Non-homogeneous random walks: Anomalous recurrence and angular asymptotics

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Joint work with
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Outline

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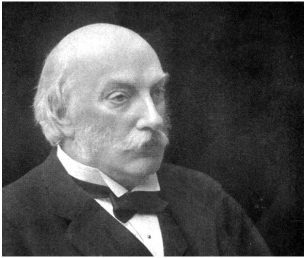
Random walk origins



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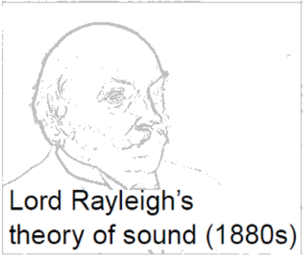

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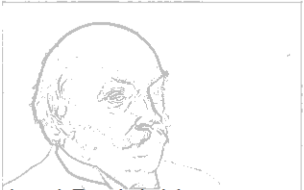

Lord Rayleigh's
theory of sound (1880s)

Louis Bachelier's

thesis on
random models
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(1900)

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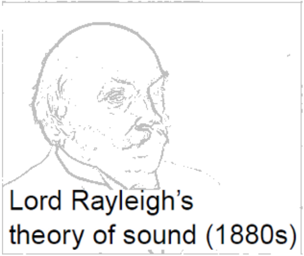

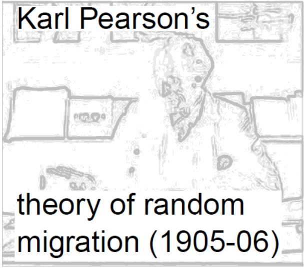

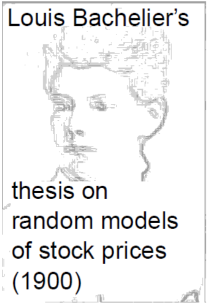

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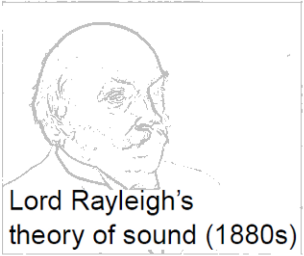

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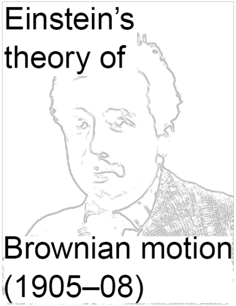
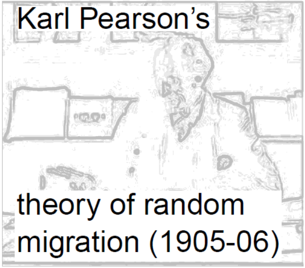
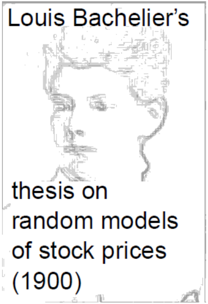


Einstein's theory of Brownian motion (1905–08)



Karl Pearson's theory of random migration (1905-06)



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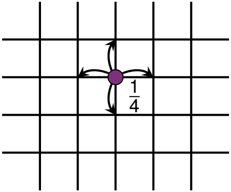
Classical zero-drift random walks

1. Symmetric simple random walk on $\mathbb{Z}^d$

- $X_n \in \mathbb{Z}^d$, $X_0 = 0$.
- Given $X_0, \ldots, X_n$, new location X_{n+1} is uniformly distributed on the $2d$ adjacent lattice sites to X_n.

Theorem (Pólya 1921)

SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.



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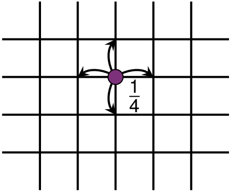

"A drunk man will find his way home, but a drunk bird may get lost forever." —Shizuo Kakutani

Classical zero-drift random walks

2. Pearson–Rayleigh random walk in $\mathbb{R}^d$

- $X_n \in \mathbb{R}^d$, $X_0 = 0$.
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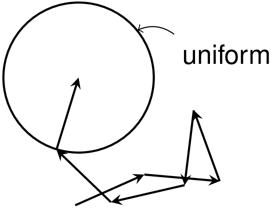

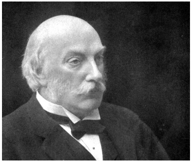

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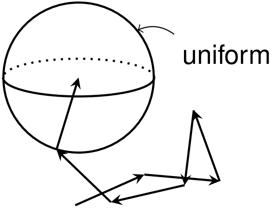

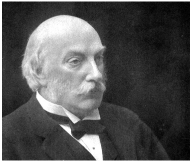

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Recurrence/transience of homogeneous random walks

Let (X_n) be a spatially homogeneous random walk in $\mathbb{R}^d$.

So X_{n+1} depends only on X_n, but $\Delta := X_{n+1} - X_n$ is independent of X_n (and n).

Let $\mu = \mathbb{E}\Delta$, the mean drift vector of the random walk.

Theorem (Chung–Fuchs)

Under mild conditions, if $\mu = 0 \in \mathbb{R}^d$, then (X_n) is

- *recurrent if $d = 1$ or $d = 2$;*
- *transient if $d \geq 3$.*

This result applies both to the symmetric simple RW and the Pearson–Rayleigh RW.

Definition

- recurrence: $\mathbb{P}$[return to (nbrhood of) origin] $= 1$.
- transience: $\mathbb{P}$[return to (nbrhood of) origin] < 1.

Non-homogeneous random walks

What if we allow Δ, the jump distribution, to depend on the current location?

Then $\mu(x) := \mathbb{E}_x \Delta := \mathbb{E}[\Delta \mid X_n = x]$ becomes a function of the current position $x \in \mathbb{R}^d$.

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For $d = 1$: yes (essentially) — zero drift implies recurrence.

For higher dimensions: no — either behaviour is possible.

Theorem

There exist non-homogeneous random walks with $\mu(x) = 0$ for all $x \in \mathbb{R}^d$ that are

- *transient in $d = 2$;*
- *recurrent in $d \geq 3$.*

Outline

Elliptical random walk ($d = 2$)

- We illustrate general phenomena with a simple family of examples.
- We modify the Pearson–Rayleigh random walk to make jumps distributed on an ellipse.

The ellipse has fixed size, but orientation depends on current position of the walk.

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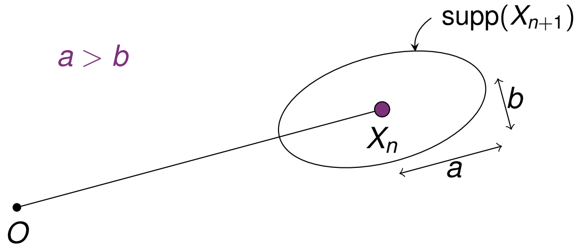

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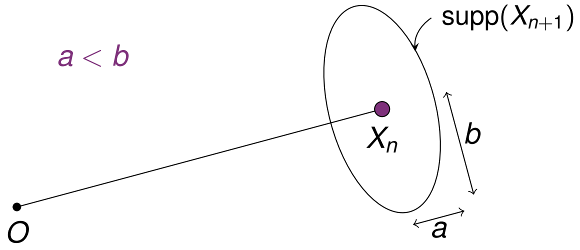
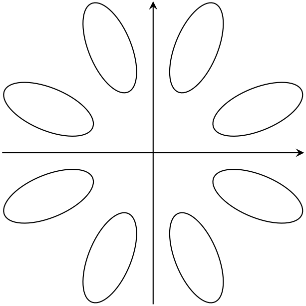


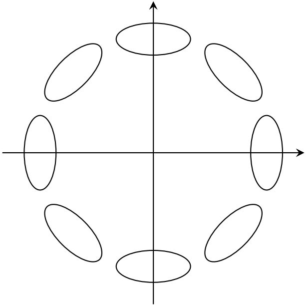
Elliptical random walk ($d = 2$)

$a > b$



radial bias

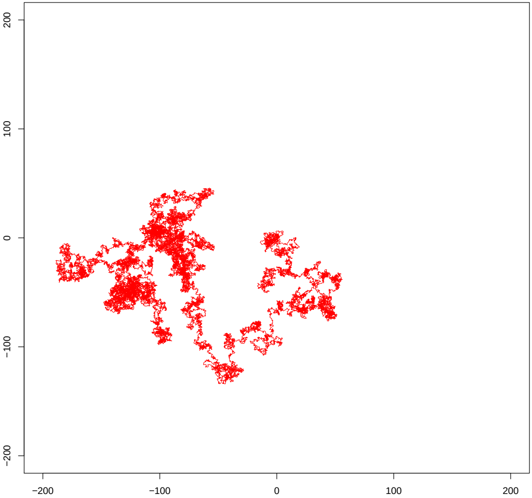
$a < b$



transverse bias

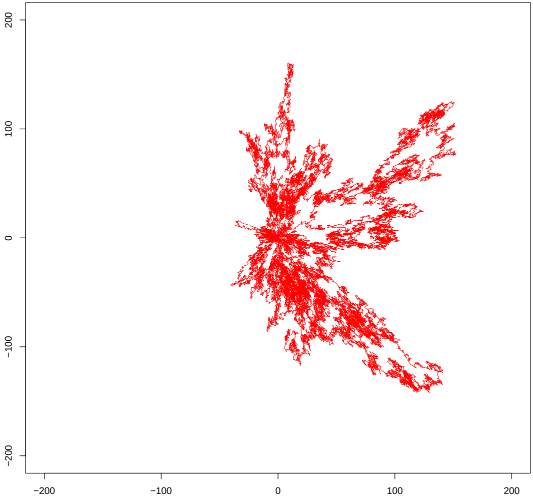
Simulations

$a = 1, b = 1$



Simulations

$a = 2, b = 1$



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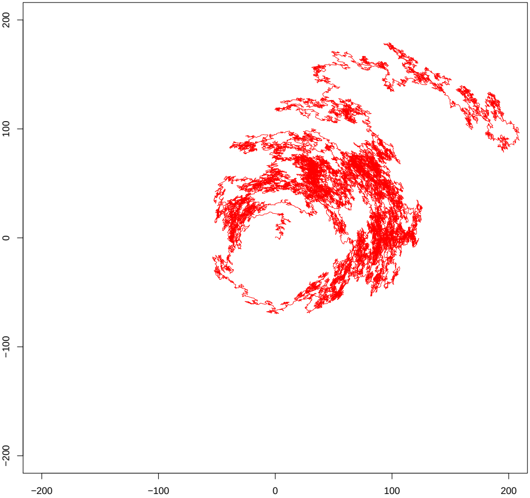
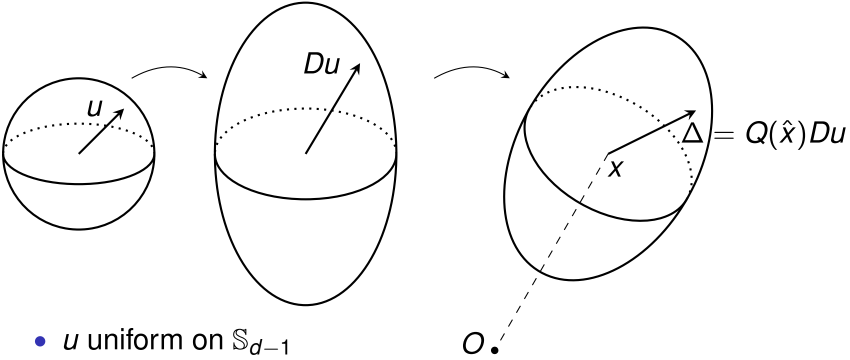


Elliptical random walk ($d \geq 2$)

Suppose $X_n = x \in \mathbb{R}^d$. Write $\hat{x}$ for unit vector in direction x.



- u uniform on $\mathbb{S}_{d-1}$
- $D = \mathrm{diag}(a, b, \ldots, b)$
- $Q(\hat{x})$ orthogonal matrix, with $Q(\hat{x})e_1 = \hat{x}$.

Increment moments

Notation: write $\mathbb{E}_x[\,\cdot\,]$ for $\mathbb{E}[\,\cdot \mid X_n = x]$ and write Δ_x for the component of $\Delta := X_{n+1} - X_n$ in direction x:

$$\Delta_x = \Delta \cdot \hat{x} = \frac{\Delta \cdot x}{\|x\|}.$$

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Therefore, by construction,

$$\mathbb{E}_x[\Delta] = 0, \quad \mathbb{E}_x[\Delta\Delta^\top] = \frac{1}{d} Q(\hat{x}) D^2 Q^\top(\hat{x}).$$

Hence,

$$\mathbb{E}_x[\Delta_x] = 0, \quad \mathbb{E}_x[\Delta_x^2] = \frac{a^2}{d}, \quad \mathbb{E}_x[\|\Delta\|^2] = \frac{a^2 + (d-1)b^2}{d}.$$

Radial component of X_n

We analyse (X_n) by considering $R_n := \|X_n\|$.

By symmetry, R_n is also Markov (R_n is a non-homogeneous random walk on $\mathbb{R}_+$).

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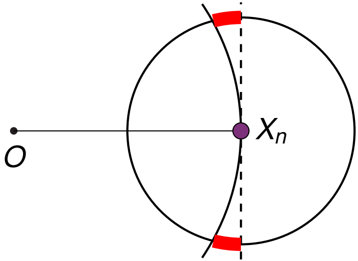
$$\mathbb{E}[R_{n+1} - R_n \mid R_n = r] \sim c/r,$$

where positive constant c depends on model parameters and ambient dimension.



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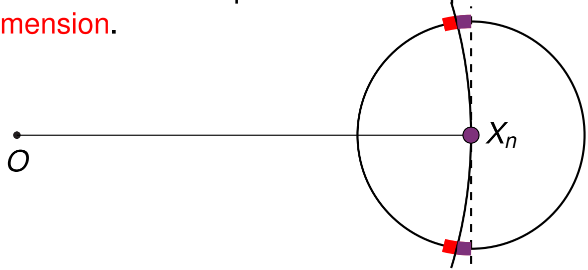
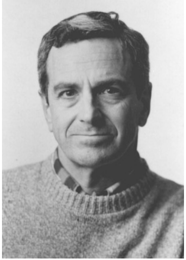


Lamperti's classification

Define $\mu_k(r) := \mathbb{E}[(R_{n+1} - R_n)^k \mid R_n = r]$.

In the early 1960s,
John Lamperti studied in detail how
the asymptotics of a stochastic process on
$\mathbb{R}_+$ are determined by the first two moment
functions of its increments, μ_1 and μ_2.



Theorem (Lamperti, 1960)

Let (R_n) be a Markov chain on $\mathbb{R}_+$.
Under mild conditions:

- *If $2r\mu_1(r) - \mu_2(r) > 0$ for all large enough r, then R_n is transient,*
- *If $2r\mu_1(r) - \mu_2(r) < 0$ for all large enough r, then R_n is recurrent.*

Recurrence/transience of elliptical random walk

Given $X_n = x$,

$$\begin{aligned} R_{n+1} - R_n &= \|x + \Delta\| - \|x\| \\ &= [\ldots \text{expand using Taylor's theorem} \ldots] \\ &= \Delta_x + \frac{\|\Delta\|^2 - \Delta_x^2}{2\|x\|} + O(\|x\|^{-2}). \end{aligned}$$

So,

$$\mu_1(r) = \frac{(d-1)b^2}{d} \frac{1}{2r} + O(r^{-2}), \quad \mu_2(r) = \frac{a^2}{d} + O(r^{-1}).$$

Theorem (GMMW 2015)

Let (X_n) be an elliptical random walk in $\mathbb{R}^d$, with parameters a and b.

- *If $(d-1)b^2 - a^2 > 0$ then (X_n) is transient.*
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- *If $(d-1)b^2 - a^2 \leq 0$ then (X_n) is recurrent.*

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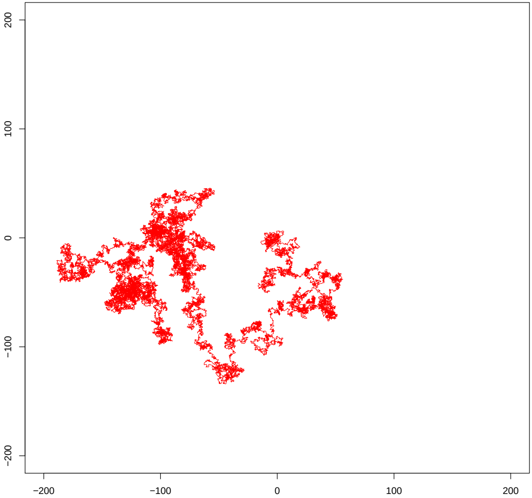

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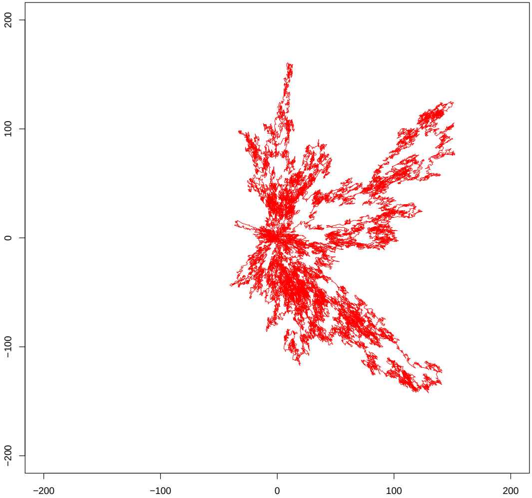

Simulations

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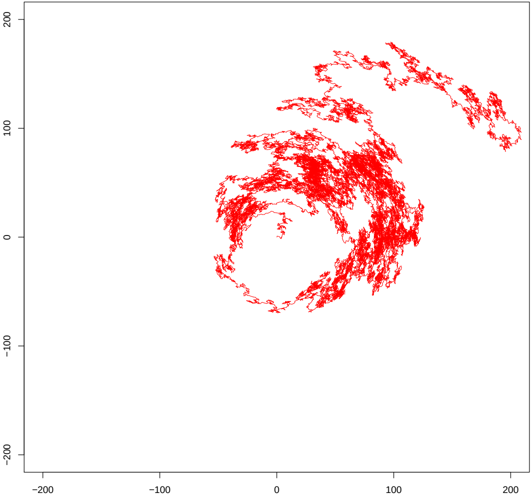
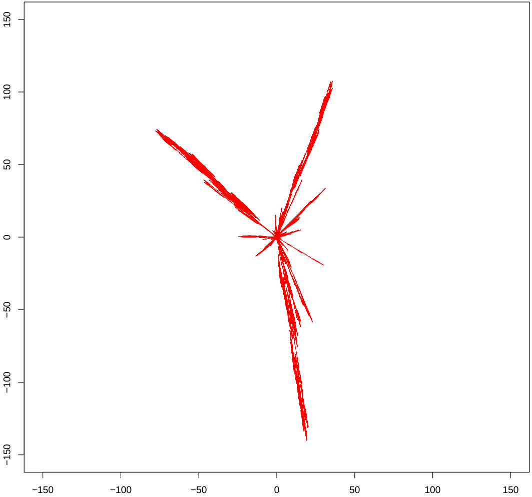


Simulations

$a = 1, b = 0.05$



Simulations

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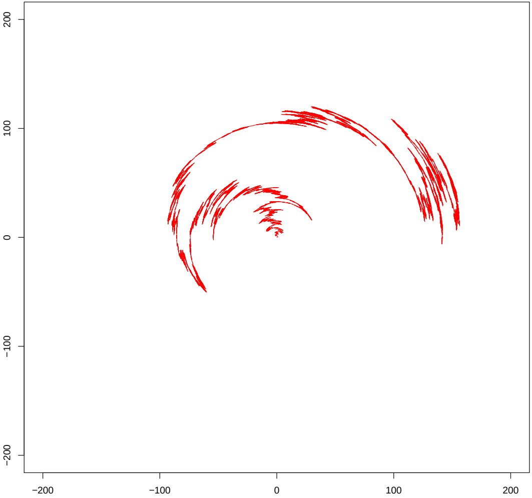

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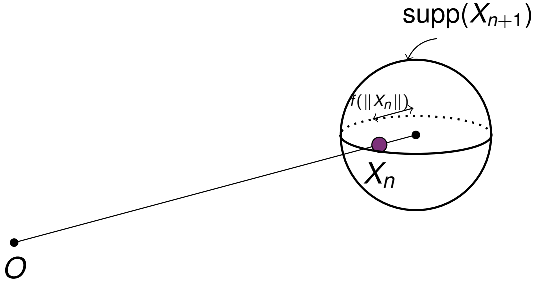
- If we impose the condition that the increment covariance is fixed throughout space, then we regain the conclusion of the Chung–Fuchs theorem (recurrence if and only if $d \leq 2$).
- In the case of a fixed increment covariance, to probe more precisely the recurrence/transience phase transition it is natural to study walks with asymptotically zero drift.

Outline

Centrally biased random walk

- Again, we illustrate general results with a concrete family of examples, the so-called centrally biased random walks.
- Again modify the Pearson–Rayleigh random walk, by shifting the centre of the sphere on which the jumps from x are supported by an amount $f(\|x\|)$ away from the origin, where $f(r) \to 0$ as $r \to \infty$.



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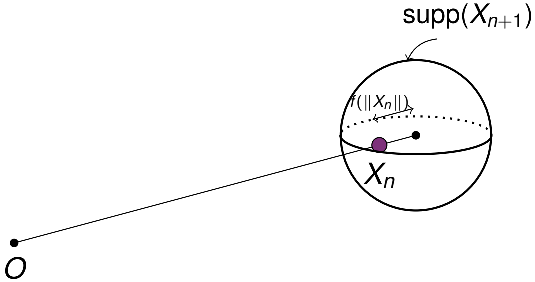

- A natural choice is $f(r) = \rho r^{-\beta}$ where $\rho \in \mathbb{R}$ and $\beta > 0$.
- Then the random walk X_n has mean drift

$$\mu(x) = \mathbb{E}[\Delta \mid X_n = x] = \rho \|x\|^{-\beta} \hat{x},$$

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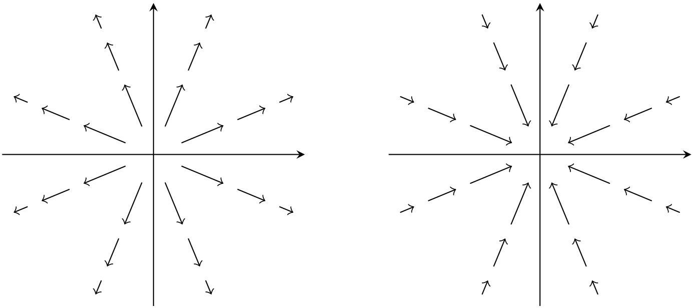
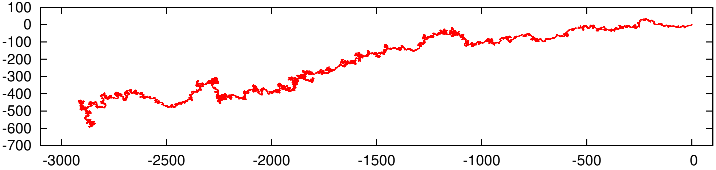


Centrally biased random walk simulation



Here is a simulation of 10^5 steps of a centrally biased random walk with $\rho = 1$ and $\beta = 1/2$.

Centrally biased walk and Lamperti's problem

- Again we consider the Lyapunov function $R_n = \|X_n\|$.
- This time

$$\mu_1(r) = \rho(1 + o(1))r^{-\beta} + \frac{d-1}{2d}(1 + o(1))r^{-1};$$
$$\mu_2(r) = \frac{1}{d}(1 + o(1)).$$

- The critical case from the point of view of recurrence/transience is when $\beta = 1$. Then

$$2r\mu_1(r) - \mu_2(r) \to 2\rho + \frac{d-1}{d} - \frac{1}{d},$$

  which is positive (and hence the walk is transient) if $\rho > \frac{2-d}{2d}$.
- So, for example, if $d = 2$ the walk is transient for any $\rho > 0$.

Angular asymptotics: Critical case

Theorem (MMW 2010)

Consider a centrally biased random walk with $\mu(x) = O(\|x\|^{-1})$. Then the walk has no limiting direction, i.e.,

$$\mathbb{P}[\lim_{n\to\infty} \hat{X}_n \text{ exists}] = 0.$$

In this case the projection of the walk onto the sphere wanders without converging, and under mild conditions visits every neighbourhood on the sphere.

Angular asymptotics: Supercritical case

If $\rho > 0$ and $\beta \in (0, 1)$, the walk is transient, and the rate of escape is super-diffusive but sub-ballistic, as shown by the following result.

Theorem (MMW 2009, MW 2009)

Suppose $\rho > 0$, $\beta \in (0, 1)$. Then X_n is transient with a limiting direction, i.e., $\hat{X}_n \to u$ a.s. for some (random) unit vector u. Moreover there is a law of large numbers

$$n^{-\frac{1}{1+\beta}} \|X_n\| \to \lambda(\rho, \beta) \quad (\textit{constant}), \; \textit{a.s.}$$

In $d = 1$, there is an accompanying central limit theorem [MW 2009] which says that

$$\frac{X_n - \lambda(\rho, \beta) n^{\frac{1}{1+\beta}}}{\sqrt{n}} \to \text{ normal.}$$

Further properties

- Centrally biased random walks in the critical case ($\beta = 1$) can be viewed as prototypical near-critical stochastic systems.
- They can be positive-recurrent, null-recurrent, or transient.

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Further properties

- Centrally biased random walks in the critical case ($\beta = 1$) can be viewed as prototypical near-critical stochastic systems.
- They can be positive-recurrent, null-recurrent, or transient.
- But even if transient, they are diffusive,...
- and even if positive-recurrent, they do not possess geometric ergodicity: return times and stationary distributions have heavy tails.

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