Non-homogeneous random walks on a semi-infinite strip

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Joint work with Andrew Wade

Aspects of Random Walks
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Durham University

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Simple random walk

Let X_n be symmetric simple random walk (SRW) on $\mathbb{Z}^d$, i.e., given $X_1, \ldots, X_n$, the new location X_{n+1} is uniformly distributed on the $2d$ adjacent lattice sites to X_n.

Theorem (Pólya, 1921)

SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.

Several proofs are available, typically using combinatorics or electrical network theory, but these classical approaches are of limited use if one wants to generalise or perturb the model slightly.

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Lamperti (1960) gave a very robust approach, based on the method of Lyapunov functions. Idea: reduce to a 1-dimensional problem by taking $Z_n = \|X_n\|$.

Lamperti's problem

$X_n = 0$ if and only if $Z_n = 0$.

But Z_n is not homogeneous (and not even Markov). However, Z_n is a stochastic process with asymptotically zero drift.

Lamperti investigated the asymptotic behaviour of these non-homogeneous random walks on $\mathbb{Z}_+$. He studied in detail how the asymptotic behaviour of the random walk is determined by the first two moment functions $\mu_1(z)$ and $\mu_2(z)$ of its increments.

Here, $\mu_k(z) = \mathbb{E}[(Z_{n+1} - Z_n)^k \mid Z_n = z]$.

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Theorem (Lamperti)

Let (Z_n) be an irreducible time-homogeneous Markov chain on $\mathbb{Z}_+$. Suppose that there exists $\varepsilon > 0$ such that

$$\sup_z \mathbb{E}[|Z_{n+1} - Z_n|^{2+\varepsilon} \mid Z_n = z] < \infty;$$

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- *If* $\liminf_{z\to\infty}(2z\mu_1(z) - \mu_2(z)) > 0$, *then* Z_n *is transient.*

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- *If* $|2z\mu_1(z)| \leq \mu_2(z) + O(z^{-\delta})$, *for some* $\delta > 0$, *then* Z_n *is null-recurrent.*

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- *If* $\limsup_{z\to\infty}(2z\mu_1(z) + \mu_2(z)) < 0$, *then* Z_n *is positive-recurrent.*

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Lamperti's classification

Typically, the result is applied when the drift $\mu_1(x)$ is asymptotically zero, decaying as $1/z$ as $z \to \infty$ and $\mu_2(z)$ is asymptotically constant (and nonzero).
In particular, for $\mu_1(z) = c/z + o(z^{-1})$ and $\mu_2(z) = s^2 + o(1)$, the results tell us that

- Z_n is transient for $2c > s^2$,
- Z_n is null-recurrent for $-s^2 < 2c < s^2$,
- Z_n is positive-recurrent for $2c < -s^2$.

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Non-homogeneous RW on semi-infinite strip

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- modulated queues (e.g., $S =$ states of servers)
- regime-switching processes (S contains market information)
- physical processes with internal degrees of freedom
  ($S =$ energy/momentum states of particle)

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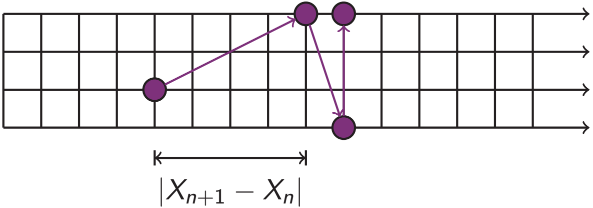
Model assumptions

Moments bound on jumps of X_n

$$(\mathrm{B}_p) \qquad \exists\ C_p < \infty \text{ s.t. } \mathbb{E}[|X_{n+1} - X_n|^p \mid \mathcal{F}_n] \le C_p$$



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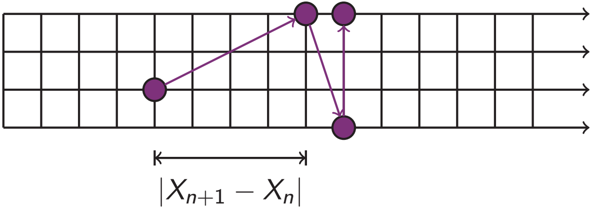

For this talk, we assume (B_p) holds for some $p > 2$.

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Model assumptions

η_n is "close to being Markov" when X_n is large

Define

$$p(x, i, y, j) = \mathbb{P}[(X_{n+1}, \eta_{n+1}) = (y, j) \mid (X_n, \eta_n) = (x, i)]$$

$$q_x(i, j) = \sum_{y \in \mathbb{Z}_+} p(x, i, y, j)$$

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$$q_x(i,j) = \sum_{y\in\mathbb{Z}_+} p(x,i,y,j)$$

(Q_∞) $q(i,j) = \lim_{x\to\infty} q_x(i,j)$ exists for all $i,j \in S$ and $(q(i,j))$ is irreducible

Markov chain with transition probabilities $q(i,j)$ is irreducible on finite state space S, so it has a stationary distribution π satisfying

$$\pi(j) = \sum_{i\in S} \pi(i) q(i,j) \text{ for all } j \in S.$$

Model assumptions

Lamperti-type moment conditions

Define

$$\mu_k(x,i) = \mathbb{E}[(X_{n+1} - X_n)^k \mid (X_n, \eta_n) = (x,i)]$$

($\mathsf{M_L}$) $\exists\ c_i, s_i \in \mathbb{R}$ for all $i \in S$ (at least one s_i nonzero) such that

$$\mu_1(x,i) = \frac{c_i}{x} + o(x^{-1}); \quad \mu_2(x,i) = s_i^2 + o(1).$$

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Recurrence/transience of X_n

With these three assumptions (B_p), (Q_∞), (M_L), we can give conditions that imply the recurrence or transience of X_n.

Note: X_n not assumed to be Markov — need to define what we mean by recurrence/transience of X_n. Here, finiteness of S helps.

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(X_n, η_n) is an irreducible Markov chain, so is either recurrent or transient. Moreover,

Lemma

(i) *If (X_n, η_n) is recurrent, then $\mathbb{P}[X_n = 0 \text{ i.o.}] = 1$.*

(ii) *If (X_n, η_n) is transient, then $\mathbb{P}[X_n = 0 \text{ i.o.}] = 0$, and $X_n \to \infty$ a.s.*

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Null- vs. positive-recurrence of X_n

We can also define null- and positive-recurrence of X_n:

Lemma

There exists a (unique) measure ν on $\mathbb{Z}_+$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}\{X_k = x\} = \nu(x) \text{ a.s.},$$

for all $x \in \mathbb{Z}_+$.

(i) *If (X_n, η_n) is null, then $\nu(x) = 0$ for all $x \in \mathbb{Z}_+$.*

(ii) *If (X_n, η_n) is positive-recurrent, then $\nu(x) > 0$ for all $x \in \mathbb{Z}_+$ and $\sum_{x\in\mathbb{Z}_+} \nu(x) = 1$.*

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Recurrence classification of X_n

Theorem (G., Wade, 2014)

Suppose that (B_p) holds for some $p > 2$ and conditions (Q_∞) and (M_L) hold. The following sufficient conditions apply.

- *If $\sum_{i \in S}(2c_i - s_i^2)\pi(i) > 0$, then X_n is transient.*
- *If $|\sum_{i \in S} 2c_i\pi(i)| < \sum_{i \in S} s_i^2\pi(i)$, then X_n is null-recurrent.*
- *If $\sum_{i \in S}(2c_i + s_i^2)\pi(i) < 0$, then X_n is positive-recurrent.*

[With better error bounds in (Q_∞) and (M_L) we can also show that the boundary cases are null-recurrent.]

This generalises Lamperti's results for walks on $\mathbb{Z}_+$ (the case of S a singleton).

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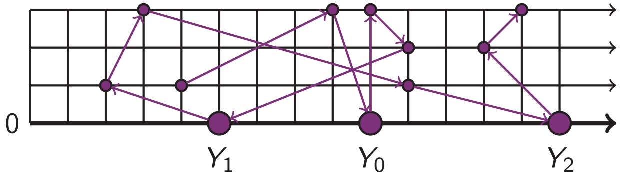
Embedded process Y_n

Label an arbitrary state $0 \in S$.
Define $\tau_0 = \min\{n \in \mathbb{Z}_+ : \eta_n = 0\}$ and for $m \geq 0$ set $\tau_{m+1} = \min\{n > \tau_m : \eta_n = 0\}$.
(Conditions (B_p) and (Q_∞) imply $\tau_m < \infty$ for all m.)

Embedded process: $Y_n = X_{\tau_n}$ on state space $\mathbb{Z}_+$



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Properties of Y_n and τ_n

Y_n is an irreducible Markov chain.

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Set $\tau = \min\{n > 0 : \eta_n = 0\}$.
Then $\tau_{n+1} - \tau_n$ conditional on $Y_n = x$ has the same distribution as τ conditional on $(X_0, \eta_0) = (x, 0)$.
This random variable is "well-behaved": it has exponential tails and all moments of τ are finite.

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(X_n) recurrent if and only if (Y_n) recurrent.
(X_n) positive-recurrent if and only if (Y_n) positive-recurrent.

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Excursion from line 0

Hence our recurrence classification will follow from an application of Lamperti's result to Y_n.
We need to calculate $\mathbb{E}[(Y_{n+1} - Y_n)^k \mid Y_n = x]$, for $k = 1, 2$.

Enough to calculate $\mathbb{E}[(X_\tau - X_0)^k \mid (X_0, \eta_0) = (x, 0)]$. For this we use the Doob decomposition of X_n.

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Doob decomposition of X_n

Write

$$X_n - X_0 = M_n + \sum_{k=0}^{n-1} \mathbb{E}[X_{k+1} - X_k \mid X_k, \eta_k],$$

where M_n is a martingale with $M_0 = 0$. Using the definition of $\mu_1(x, i)$,

$$\begin{aligned} X_n - X_0 &= M_n + \sum_{k=0}^{n-1} \mu_1(X_k, \eta_k) \\ &= M_n + \sum_{i \in S} \sum_{k=0}^{n-1} \mu_1(X_k, i) \mathbf{1}\{\eta_k = i\} \end{aligned}$$

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Moment calculations

So,

$$X_\tau - X_0 = M_\tau + \sum_{i\in S}\sum_{k=0}^{\tau-1} \mu_1(X_k, i)\mathbf{1}\{\eta_k = i\}$$

Optional Stopping Theorem: $\mathbb{E}[M_\tau \mid (X_0, \eta_0) = (x, 0)] = M_0 = 0.$

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Then

$$\mathbb{E}_{x,0}[X_\tau - X_0] = \sum_{i \in S} \mathbb{E}_{x,0}\left[\sum_{k=0}^{\tau-1} \mu_1(X_k, i) \mathbf{1}\{\eta_k = i\}\right]$$

where $\mathbb{E}_{x,0}[\,\cdot\,]$ is short for $\mathbb{E}[\,\cdot \mid (X_0, \eta_0) = (x, 0)]$.

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where $\mathbb{E}_{x,0}[\,\cdot\,]$ is short for $\mathbb{E}[\,\cdot \mid (X_0, \eta_0) = (x, 0)]$.

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Moment calculations

We need one more approximation:

$$\mathbb{E}_{x,0}\left[\sum_{k=0}^{\tau-1}\mathbf{1}\{\eta_k = i\}\right] = \frac{\pi(i)}{\pi(0)} + o(1).$$

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Combining these with $\mu_1(x,i) = c_i/x + o(x^{-1})$ we get

$$\mathbb{E}[X_\tau - X_0 \mid (X_0,\eta_0) = (x,0)] = \frac{1}{\pi(0)}\sum_{i\in S}\frac{c_i\pi(i)}{x} + o(x^{-1}).$$

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Similar reasoning using the Doob decomposition for X_n^2 yields the second moment:

$$\mathbb{E}[(X_\tau - X_0)^2 \mid (X_0,\eta_0) = (x,0)] = \frac{1}{\pi(0)}\sum_{i\in S} s_i^2\pi(i) + o(1).$$

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Moments for Y_n

In terms of Y_n we have:

Lemma

$$\mathbb{E}[Y_{n+1} - Y_n \mid Y_n = x] = \frac{1}{\pi(0)} \sum_{i \in S} \frac{c_i \pi(i)}{x} + o(x^{-1});$$

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- *If $\sum_{i \in S} (2c_i + s_i^2)\pi(i) < 0$, then X_n is positive-recurrent.*

The missing details

The proof relied on the following:

- Random variable τ has exponential tails.
- Control of $X_k - X_0$ for $k \leq \tau$.
- $\lim_{x\to\infty} \mathbb{E}_{x,0} \sum_{k=0}^{\tau-1} \mathbf{1}\{\eta_k = i\} = \frac{\pi(i)}{\pi(0)}$

All these follow from a coupling of (X_n, η_n) with $(\eta_n^\star)$ the Markov chain on S with transition matrix $(q(i,j))$.

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All these follow from a coupling of (X_n, η_n) with $(\eta_n^\star)$ the Markov chain on S with transition matrix $(q(i,j))$.

E.g. if $\tau^\star = \min\{n > 0 : \eta_n^\star = 0\}$, then conditional on η_n and $\eta_n^\star$ remaining coupled up to time m we have $\tau \leq m$ if and only if $\tau^\star \leq m$.

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Example: persistent random walk on $\mathbb{Z}_+$

Nearest-neighbour random walk (X_n) on $\mathbb{Z}_+$ where the distribution of X_{n+1} depends on the current position X_n and the current direction $X_n - X_{n-1}$. Setting $\eta_n = X_n - X_{n-1}$, we can model this as a Markov chain (X_n, η_n) on $\mathbb{Z}_+ \times S$, where $S = \{+1, -1\}$.

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Nonzero transition probabilities are $p(x, i, x + j, j) = q_x(i, j)$ with

$$q_x(i,j) = \begin{cases} \frac{1}{2} + \frac{ic}{2x} + o(x^{-1}) & \text{if } j = i \\ \frac{1}{2} - \frac{ic}{2x} + o(x^{-1}) & \text{if } j \neq i \end{cases}$$

For $c > 0$ the walk has a marginal preference to continue in the positive direction, and a marginal aversion to continuing in the negative direction. (For large x the local behaviour is approx like SRW on $\mathbb{Z}_+$.)

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Persistent random walk on $\mathbb{Z}_+$

We calculate the moments

$$\mu_1(x,i) = \frac{c}{x} + o(x^{-1}) \text{ and } \mu_2(x,i) = 1 \text{ for } i \in S.$$

Hence, our results tell us that

- X_n is transient if $c > 1/2$,
- X_n is null-recurrent if $-1/2 < c < 1/2$,
- X_n is positive-recurrent if $c < -1/2$.

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