Branching random walks and fractals

Ben Hambly

(joint with David Croydon, Philippe Charmoy)

Mathematical Insitute

University of Oxford

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- Random self-similar fractals and branching processes
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Fractals

A *fractal* is a set with some form of self-similarity.

Mathematical Examples:

- Self-similar sets such as the Cantor set, Sierpinski gasket or carpet.
- Random objects such as the sample paths of Brownian motion or Levy processes.
- Scaling limits of critical statistical mechanics models
- Attractors from dynamical systems such as Julia sets.

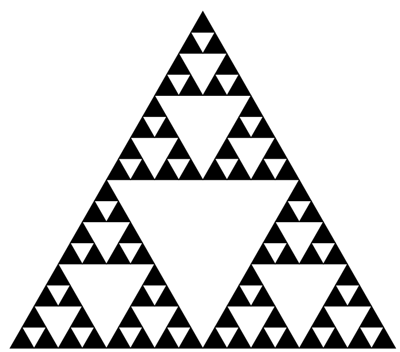
Of course, according to Mandelbrot, they are ubiquitous in nature!

Self-similar sets and trees

A self-similar set K is the fixed point of a family $\phi_i, i = 1, \ldots, n$ of contraction maps

$$K = \bigcup_{i=1}^{n} \phi_i(K).$$

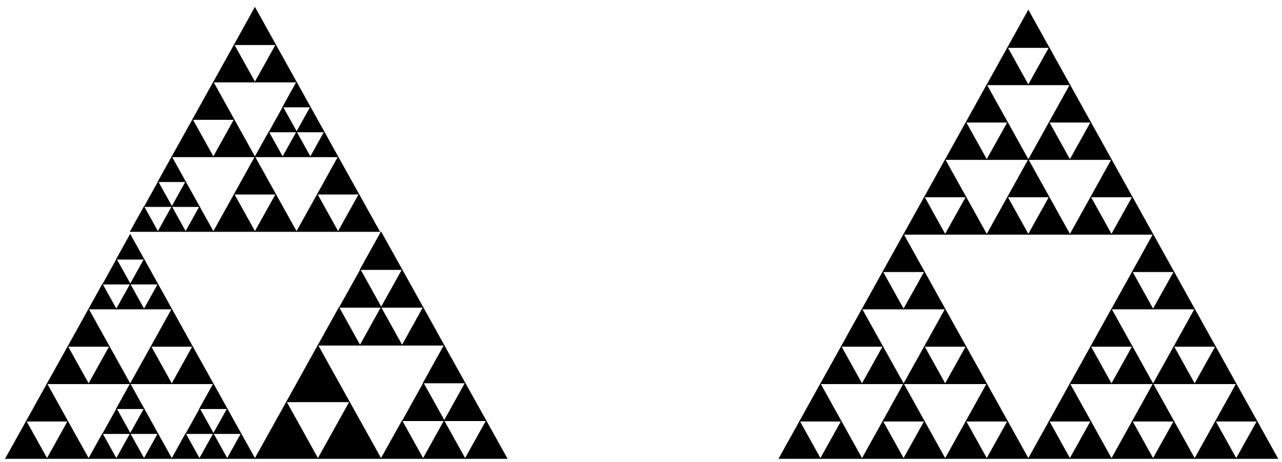
Each scaled copy of the whole has an address $\mathbf{i} = i_1 i_2 \cdots_k$ so that $K_{\mathbf{i}} = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(K)$. Each address is a point in the tree $\{1, \ldots, n\}^{\mathbb{N}}$. The Sierpinski gasket:



The fractal dimension $\log 3 / \log 2$ is given by the rate of growth of the tree.

Random self-similar sets

Two possible simple randomizations of the Sierpinski gasket:

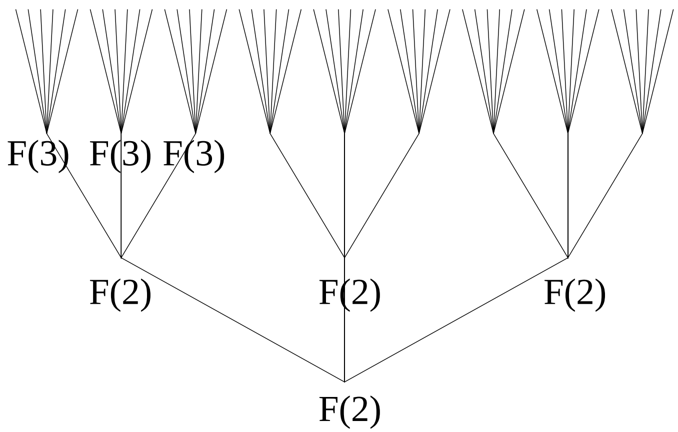


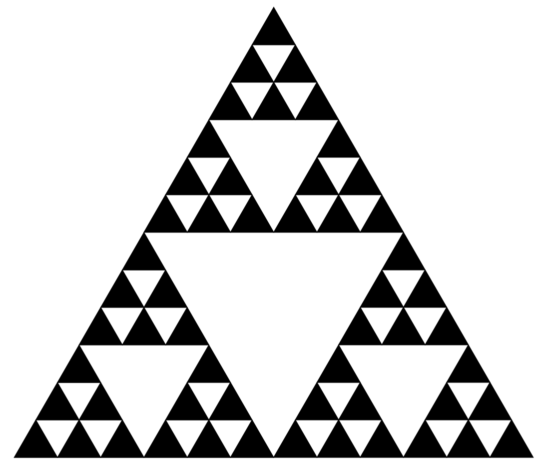
The LHS is a random recursive fractal, in that each triangle is randomly subdivided into 3 or 6.

The RHS is a homogeneous random fractal, in that at each scale we choose randomly to divide all triangles into 3 or 6.

The homogeneous random tree

The first stages and the tree for a homogeneous random gasket where at each level 2 or 3 is independently chosen with probability $p, 1-p$



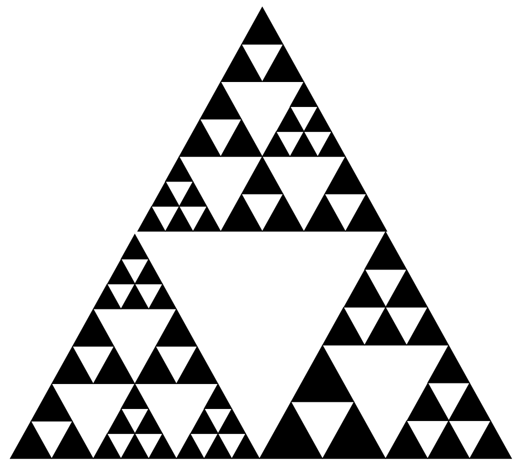


The growth rate is $3^k 6^{n-k}$ where k is the number of 2s in the construction sequence. The fractal dimension is

$$d_f = \frac{p\log 3 + (1-p)\log 6}{p\log 2 + (1-p)\log 3}.$$

The random recursive tree

The first stages of a random recursive gasket where each 2, 3 is independently chosen with probability $p, 1-p$ within each triangle.



The tree of cell addresses is now a Galton-Watson branching process.

However we need a more sophisticated model to compute the dimension.

General (CMJ) branching processes

To tackle a range of examples like this we use a branching process description.

An individual x in a general branching process has

1. offspring whose birth times are a point process ξ_x on $(0, \infty)$,
2. a lifetime which is a non-negative random variable L_x,
3. a characteristic which is a (possibly random) càdlàg function ϕ_x on $\mathbb{R}$.

We make no assumption on the joint distribution of (ξ_x, L_x, ϕ_x) and allow ϕ_x to depend on the progeny of x. Each individual evolves independently.

Let

$$\xi(t) = \xi((0,t]),\ \nu(dt) = \mathbb{E}\xi(dt),\ \xi_\gamma(dt) = e^{-\gamma t}\xi(dt),\ \nu_\gamma(dt) = \mathbb{E}\xi_\gamma(dt).$$

We assume that the GBP is super-critical in that $\nu(\infty) > 1$.

Then there exists a Malthusian parameter $\gamma \in (0, \infty)$ such that

$$\nu_\gamma(\infty) = 1.$$

Let $\mu = \int_0^\infty t\nu_\gamma(dt)$.

The individuals of the population are counted using the characteristic ϕ through the characteristic counting process Z^ϕ defined by

$$Z^\phi(t) = \sum_{x \in \mathcal{T}} \phi_x(t - \sigma_x) = \phi_\emptyset(t) + \sum_{i=1}^{\xi_\emptyset(\infty)} Z_i^\phi(t - \sigma_i),$$

where σ_x is the birth time of the individual x, $\mathcal{T}$ is the ancestral tree and Z_i^ϕ are i.i.d. copies of Z^ϕ.

Counting with characteristics

- The population size:

$$\phi(t) = I_{0 \le t \le L},$$

then $Z^\phi(t)$ corresponds to the number of individuals in the population alive at time t.

- For the calculation of the Minkowski dimension

$$\phi(t) = \xi(\infty) - \xi(t),$$

then $\phi(t)$ corresponds to the number of offspring born after time t to parents born up to time t.

- Later we will use characteristic functions whose corresponding counting process contains information about the Minkowski content, the spectral counting function or the heat content of the set.

Random recursive fractals

A random recursive fractal is a compact subset K of $\mathbb{R}^d$ determined by a random number N and random contracting similitudes $\Phi_1, \ldots, \Phi_N$, with contraction ratios $R_1, \ldots, R_N$. The set K is such that

$$K = \bigcup_{i=1}^{N} \Phi_i(K_i), \quad \text{a.s.},$$

where $K_1, \ldots, K_N$ are i.i.d. copies of K.

<u>Theorem:</u> Let K be a non-empty random recursive fractal with int$(K_i) \cap$ int$(K_j) = \emptyset$ for all i, j. Write $(N, R_1, \ldots, R_N)$ for the random variable of number of similitudes and their ratios, then a.s.

$$\dim K = \alpha := \inf \left\{ s : \mathbb{E}\left(\sum_{i=1}^{N} R_i^s \right) \leq 1 \right\}.$$

Connection with the GBP

The general branching process for a random recursive fractal has

$$\xi_x = \sum_{i=1}^{N_x} \delta_{-\log R_{x,i}}.$$

For the first generation of offspring this means that

$$e^{-\sigma_i} = R_i.$$

The offspring x born around time t correspond to compact sets K_x of size around e^{-t}. As

$$\mathbb{E}\int_0^\infty e^{-sx}\xi(dx) = \mathbb{E}\left(\sum_{i=1}^{N} R_i^s\right),$$

the Malthusian parameter of the underlying general branching process is equal to the almost sure Hausdorff/Minkowski dimension of the set K.

Renewals and martingale

Two key ideas for the GBP:

1. The functions $z^\phi(t) = e^{-\gamma t}\mathbb{E}Z^\phi(t)$ and $u^\phi(t) = e^{-\gamma t}\mathbb{E}\phi(t)$, satisfy the renewal equation
$$z^\phi(t) = u^\phi(t) + \int_0^\infty z^\phi(t-s)\nu_\gamma(ds).$$
2. Let $\mathcal{F}_x = \sigma(\{(\xi_y, L_y) : \sigma_y \leq \sigma_x\}), \quad \mathcal{F}_t = \sigma(\mathcal{F}_x, \sigma_x \leq t)$ and $\Lambda_t = \{x \in \mathcal{T} :\ x = yi$ for some $y \in \mathcal{T}, i \in \mathbb{N},$ and $\sigma_y \leq t < \sigma_x\}$. The process M defined by
$$M_t = \sum_{x\in\Lambda_t} e^{-\gamma\sigma_x}$$
is a non-negative càdlàg $\mathcal{F}_t$-martingale and hence converges to M_∞ a.s. which is non-degenerate under an $x\log x$ condition.

A strong law for GBP

An analogue of the supercritical GW process convergence theorem:

<u>Theorem</u> (Nerman)
Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ, where $\phi \geq 0$ and $\phi(t) = 0$ for $t < 0$. Assume that ν_γ is non-lattice. Assume there exist non-increasing bounded positive integrable càdlàg functions g and h on $[0, \infty)$ such that

$$\mathbb{E}\left[\sup_{t\geq 0} \frac{\xi_\gamma(\infty) - \xi_\gamma(t)}{g(t)}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\sup_{t\geq 0} \frac{e^{-\gamma t}\phi(t)}{h(t)}\right] < \infty.$$

Then,

$$z^\phi(t) \to z^\phi(\infty) = \mu^{-1} \int_0^\infty u^\phi(s)ds,$$

and as $t \to \infty$,

$$e^{-\gamma t} Z^\phi(t) \to z^\phi(\infty) M_\infty, \text{ a.s.}$$

CLT for GBP

Centring and scaling:

Let $\bar{Z}$ be a version of Z^{ϕ} which satisfies

$$\bar{Z}(t) = \sum_{x \in \mathcal{T}} \bar{\zeta}_x(t - \sigma_x),$$

where the functions $\bar{\zeta}_x$, which may depend on the progeny of x, are chosen so that $\mathbb{E}\bar{Z}(t) = 0$.

Let $\tilde{Z}$ of $\bar{Z}$, be

$$\tilde{Z}(t) = e^{-\gamma t/2}\bar{Z}(t) = \tilde{\zeta}_{\emptyset}(t) + \sum_{i=1}^{\xi(\infty)} e^{-\gamma\sigma_i/2}\tilde{Z}_i(t - \sigma_i),$$

where $\tilde{\zeta}(t) = e^{-\gamma t/2}\bar{\zeta}(t)$.

Variance

Define

$$V(t) = \bar{Z}(t)^2 = \rho_\emptyset(t) + \sum_{i=1}^{\xi(\infty)} V_i(t - \sigma_i),$$

where

$$\rho_\emptyset(t) = \bar{\zeta}_\emptyset(t)^2 + 2\bar{\zeta}_\emptyset(t) \sum_{i=1}^{\xi(\infty)} \bar{Z}_i(t - \sigma_i) + 2 \sum_{i=1}^{\xi(\infty)} \sum_{j<i} \bar{Z}_i(t - \sigma_i)\bar{Z}_j(t - \sigma_j).$$

We will use the notation

$$v(t) = e^{-\gamma t}\mathbb{E}V(t) \quad \text{and} \quad r(t) = e^{-\gamma t}\mathbb{E}\rho(t).$$

As before, v and r satisfy the renewal equation

$$v(t) = r(t) + \int_0^\infty v(t - s)\nu_\gamma(ds).$$

Conditions

The central limit theorem requires two technical conditions.

Condition A:

There exists $\epsilon \in (0, 1/2)$ such that

$$e^{-\gamma t/2} \sum_{\sigma_x \le \epsilon t} \bar{\zeta}_x(t - \sigma_x) \to 0, \text{ in probability,}$$

as $t \to \infty$.

Condition B:

There exists $\alpha \in (0, \infty)$ such that

$$\sup_{t \in \mathbb{R}} \mathbb{E}\{|\tilde{Z}(t)|^{2+\alpha}\} < \infty.$$

The CLT

Theorem:

Let $(\xi_x, L_x, \phi_x)_x$ be a general branching process with Malthusian parameter γ. Assume that v is bounded and that

$$v(t) \to v(\infty),$$

some finite constant, as $t \to \infty$. Assume further that Conditions A and B hold. Then,

$$\tilde{Z}(t) \to \tilde{Z}_\infty, \text{ in distribution,}$$

as $t \to \infty$, where the distribution of $\tilde{Z}_\infty$ is characterised by

$$\mathbb{E}\left[e^{i\theta \tilde{Z}_\infty}\right] = \mathbb{E}\left[e^{-\frac{1}{2}\theta^2 v(\infty) M_\infty}\right].$$

Applictions

In applications, we generally have $e^{-\gamma t} Z^{\phi}(t) \to z^{\phi}(\infty) M_{\infty},$ in probability, as $t \to \infty$. To understand the fluctuations around the limiting behaviour, we study the expression

$$\begin{aligned} & e^{\gamma t/2}\left[e^{-\gamma t} Z^{\phi}(t) - z^{\phi}(\infty) M_{\infty}\right] \\ = \quad & e^{-\gamma t/2}\left[Z^{\phi}(t) - e^{\gamma t} z^{\phi}(t) M_{\infty}\right] + e^{\gamma t/2}[z^{\phi}(t) - z^{\phi}(\infty)] M_{\infty}. \end{aligned} \qquad (1)$$

The first term on the right hand side suggests centring Z using

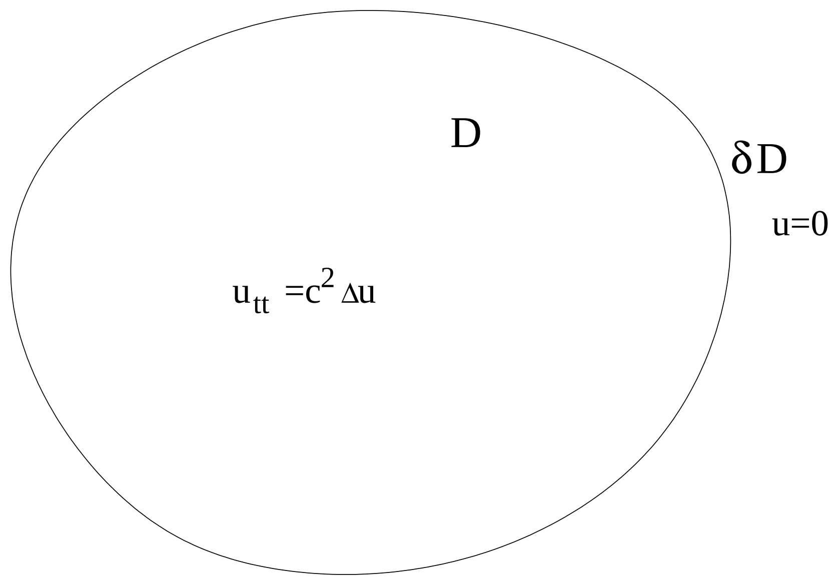
$$\bar{Z}(t) = Z^{\phi}(t) - e^{\gamma t} z^{\phi}(t) M_{\infty} = \bar{\zeta}_{\emptyset}(t) + \sum_{i=1}^{\xi(\infty)} \bar{Z}_i(t - \sigma_i), \qquad (2)$$

where

$$\bar{\zeta}_{\emptyset}(t) = \phi_{\emptyset}(t) + \sum_{i=1}^{\xi(\infty)} e^{\gamma(t-\sigma_i)}[z^{\phi}(t - \sigma_i) - z^{\phi}(t)] M_i(\infty). \qquad (3)$$

Drums

A vibrating membrane, fixed on its boundary, satisfies the wave equation



We can find the pure tones of the drum by substituting $u(x,t) = F(x)e^{i\omega t}$ into the wave equation. This gives, setting $c^2 = 1$,

$$\Delta F = -\omega^2 F.$$

Spectral asymptotics

- The standard Laplacian on a bounded domain $D \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions has a discrete spectrum consisting of eigenvalues $0 < \lambda_1^D < \lambda_2^D \leq \ldots$. That is λ_i satisfies for some u

$$\begin{cases} -\Delta u = \lambda_i u & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

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- Weyl's Theorem of 1912 states that the eigenvalue counting function

$$N(\lambda) = |\{\lambda_i : \lambda_i \leq \lambda\}|$$

satisfies

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{B_d}{(2\pi)^d} |D|$$

where $|D|$ is the Lebesgue measure of D. and B_d the volume of the unit ball in $\mathbb{R}^d$.

A heuristic proof

Consider the Dirichlet heat kernel on the domain. Mercer's theorem gives

$$p_t^a(x,y) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(y),$$

where ϕ_i are an orthonormal set of eigenfunctions, eigenvalue λ_i.

The trace of the heat semigroup, or the partition function, satisfies

$$\int_D p_t^a(x,x)dx = \sum_{i=1}^{\infty} e^{-\lambda_i t} = \int_0^{\infty} e^{-st}N(ds).$$

Thus information about the spectrum can be recovered from Tauberian theorems, if we understand the short time heat kernel asymptotics, and vice versa.

In D the heat kernel with Dirichlet or Neumann boundary conditions will be like the free space heat kernel for small times - The ‘principle of not feeling the boundary’;

$$\int_D p_t(x,x)dx \approx \int_D p_t^F(x,x)dx = \frac{|D|}{(4\pi t)^{d/2}}.$$

Thus

$$\int_0^\infty e^{-st}N(ds) \approx \frac{|D|}{(4\pi)^{d/2}} t^{-d/2},$$

and a standard Tauberian theorem gives

$$N(\lambda) \asymp \frac{B_d|D|}{(2\pi)^d}\lambda^{d/2}, \quad \lambda \to \infty.$$

The Weyl-Berry conjecture

- In the case where D is a manifold with a smooth boundary ∂D (under a billiard condition) we have

$$N(\lambda) = \frac{B_d}{(2\pi)^d}|D|\lambda^{d/2} - \frac{1}{4}\frac{B_{d-1}}{(2\pi)^{d-1}}|\partial D|\lambda^{(d-1)/2} + o(\lambda^{(d-1)/2}).$$

- 1979 Berry conjectured that if the boundary was fractal, then the second term would have as exponent the Hausdorff dimension of the boundary.
- Brossard and Carmona showed this was not true - the Hausdorff dimension should be replaced by the Minkowski dimension.
- This led to a modified Weyl-Berry conjecture.

- The modified Weyl-Berry conjecture was that

$$N(\lambda) = \frac{B_d}{(2\pi)^d}|D|\lambda^{d/2} - c_{d,d_m}\mathcal{M}(d_m, \partial D)\lambda^{d_m/2} + o(\lambda^{d_m/2}).$$

where the (upper) Minkowski dimension of the boundary

$$d_m = \inf\{\alpha : \mathcal{M}^*(\alpha, \partial D) = \limsup_{\epsilon \to 0} \epsilon^{-(d-\alpha)}|\partial D_\epsilon \cap D| < \infty\}.$$

The Minkowski content $\mathcal{M}(d_M, \partial D)$ exists if the limit in the definition of $\mathcal{M}^*(\alpha, \partial D)$ exists.

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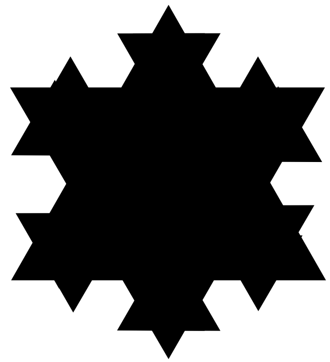
- Lapidus and Pomerance showed the modified Weyl-Berry conjecture was false in $\mathbb{R}^d$ for $d > 1$.
- It does hold for $d = 1$ and the inverse spectral problem is related to the Riemann hypothesis.

The snowflake domain

In the case of the snowflake domain



using the self-similarty gives the existence of a periodic function such that

$$N(\lambda) = \frac{B_2}{(2\pi)^2}|D|\lambda - p(\ln\lambda)\lambda^{\log 4/2\log 3} + o(\lambda^{\log 4/2\log 3}).$$

In fact the higher order term can be expressed more explicitly.

It has still not been proven that the periodic function p is not constant.

Subsequent work has focused on the heat content for different snowflake domains (van den Berg and den Hollander).

Heat content

We can consider the heat content of a domain. That is we let $u(t,x)$ be the solution to the heat equation in the domain with unit boundary condition and 0 initial condition;

$$u_t = \Delta u, \;\; x \in D, \;\; u(t,x) = 1, \;\; x \in \partial D,$$

with $u(0,x) = 0$ for all $x \in D$. The heat content is

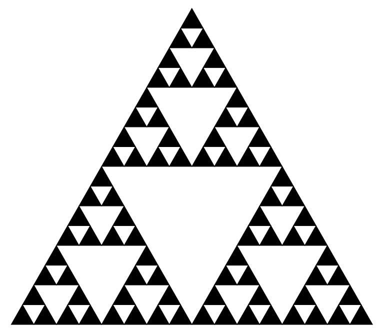
$$E(t) = \int_D u(t,x) dx.$$

This quantity does not have the leading order term of the partition function. Instead, for small times, it is determined by the behaviour of the solution to the heat equation at the boundary.

This has a nice probabilistic representation as $u(t,x) = \mathbb{P}^x(\tau_D < t)$, where τ_D is the exit time from the domain.

Fractal drums

What happens if the set itself is fractal? For example the Sierpinski gasket.



This is not a domain and the set itself is self-similar.

1. There is a Laplace operator defined as a renormalized limit of discrete Laplacians which has a discrete spectrum.

2. The spectral dimension is the exponent describing the growth of the eigenvalue counting function

$$d_s = 2 \lim_{\lambda \to \infty} \frac{\log N(\lambda)}{\log \lambda} = \frac{2 \log 3}{\log 5} \neq d_f = \frac{\log 3}{\log 2}.$$

For the Sierpinski gasket (and other nested fractals) we have

$$N(\lambda) = \lambda^{d_S/2}(G(\ln \lambda) + o(1)), \qquad \text{as } \lambda \to \infty,$$

where G is a periodic function (Fukushima-Shima, Barlow-Kigami).

This is due to the symmetry and exact self-similarity of the set.

We can construct strictly localized eigenfunctions on this set and use the self-similarity and symmetry to construct other eigenfunctions. Thus there are eigenvalues with very high multiplicity.

For self-similar sets with less symmetry but finite ramification (p.c.f fractals), if the logarithms of the scaling ratios are not rationally related, then (Kigami-Lapidus)

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d_s/2}} = C.$$

Spectral asymptotics for random gaskets

For the random recursive Sierpinski gasket, where each 2, 3 is independently chosen with probability $p, 1-p$ for each triangle

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d_s/2}} = W, \quad a.s.$$

where $d_s = 2\alpha/(\alpha+1)$ and α satisfies $p3(\frac{3}{5})^\alpha + (1-p)6(\frac{7}{15})^\alpha = 1$.

For the homogeneous random gasket, where each 2, 3 is independently chosen with probability $p, 1-p$ for each scale there are constants s.t.

$$0 < \limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d_s/2}\phi(\lambda)^{c_1}} \quad \limsup_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d_s/2}\phi(\lambda)^{c_2}} < \infty, \quad a.s.$$

where

$$\frac{d_s}{2} = \frac{p\log 3 + (1-p)\log 6}{p\log 5 + (1-p)\log 90/7}, \quad \phi(t) = \exp(\sqrt{\log\lambda \log\log\log\lambda}).$$

The Brownian string

A natural random fractal string can be generated by Brownian motion.

Take Brownian motion started from 0 in $\mathbb{R}$ run for unit time. The path can be viewed as a sequence of excursions away from 0. The zero set is a Cantor set (perfect and nowhere dense) and so divides the time axis into a countable number of intervals. Thus we have a decomposition of the unit interval - a fractal string.

For the Dirichlet counting function

$$N(\lambda) = \frac{1}{\pi}\lambda^{1/2} - L\zeta(1/2)\lambda^{1/4} + o(\lambda^{1/8+\epsilon}).$$

where L is the local time at 0 of the Brownian motion and ζ is the Riemann zeta function (H-Lapidus).

Our aim is to understand higher order terms for some random fractal strings and the continuum random tree.

Random strings

Let $K_0 = [0, 1]$, let $T_1, \ldots, T_n$ be non-negative random variables such that $T_1 + \cdots + T_n = 1$ and let $\gamma \in (0, 1)$. Now put

$$K^{(1)} = [a_1, b_1] \cup \cdots \cup [a_n, b_n],$$

where $a_1 = 0, \quad b_n = 1, \quad b_i - a_i = T_i^{1/\gamma}$ and $a_{i+1} - b_i = a_{i+2} - b_{i+1}$.
Iterating and putting

$$K = \bigcap_{n \in \mathbb{N}} K^{(n)}$$

produces a random Cantor type subset of $[0, 1]$.

Letting $R_i = T_i^{1/\gamma}$, the Malthusian parameter of the branching process associated with K is γ. The martingale $M \equiv 1$.

We consider one simple example from this collection.

The geometry

Let $n = 3$ and (T_1, T_2, T_3) follow a Dirichlet-$(1/2, 1/2, 1/2)$ distribution.
We write $S = S_1 \cup J_1 \cup S_2 \cup J_2 \cup S_3$, so the intervals forming the string are J_1, J_2.
The Hausdorff and Minkowski dimensions of S, the boundary of the string, are both γ almost surely.
We now use the general branching process to look at the volume of the inner-ϵ-neighbourhood $\mu(\epsilon)$ of ∂S.

Theorem:
For the fractal string S, we have

$$\epsilon^{\gamma-1}\mu(\epsilon) \to \mathcal{M}, \text{ a.s.,}$$

as $\epsilon \to 0$, for some positive constant $\mathcal{M}$.

Proof idea

Notice that

$$\mu(\epsilon) = \sum_{i=1}^{2} \mu_{J_i}(\epsilon) + \sum_{i=1}^{3} \mu_{S_i}(\epsilon).$$

Putting

$$Z^\phi(t) = e^t \mu(e^{-t}) \quad \text{and} \quad \phi(t) = e^t[\mu_{J_1}(e^{-t}) + \mu_{J_2}(e^{-t})],$$

by scaling $\mu_{S_i}(\epsilon) = R_i \mu_i(R_i^{-1}\epsilon)$, with $\mu_i = \mu$ in distribution,

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{3} Z_i^\phi(t - \sigma_i),$$

where the Z_i^ϕ are i.i.d. copies of Z^ϕ and ϕ is bounded.

Now apply the LLN for the GBP.

The CLT

Theorem:

For the string, we have

$$\epsilon^{-\gamma/2}\left[\epsilon^{\gamma-1}\mu(\epsilon)-\mathcal{M}\right]\to N(0,\sigma_1^2),\ \text{in distribution,}$$

as $\epsilon \to 0$, for some strictly positive constant σ_1

The proof uses the explicit form of the Laplace transform of the offspring distribution. This enables us to control $|z^{\phi}(t)-z^{\phi}(\infty)|$.

The counting function

For the boundary term in the asymptotics

<u>Theorem:</u>

For the fractal string S, we have

$$\lambda^{-\gamma/2}\left[\pi^{-1}\lambda^{1/2} - N(\lambda)\right] \to C, \text{ a.s.,}$$

as $\lambda \to \infty$, for some positive constant C

Proof set up

Let $\bar{N}_D(\lambda) = \pi^{-1}\text{vol}(D)\lambda^{1/2} - N_D(\lambda)$ for an interval D. By scaling $\bar{N}_{rD}(\lambda) = \bar{N}_D(r\lambda)$.

Thus we have if $X = (1 - R_1 - R_2 - R_3)/2$

$$\bar{N}_S(\lambda) = 2\bar{N}_{X[0,1]}(\lambda) + \sum_{i=1}^{3} \bar{N}_{R_i S_i}(\lambda).$$

Putting $\phi(t) = 2e^{-\gamma t/2}\bar{N}_{[0,1])}(X^2 e^t)$ and $Z^\phi(t) = e^{-\gamma t/2}\bar{N}(e^t)$ we have

$$Z^\phi(t) = \phi(t) + \sum_{i=1}^{\xi(\infty)} Z_i^\phi(t - \sigma_i),$$

where the Z_i^ϕ are i.i.d. copies of Z^ϕ.

The CLT

For the second term

<u>Theorem:</u>
For the string S, we have

$$\lambda^{\gamma/4}\{\lambda^{-\gamma/2}\left[\frac{1}{\pi}\lambda^{1/2} - N(\lambda)\right] - C\} \to N(0, \sigma^2), \text{ in distribution,}$$

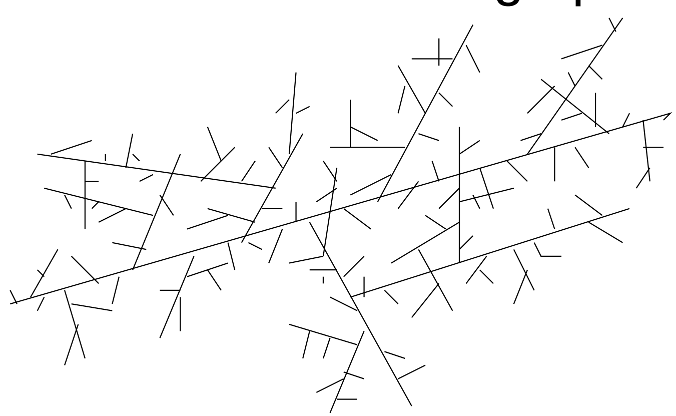
as $\lambda \to \infty$, for some positive constant σ.

The continuum random tree

The continuum random tree, initially constructed by Aldous, arises as

- the scaling limit of uniform random trees on n vertices.
- a random real tree defined as the contour process of Brownian excursion.
- A third view is that it is a random recursive self-similar set.

It is closely related to mean field limits for critical percolation on graphs, in particular high dimensional critical percolation on $\mathbb{Z}^d$ and limit models arising in the critical window of the random graph model.

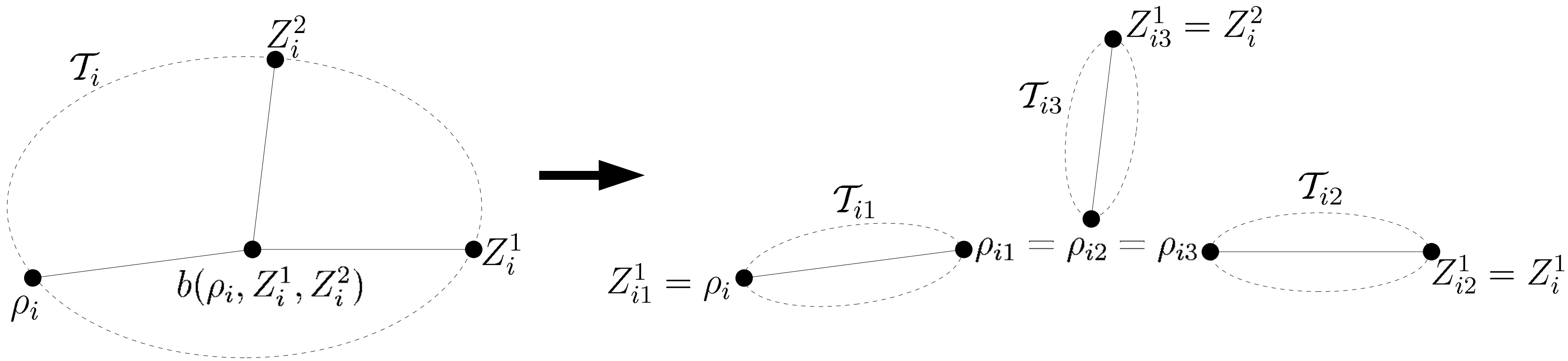


Self-similar decomposition

Let Z^1, Z^2 be two $\mu_{\mathcal{T}}$-random vertices of $\mathcal{T}$. There exists a unique branch-point $b^{\mathcal{T}}(\rho, Z^1, Z^2) \in \mathcal{T}$ of these three vertices. Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ the components containing ρ, Z^1 and Z^2. For $i = 1, 2, 3$, we define a metric $d_{\mathcal{T}_i}$ and probability measure $\mu_{\mathcal{T}_i}$ on $\mathcal{T}_i$ by setting

$$d_{\mathcal{T}_i} := \Delta_i^{-1/2} d_{\mathcal{T}}|_{\mathcal{T}_i \times \mathcal{T}_i}, \qquad \mu_{\mathcal{T}_i}(\cdot) := \Delta_i^{-1} \mu(\cdot \cap \mathcal{T}_i),$$

where $\Delta_i := \mu_{\mathcal{T}}(\mathcal{T}_i)$.

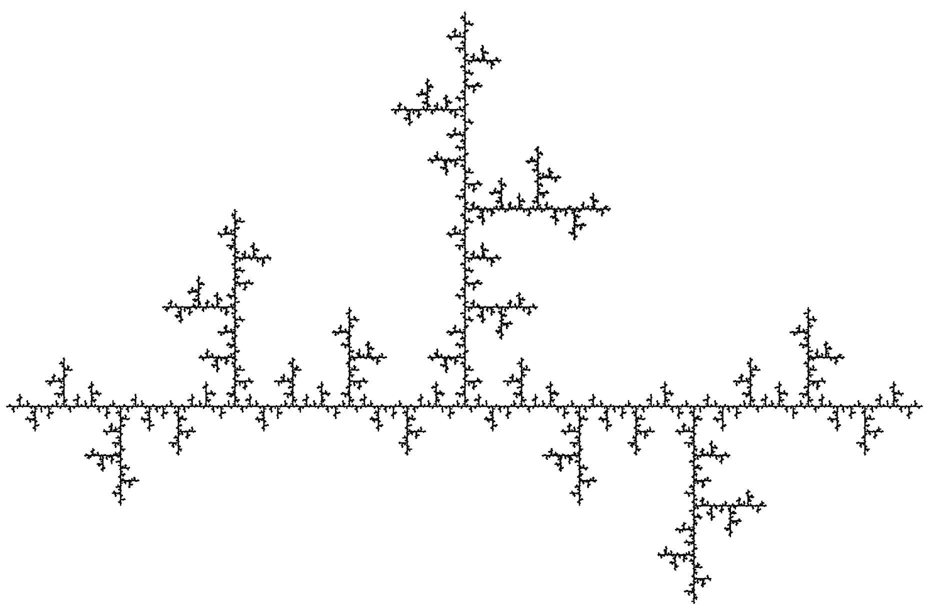


Random recursive fractal

<u>Lemma</u>

The collections $(\mathcal{T}_i, d_{\mathcal{T}_i}, \mu_{\mathcal{T}_i}, \rho_i, Z_i^1, Z_i^2)$, $i = 1, 2, 3$, are independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \rho, Z^1, Z^2)$, and moreover, the entire family of random variables is independent of $(\Delta_i)_{i=1}^3$, which has a Dirichlet-$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution.

The CRT is isomorphic to a deterministic self-similar set with a random metric



The Dirichlet form

The natural Laplace operator on $\mathcal{T}$ is defined via its Dirichlet form.

$\mathbf{P}$-a.s. there exists a local regular Dirichlet form $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ on $L^2(\mathcal{T}, \mu)$, which is associated with the Laplace operator $\mathcal{L}_{\mathcal{T}}$ via for $f, g \in \mathcal{F}_{\mathcal{T}}$

$$\mathcal{E}_{\mathcal{T}}(f, g) = -(\mathcal{L}_{\mathcal{T}} f, g).$$

and the metric $d_{\mathcal{T}}$ through, for every $x \neq y$,

$$d_{\mathcal{T}}(x, y)^{-1} = \inf\{\mathcal{E}_{\mathcal{T}}(f, f) :\ f \in \mathcal{F}_{\mathcal{T}},\ f(x) = 0,\ f(y) = 1\}.$$

A Neumann eigenvalue λ with eigenfunction u satisfies $\mathcal{E}_{\mathcal{T}}(f, u) = \lambda(f, u)$ for all $f \in \mathcal{F}_{\mathcal{T}}$.

We work with the eigenvalue counting function defined from $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$.

CRT results

<u>Theorem</u>

Suppose $(N_{\mathcal{T}}(\lambda))_{\lambda\in\mathbb{R}}$ is the eigenvalue counting function for the natural Laplacian on the continuum random tree. As $\lambda \to \infty$:

- $$\mathbb{E}N_{\mathcal{T}}(\lambda) = C_0\lambda^{2/3} + O(1).$$

CRT results

Theorem

Suppose $(N_{\mathcal{T}}(\lambda))_{\lambda\in\mathbb{R}}$ is the eigenvalue counting function for the natural Laplacian on the continuum random tree. As $\lambda\to\infty$:

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- $\mathbb{P}$-a.s., for $\epsilon > 0$,
$$N_{\mathcal{T}}(\lambda) = C_0\lambda^{2/3} + o(\lambda^{1/3+\epsilon}).$$

CRT results

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- $$\frac{N_{\mathcal{T}}(\lambda) - C_0\lambda^{2/3}}{\lambda^{1/3}} \to N(0, y(\infty)), \text{ in distribution.}$$

CRT results

Theorem

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- $$\frac{N_{\mathcal{T}}(\lambda) - C_0\lambda^{2/3}}{\lambda^{1/3}} \to N(0, y(\infty)),\ \text{in distribution.}$$
- Health warning... we have not yet proved $y(\infty) > 0$!

Percolation

For percolation clusters in high dimensions the Alexander-Orbach conjecture has been proved by Kozma and Nachmias. This shows that for random walk on the incipient infinite cluster at criticality we have $d_s = 4/3$. This is established for the on-diagonal decay of the heat kernel on the graph.

This scaling is observed in other mean field models including the critical random graph.

The question of the spectral asymptotics for the CRG will be determined by the spectral asymptotics for random self-similar trees.

Scaling limit of the CRG

Let $G(N,p)$ be the Erdös-Renyi random graph. The critical window is $p = N^{-1} + \nu N^{-4/3}$ for a fixed $\nu \in (-\infty, \infty)$.

Addario-Berry, Broutin and Goldschmidt construct the scaling limit: Conditioned on the number of connections $J = j$ we have (for $j \geq 2$) that $\mathcal{M}$ is constructed by

- taking a random 3 regular graph on $2(j-1)$ vertices
- generate $(\alpha_1, \ldots, \alpha_{3(j-1)})$ according to a Dirichlet $(\frac{1}{2}, \ldots, \frac{1}{2})$ distribution.
- construct $3(j-1)$ size α_j CRTs with root plus a randomly chosen vertex.
- replace the edges in the graph with the trees linked at the roots and randomly chosen vertices.

Dirichlet-Neumann bracketing allows us to compare eigenvalues of $\mathcal{M}, \mathcal{T}$.

<u>Theorem</u>

Suppose $(N_{\mathcal{M}}(\lambda))_{\lambda \in \mathbb{R}}$ is the eigenvalue counting function for the natural Laplacian on the scaling limit of the giant component of the critical random graph $\mathcal{M}$, and Z_1 is the mass of $\mathcal{M}$ with respect to its canonical measure $\mu_{\mathcal{M}}$. Then, as $\lambda \to \infty$:

- $\mathbb{E}N_{\mathcal{M}}(\lambda) = C_0 \mathbb{E} Z_1 \lambda^{2/3} + O(1).$
- $\lambda^{-2/3} N_{\mathcal{M}}(\lambda) \to C_0 Z_1.\ \ \mathbb{P} - a.s.$
- $\frac{N_{\mathcal{M}}(\lambda) - a Z_1 \lambda^{2/3}}{Z_1^{1/2} \lambda^{1/3}} \to N(0, y(\infty))$ in distribution.