation in Eq. 1 for (random) component states at time $t$.
If we assume that a component is non-repairable, i.e. it remains failed once failed, description of the stochastic process of its state simplifies greatly. In such case, it suffices to model time to failure (TTF) of a component as a positive random variable, say $T_i$. The probability that a component functions at an arbitrary positive time $t$ is then equivalent to the probability that the component survived up to time $t$, i.e. probability of event $\{T_i > t\}$. This probability is modelled by a survival function $R_i(t) := Pr(T_i > t)$. We will here-on consider systems with non-repairable components.

## 2.2 Survival signatures

Evaluation of Eq. 1, requires us to sum over the set of all possible component states, the cardinality of which increases exponentially with the number of components. This subsection introduces a methodology, which allows us to simplify this calculation for a specific, but general enough, class of scenarios.

By the law of total expectation, for arbitrary RVs $A, B$ on common sample space, the expectation of RV $A$ can be equivalently expressed as the expectation of the conditional expectation $\mathbb{E}\{A|B\}$, i.e. $\mathbb{E}\{A\} = \mathbb{E}\{\mathbb{E}\{A|B\}\}$. The key idea is to introduce a convenient auxiliary RV.
Consider a system consisting of $K$ types of components s.t. TTFs of components of the same time are exchangeable (or i.i.d.). We define a $K$-dimensional random vector (process) $L(t)$, elements of which will represent the number of functioning components of respective types. With such a choice, the conditional expectation from the law of total expectation $\mathbb{E}\{X_S|L\}$, represents probability that a system is functioning conditioned upon there being exactly $l_j$ components of type $j$ functioning. This special conditional expectation was named Survival Signature [4] and we will denote it as $\Phi_S(l)$. For every vector $l$ its value can be computed as a ratio of functioning states among all the states for which $L$ holds, i.e.:

$$\Phi_S(l) := \frac{|\{x \in \Omega_X : l \text{ holds } \wedge \varphi(x) = 1\}|}{|\{x \in \Omega_X : l \text{ holds}\}|}, \qquad (2)$$

where $\Omega_X := \{0, 1\}^N$ is the state of possible values of combinations of component states.

Furthermore, a system is considered coherent if its structure function satisfies:

- $\varphi(\vec{0}) = 0$, system is failed if all its components are failed,
- $\varphi(\vec{1}) = 1$, system is functional if all its components are functional,
- $\varphi$ is a non-decreasing function, i.e. improving state of a component does not worsen the state of the system.

For a coherent system composed of non-repairable components, we may equivalently describe its probability of being functional by describing the distribution of a RV $T_S$, representing the failure time of the system. For system a composed of $K$ distinct types with exchangeable failure types in each group and total amount of $M_j$ component of type $j$ in the system, distribution of system failure time can be expressed via its Survival Signature as:

$$P(T_S > t) = \sum_{l_1=0}^{M_1} \ldots \sum_{l_K=0}^{M_K} \Phi_S(\vec{l}) Pr(L(t) = \vec{l}), \qquad (3)$$

where

$$Pr(L(t) = \vec{l}) = \prod_{j=1}^{K} \left[ \binom{M_j}{l_j} [R_j(t)]^{l_j} [1 - R_j(x)]^{M_j - l_j} \right], \qquad (4)$$

where $R_j(.)$ denotes the survival function common to components of type $j$.

Once we can model a system solely by modelling its TTF with a survival function $R_S(.)$, the event describing successful mission completion can be simplified to an equivalent event $X_M = \{T_S > T_M\}$. Mission reliability can then be assessed as:

$$Rel_S = R_S(T_M). \qquad (5)$$

## 2.3 Survival signatures for PM

A phased mission consists of a series of $W$ phases. The whole system mission time $T_M$ may be divided into intervals $[\tau_{i-1}, \tau_i]$, for $i = 1, \ldots, W$ representing time frames of the mission phases. For each of the phases we have to consider that:

- different components might be used in different ways, which can be modelled by separate structure functions for each of the phases, and
- components might be subjected to different conditions, which affects their deterioration rates and, therefore, failure probabilities.

A phased mission is considered successful if it successfully completes all its phases. We can reflect this by constructing a structure function for the whole mission. Taking the time evolution of (non-repairable) component states into account, the structure function of a mission can be expressed as:

$$\varphi_{PM}(x(\tau_1), \ldots, x(\tau_W)) = \prod_{i=1}^{W} \varphi_i(x(\tau_i)), \qquad (6)$$

where $\varphi_i$ is structure function describing requirements on phase $i$ and vector $x(\tau_i)$ represents states of components at the end of phase $i$.

To simplify PM reliability computation in the way we did with single phase systems via Survival Signatures, just minor differences have to be considered.

**Phased mission state space**

PM structure function is now a function of component states at the end of the phases. This means that every component may be again functional or not at each of those. Care must be taken to include our assumption of non-repairability of the components, i.e. to exclude states for which some component is non-functional at one time and functional later. The new component state evolution space, $\Omega_X$, will be a subset of $\{0, 1\}^{K \cdot W}$ which excludes these.

**Component grouping**

Grouping of components into types with exchangeable lifetimes have to take into account whether components of the group are stressed in exactly the same way throughout the whole mission. In simplified case, where all the components are stressed in all the phases, we may employ the same discrimination as in single phased systems, i.e. grouping them by they physical type. Further discussion on component grouping is provided in [5].

**State space decomposition**

The auxiliary RV $L$, introduced via the law of total expectation for convenient state space decomposition, will just change form reflecting the change of state space and component grouping. It will, again, be a vector of values which represent the number of functioning components in a group, now also for end of each phase. Dimension of the vector will therefore be $K \cdot W$, where $K$ is the number of component groups and $W$ the number of phases. The ordering can be made arbitrary, e.g.:

$$\vec{l} = (\vec{l}_1, \ldots, \vec{l}_W),$$

where $\vec{l}_i$ is a $K$-dimensional vector, elements of which represents number of respective functioning components at the end of phase $i$.

**Survival signature**

Taking the changes introduced earlier, the survival signature is defined exactly in the same way. We only need to reflect the change of $\Omega_X$ and what does it mean that $l$ holds (i.e. $x$ is such that $l_{ij}$ components of type $j$ function at the end of phase $i$).

**Phased mission reliability**

The PM reliability can be computed as the expectation of the structure function for RV $\vec{X} := (X(\tau_1), \ldots, X(\tau_W))$. We can decompose the expectation via law of total expectation through the augmented auxiliary RV $L$ as:

$$\begin{aligned} Rel_{PM} &:= \mathbb{E}\left\{\varphi_{PM}(\vec{X})\right\} \\ &= \sum_{l_1=0}^{M_1} \cdots \sum_{l_{KW}=0}^{M_{KW}} \Phi_{PM}(\vec{l}) Pr(\vec{L} = \vec{l}), \end{aligned} \qquad (7)$$

where the mixing probability $Pr(\vec{L} = \vec{l})$ represents the probability that $l_i^j - l_{i-1}^j$ failures of component of group $j$ occur in phase $i$. In the case where all the components are present in all the phases, it can be calculated as:

$$Pr(\vec{L} = \vec{l}) = \prod_{i=1}^{W} \prod_{j=1}^{K} \binom{l_{i-1}^j}{l_i^j} p_{ij}^{l_i^j} (1 - p_{ij})^{(l_{i-1}^j - l_i^j)}, \quad (8)$$

where $p_{ij}$ denotes probability that a single component of group $j$ survives phase $i$, given that it has been functioning at the beginning of the phase, and $l_0^j$ is the total amount of components of group $j$ in the system (i.e. the amount functioning at the beginning of the first phase).

# 3 Imprecision

Engineering, and also other applied fields, have to take uncertainties in our models into account in order to produce reliable conclusions. The first step is to reflect possible uncertainties in the value of parameters of our models. Although the uncertainty

may be modelled by stochastic models and inferred via statistical methods, often we face a situation in which we need to include an additional, strong, assumption to be able to carry out the inference. This may bias our analyses and make them unreliable. In cases of limited information (or also for the purpose of sensitivity analysis), interval models may be employed, which model our uncertainty of a parameter value by an interval of its possible range. Imprecisions in parameters of stochastic models are studied within the theory of Imprecise Probabilities [1, 3].
In this section, we will derive novel results which take into account interval uncertainty in the duration of mission phases and component failure laws. We will restrict ourselves to the specific scenario in which the sub-systems, representing individual phases, are coherent and the failure probability of components is modelled by exponential distributions, i.e. a constant failure rate law. Restriction to constant failure rate models provides us with a necessary simplification of the expression for PM system reliability while still allowing us to address a wide class of real world situations.

## 3.1 Study case

We will here-on assume that the:

- structure function of each of the phases corresponds to a structure function of some coherent system. I.e. each $\varphi_i(.)$ is non-decreasing in $x$ and therefore $\varphi_{PM}(.)$, a product of non-decreasing functions, is also non-decreasing in $(x(\tau_1), \ldots, x(\tau_W))$ (in the sense of partial ordering of component states),
- hazard rate of components of group $j$ in phase $i$ is constant and equal to $\lambda_{ij}$. I.e. the conditional probability from Eq. 8 will take the form $p_{ij} = \exp(-\Delta_i \cdot \lambda_{ij})$, where $\Delta_i := (\tau_i - \tau_{i-1})$ is the duration of phase $i$.

## 3.2 Monotonicity properties

**Theorem 1** *Reliability of a PM (Eq. 7) satisfying assumptions from Sect. 3.1 is non-increasing function of* $\lambda := (\lambda_{11}, \ldots, \lambda_{WK})$.

**Proof 1** *(Thm. 1)*

*§1. All the RVs representing component states are binary, therefore:*

$$Pr(X > x) = \begin{cases} 0 & ; x \geq 1 \\ 1 & ; x < 0 \\ Pr(X = 1) & ; x \in [0, 1) \end{cases}$$

*§2. For each component $i$*

$$Pr(X_i(t) = 1) = Pr(T_i > t),$$

*where $T_i$ is the component failure time and $Pr(T_i > t) = \exp(-H_i(t))$, where $H_i(t)$ is the hazard function of component $i$.*

*§3. $H_i(t) = \sum_{j=0}^{\rho(t)} \lambda_{j,c(i)} \Delta_j$ for each time, which coincide with end of some of the mission phases. $\rho(.)$ is a mapping selecting which one is that, and $c(.)$ select to which group the component belongs.*

*§4. Hazard rate is strictly increasing in $\lambda_{j,c(i)}$, and, therefore, $Pr(X_i(t))$ is strictly decreasing.*

*§5. Given two processes $\tilde{X}(t), X(t)$ of component states, subjected to failure rates $\tilde{\lambda}, \lambda$, respectively, s.t. $\tilde{\lambda} \geq \lambda$, $\tilde{X}(t)$ is stochastically dominated by $X(t)$ at each time corresponding to an end of a phase.*

*§6. Since PM reliability is an expectation of a monotone function, the expectation is lower or equal for the dominated process.* □

**Theorem 2** *Reliability of a PM (Eq. 7) satisfying assumptions from Sect. 3.1 is non-increasing function of* $\Delta := (\Delta_1, \ldots, \Delta_W)$.

**Proof 2** *(Thm. 2)*

*§1. The reliability of a PM effectively depends on phase time through the conditional probability of component failure during a phase, $p_{ij} = \exp(-\Delta_i \cdot \lambda_{ij})$ (from Eq 8).*

*§2. For any altered system with $\tilde{\Delta}_i = \alpha_i \Delta_i$, we can construct an auxiliary PM system with equivalent expression for reliability by, instead, assuming that $\tilde{\Delta}_i = \Delta_i$ and $\tilde{\lambda}_{ij} = \alpha_i \lambda_{ij}, \forall j$.*

*§3. $\forall i : \alpha_i \geq 1 \Rightarrow \forall i, j : \tilde{\lambda}_{ij} \geq \lambda_{ij} \Rightarrow \tilde{\lambda} \geq \lambda$.*

*§4. Reliability of PM is non-increasing in $\lambda$, therefore reliability of the altered system is lesser or equal to that of the former one.* □

## 3.3 Implications to interval analysis

We have proved, for our specific scenario, that the reliability of such a PM is monotone in both failure rates and phase durations, so interval analysis can be carried out at the cost of just two precise analyses.

If we are to assess reliability of such PM across a set of plausible failure rates $\lambda \in \Omega_\lambda$ and phase durations $\Delta \in \Omega_\Delta$, the lower bound can be attained in the upper extreme corner of $\Omega_\lambda \times \Omega_\Delta$, i.e.:

$$\min_{(\lambda,\Delta) \in \Omega_\lambda \times \Omega_\Delta} Rel_{PM}(\lambda, \Delta) = Rel_{PM}\left(\overline{\lambda}, \overline{\Delta}\right), \quad (9)$$

where $\overline{\lambda} := \max\{\lambda \in \Omega_\lambda\}$ and $\overline{\Delta} := \max\{\Delta \in \Omega_\Delta\}$ Mutatis mutandis for the upper bound.

# 4 Ordering of the phases

In robust design, we are interested in how to design our system to increase their performance, one of which is their reliability. Sometimes, we require a system to perform multiple actions, but we might not be dogmatic about which order it does it. This section addresses the question of whether, and how does the reliability of a PM depend on the ordering of the phases.

## 4.1 Examples

In this section, we will introduce examples which show that the order matters. The presented figures depict mission reliability for an arbitrary time $t \in [0, T_M]$ (the means of conducting such analyses is described in [5] and will be omitted here).

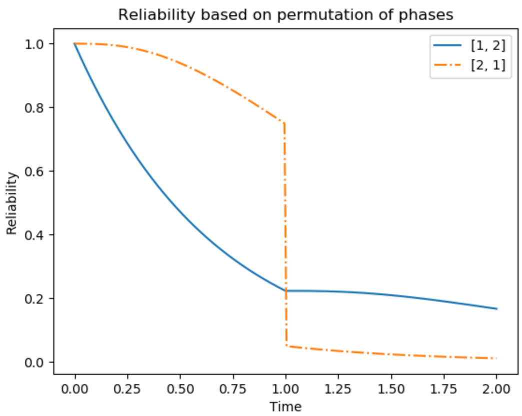


Figure 1: Survival function of a simple PM with serial (1) and parallel (2) phases for both possible orderings of the phases.

The first example depicts (in Fig. 1) differences in reliability functions for a PM consisting of 3 components with exchangeable TTF distributions (i.e. K=1) and equal failure rate in each of the phases. One of the phases requires all the components to be functional, the other just one of them (serial and parallel systems, respectively). Reliability of the whole PM is equal to reliability at the time $t = 2$, the end of mission. Even from this simple example it is apparent, that ordering of the phases plays a significant role in system design.

Second example (in Fig. 2) does the similar, just to provide a slightly more complex result for comparison. Now, the PM consists of 3 phases with the same component failure rates. Structure functions for each phase are described by reliability block diagrams depicted in Fig. 3.

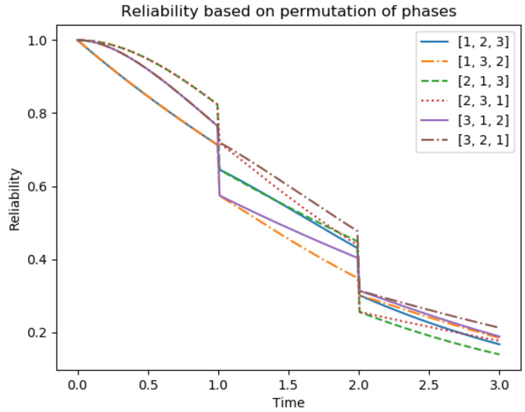


Figure 2: Survival function of a PM with three phases for all the possible permutations of phase ordering.

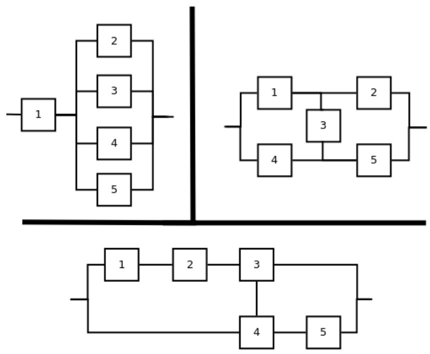

Figure 3: Reliability block diagrams for mission phases. Top-left: phase 1, top-right: phase 2, bottom: phase 3. The numbers identify individual physical components.

The examples show a phenomenon which was observed irrespective of the actual structure functions of the phases. The survival function of the PM systems differs greatly for possible orderings. The same applies for the mission reliability (survival function at the mission time $T_M$). Therefore, the phase ordering should be optimised whenever it is possible.

## 4.2 Properties of the ordering

No mathematical relations, that could easily decide which ordering is better has been found. In order to find the best ordering, a full optimisation process has to be conducted over all possible permutations of phase orders. This means, that for each of the permutation, we need to calculate the PM system survival signature.

Nevertheless, a simple relation has been found, which can either serve as a heuristics to avoid necessity of calculating signatures for each ordering, or as an tool to restrict our search space.

A simple bound on PM system reliability can be obtained by applying the chain rule. Since:

$$Rel_{PM} = Pr(F_1)Pr(F_2|F_1)\dots Pr(F_W|F_1,\dots,F_{W-1}),$$

where event $F_i$ represents successful completion of phase $i$.
Given that all $Pr(F_i|F_{1:i-1})$ are probabilities (i.e. $\leq 1$), the PM reliability can be bounded by $Pr(F_1)$, and consequently also by $Pr(F_i)$ for any $i$. Therefore:

$$Rel_{PM} \leq \min_{i\in 1,\dots,W} Pr(F_i), \tag{10}$$

where $Pr(F_i) = \mathbb{E}\{\varphi_i(X(\tau_i))\}$.

### 4.3 Implications

The derived bound (Eq. 10) depends on the ordering of the phases only through the hazard function which influences the probability that a component is functional at time $\tau_i$ and can be computed without taking into account the actual structure of a PM. This means that these partial (smaller and easier) survival signatures can be calculated only once before the optimisation process and used to discard some orderings without the need to construct the (more expensive) survival signature of the whole PM system.

Consequently, the bound may also be used for preliminary reliability analysis to discard inadmissible designs or identify critical phases and components.

## 5 Conclusions

We have reviewed survival signatures for system reliability modelling [4] and its generalisation for phased missions (PMs) [5].
We have provided theorems which allows us to assess reliability of phased missions subjected to epistemic uncertainty in both component failure rates and phase durations. Reliability of a special case of phased mission was shown to be a monotone function of these both (Thms. 1,2). This allows us to calculate lower and upper probabilities of successful completion of a mission with almost no further computational effort (Eq. 9). The main contribution to computational complexity arises from calculating the survival signature and has to be done only once, i.e. the same amount as for assessing reliability of precisely specified system.
Further, we have introduced preliminary work on dependency of PM reliability on ordering of the phases. Examples provided (Figs. 1, 2) clearly show that the ordering matters, which introduces further possibilities for designing reliable systems. So far, in order to find the most reliable phase ordering, an optimisation has to be run over the set of all permutations of mission phase orderings.
A simple heuristic method has been introduced which aims to help with PM system design (Thm. 10). It provides no guarantees of optimality, but may be used as a heuristics, or to restrict the search space for reliability optimisation.

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