Reflecting random walks in curvilinear wedges

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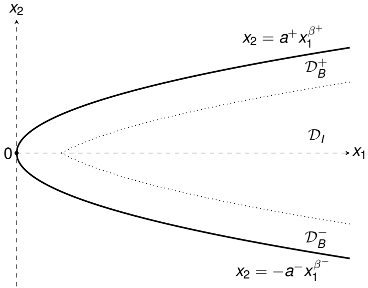
Model: domain

An unbounded planar domain (a generalized parabola)

$$\mathcal{D} := \left\{ (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : -a^- x_1^{\beta^-} \le x_2 \le a^+ x_1^{\beta^+} \right\},$$

for constants $a^+, a^- > 0$ and $\beta^+, \beta^- \in (0, 1)$.



The domain is partitioned into a (thickened) boundary $\mathcal{D}_B$ and the interior $\mathcal{D}_I$.

The $a^+, a^- > 0$ are not important. The β^+, β^- are crucial, although we could allow some small perturbations.

Model: process

- Our process is $\xi := (\xi_0, \xi_1, \ldots)$ a discrete-time, time-homogeneous Markov chain in $\mathcal{D}$.
- Everywhere: increments have uniformly bounded *p*th moments, $p > 2$.
- Interior: process has zero drift and a fixed, positive-definite increment covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho \\ \rho & \sigma_2^2 \end{pmatrix}.$$

- Boundary: process "reflects" with drift at angle α relative to the inwards pointing normal. Positive α is anticlockwise.

History: Reflecting random walks in quadrants (Kingman, 1961, Malyshev, 1970, Fayolle *et al.*, 1991, ...); reflecting diffusions in wedges (Varadhan & Williams, 1985, ...); reflecting diffusions in generalized cones (Pinsky, 2009); many more.

Model: picture

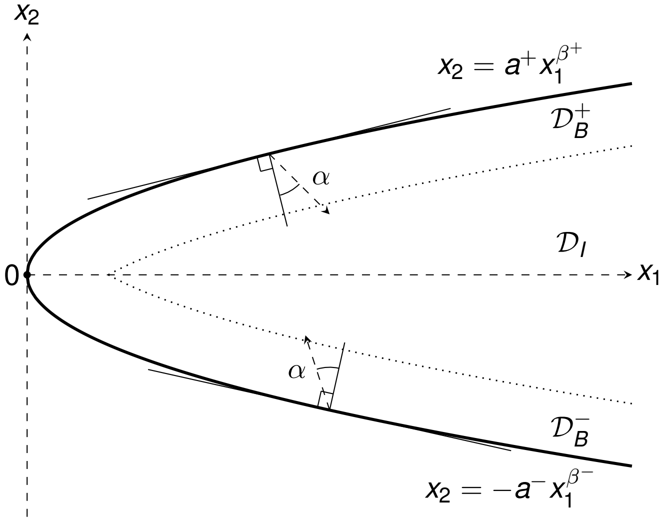


Figure: An illustration of the model parameters.

Main question

Key quantities: β^+, β^-, (domain growth) $\sigma_1^2, \sigma_2^2, \rho$ (interior covariance), α (reflection angle).

Main question: Is the process recurrent or transient?

Note that by choosing the same α top and bottom, the reflection is asymptotically opposite from top and bottom. If this is not the case, then there is no phase transition between recurrence and transience.

Special case where $\alpha = 0$ is normal reflection. This case was studied by PINSKY (2009) for reflecting Brownian motion with canonical covariance. In that case there is no phase transition, but if one permits general covariance then even the $\alpha = 0$ case shows a phase transition.

Main results: normal reflection

Theorem

Suppose that $\alpha = 0$*. Let* $\beta := \max(\beta^+, \beta^-)$*.*

- *If* $\beta \leq \sigma_1^2/\sigma_2^2$*, then* ξ *is recurrent.*
- *If* $\sigma_1^2/\sigma_2^2 < \beta < 1$*, then* ξ *is transient.*

Remarks

- Here ρ does not play a role, but it will for $\alpha \neq 0$.
- If $\sigma_1^2/\sigma_2^2 < 1$, there is non-monotonicity: there exist regions $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3$ such that the reflecting random walk is recurrent on $\mathcal{D}_1$ and $\mathcal{D}_3$, but transient on $\mathcal{D}_2$. This does not occur in the case of canonical covariance (cf. PINSKY, 2009).

Main results: normal reflection

In the recurrent cases, we quantify recurrence by looking at moments of return times. Let τ be the return time to a compact set. Define

$$s_0 := s_0(\Sigma, \beta) := \frac{1}{2}\left(1 - \frac{\sigma_2^2 \beta}{\sigma_1^2}\right).$$

Theorem

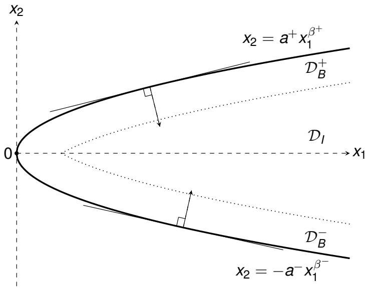
Suppose that $\alpha = 0$. Let $\beta := \max(\beta^+, \beta^-)$. If $\beta < \sigma_1^2/\sigma_2^2$, then $\mathbb{E}(\tau^s) < \infty$ for all $s < s_0$ but $\mathbb{E}(\tau^s) = \infty$ for all $s > s_0$.

Remark

- It's no surprise that $s_0 < 1/2$, so every case is less recurrent than one-dimensional simple random walk.

Main results: some intuition

The picture for orthogonal reflection:



Horizontal component of drift is 0 in interior and about $x_1^{\beta-1}$ when at the boundary.

At horizontal position x_1, one would expect to be at the boundary with probability about $x_1^{-\beta}$.

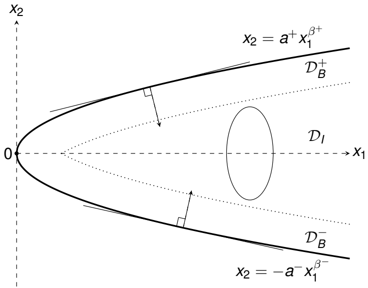
So effective drift is of order $x_1^{\beta-1} \cdot x_1^{-\beta} = 1/x_1$.

The is the critical regime for recurrence/transience, cf. LAMPERTI (1960).

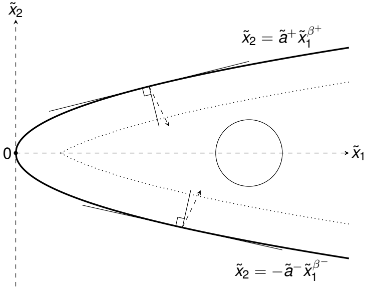
Main results: some intuition

Why is increasing σ_2^2 relative to σ_1^2 good for transience?

The ellipse represents the elongated covariance matrix when $\sigma_2^2 > \sigma_1^2$:



Linear transformation to recover a canonical covariance increases the angles a little:



Main results: opposed reflection

Define

$$\beta_{\mathrm{c}} := \beta_{\mathrm{c}}(\Sigma, \alpha) := \frac{\sigma_1^2}{\sigma_2^2} + \left(\frac{\sigma_2^2 - \sigma_1^2}{\sigma_2^2} \right) \sin^2 \alpha + \frac{\rho}{\sigma_2^2} \sin 2\alpha. \tag{1}$$

Theorem

Suppose that $\alpha \in (-\pi/2, \pi/2)$. Let $\beta := \max(\beta^+, \beta^-)$.

- *If $\beta \leq \beta_{\mathrm{c}}$, then ξ is recurrent.*
- *If $\beta > \beta_{\mathrm{c}}$, then ξ is transient.*

Remarks

- The threshold (1) is invariant under $(\alpha, \rho) \mapsto (-\alpha, -\rho)$.
- In the case where $\sigma_1^2 = \sigma_2^2$ and $\rho = 0$, then $\beta_{\mathrm{c}} = 1$, so recurrence is certain for all $\beta^+, \beta^- < 1$ and all α.

Main results: opposed reflection

Define

$$s_0 := s_0(\Sigma, \alpha, \beta) := \frac{1}{2}\left(1 - \frac{\beta}{\beta_c}\right).$$

Theorem

Suppose that $\alpha \in (-\pi/2, \pi/2)$*. Let* $\beta := \max(\beta^+, \beta^-)$*. If* $\beta < \beta_c$*, then* $\mathbb{E}\left(\tau^s\right) < \infty$ *for all* $s < s_0$ *but* $\mathbb{E}\left(\tau^s\right) = \infty$ *for all* $s > s_0$*.*

Final remarks

Proofs use Foster–Lyapunov criteria.

One expects the same behaviour for the diffusion case.

Thank you!

References

- J. Lamperti, Criteria for the recurrence or transience of stochastic processes I, *J. Math. Anal. Appl.* **1** (1960) 314–330.
- M.V. Menshikov, A. Mijatović, & A.R. Wade, Reflecting random walks in curvilinear wedges. Preprint (2020) arXiv:2001.06685.
- R.G. Pinsky, Transience/recurrence for normally reflected Brownian motion in unbounded domains. *Ann. Probab.* **37** (2009) 676–686.