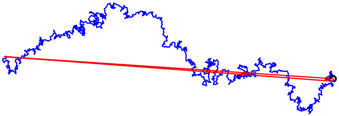
Random walk avoiding its convex hull with a finite memory



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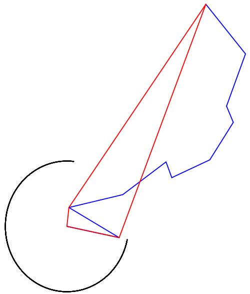
Introduction

Fix $d \geq 2$ (ambient dimension) and $k \geq d - 1$ an integer (the 'memory').

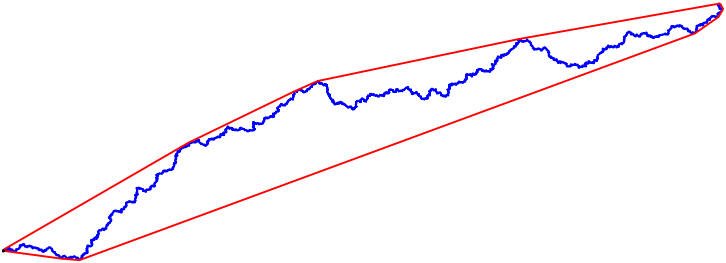
We study the process $X = (X_0, X_1, X_2, \dots)$ in $\mathbb{R}^d$, where, roughly speaking, given $X_0, \dots, X_n$, the next position X_{n+1} is uniform on the unit ball centred at X_n but conditioned so that the line segment from X_n to X_{n+1} does not intersect the convex hull of $\{0, X_{n-k}, X_{n-k+1}, \dots, X_n\}$ at any point other than X_n.



10 steps with $d = 2$, $k = 2$.

Introduction

Mathematical antecedent: The case '$k = \infty$' is a variation on a model of Angel *et al.* (2003) which avoids the convex hull of its entire previous trajectory.



10^4 *steps of the '$k = \infty$' walk in $d = 2$.*

Motivation

For the 'infinite memory' model of ANGEL *et al.* (2003):

The frontiersman: "A frontier rancher who is walking about and at each step increases his ranch by dragging with him the fence that defines it, so that the ranch at any time is the convex hull of the path traced until that time."

Extremal investor: Investment decisions driven by previous maximum and minimum fund values.

Motivation

More generally, our process is a self-interacting random walk where the self-interaction is mediated by some occupation statistic of the previous trajectory.

In general, such self-interaction may be

- local, such as in reinforced or excited random walks, where the walker's motion is biased by its occupation measure in the immediate vicinity, or
- global, such as for processes whose self-interaction is mediated via some global functional of the past trajectory, such as its centre of mass or convex hull.

In either case, the self-interaction can be attractive or repulsive.

There is an important distinction between static models, such as self-avoiding walk, and dynamic models that are genuine stochastic processes. Our model is of the latter type.

Motivation

Self-interacting processes are typically non-Markovian, and arise naturally in systems where there is learning, resource depletion, or physical interaction.

Two examples that could fit with our model are

- a roaming animal performing a random walk may tend to avoid previously visited regions in the hunt for new resources [S];
- our trajectory shares properties with linear chain polymer molecules in the extended phase [B].

[S] P.E. Smouse *et al.*, *Phil. Trans. Roy. Soc. B* **365** (2010) 2201–2211.

[B] M.N. Barber & B.W. Nimham, *Random and Restricted Walks: Theory and Applications.* Gordon and Breach, New York, 1970.

Main phenomena

The '$k = \infty$' model is conjectured to be ballistic.

This means it is expected to have

- a positive limiting speed, and
- a limiting direction.

Formally, it is believed that, a.s., for a constant $v_{d,\infty} > 0$ and a random $\ell \in \mathbb{S}^{d-1}$,

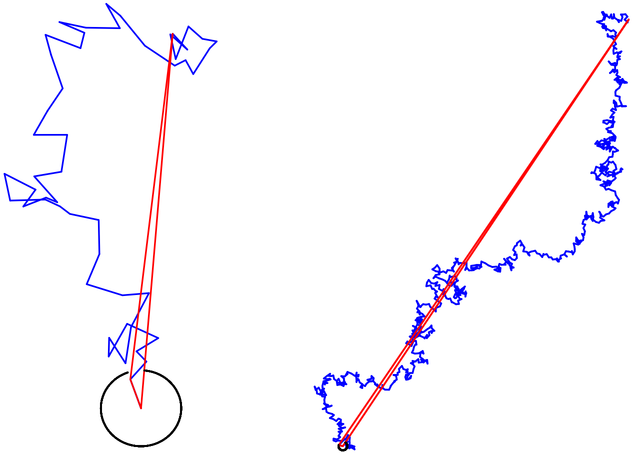
$$\lim_{n\to\infty} n^{-1}\|X_n\| = v_{d,\infty}, \text{ and } \lim_{n\to\infty} \hat{X}_n = \ell,$$

where $\hat{x} := x/\|x\|$.

For the '$k = \infty$' model it is known (ZERNER, 2005) only that $\liminf_{n\to\infty} n^{-1}\|X_n\| \geq c > 0$.

We establish ballisticity for the finite memory model.

Simulations



The walk with $d = 2$, $k = 1$ *for 50 (left) and 1000 steps (right).*

Outline

Main results for finite memory

Recall that $k \geq d - 1$. The following is our ballisticity result.

Theorem

There exist a positive constant $v_{d,k}$ and a random unit vector ℓ such that

$$\lim_{n \to \infty} n^{-1} X_n = v_{d,k} \ell, \ \textit{a.s.}$$

The constants $v_{d,k}$ seem hard to compute in general, but:

Theorem

If $d = 2$ and $k = 1$, then

$$v_{2,1} = \frac{8}{9\pi^2} \approx 0.09006327.$$

Open problems

Simulations suggest the following:

Conjecture

We have $v_{d,k} \leq v_{d,k+1}$.

This would imply that $\lim_{k\to\infty} v_{d,k}$ exists. It is tempting to believe:

Conjecture

We have $\lim_{k\to\infty} v_{d,k} = v_{d,\infty}$, where $v_{d,\infty}$ is the (conjectural) speed of the '$k = \infty$' model.

Simulations are reasonably consistent with this, but not entirely convincing.

Outline

Some geometrical facts

Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$.

- Given $\mathcal{F}_n$, we have that the support of X_{n+1} has volume bounded between $v_d/2$ and v_d, uniformly over histories of the process, where v_d is the volume of the unit radius ball.
- It is not hard to show that $\mathbb{E}\left[(X_{n+1} - X_n) \cdot \hat{X}_n \mid \mathcal{F}_n\right] \geq 0$, a.s.
- Over one step, one cannot do better than this, but with a bit more work one can show that, for some $m := m(d, k) \in \mathbb{N}$ and $c := c(d, k) > 0$, $\mathbb{E}\left[\|X_{n+m}\| - \|X_n\| \mid \mathcal{F}_n\right] \geq c$, a.s.
- It follows from this and the one-sided Azuma–Hoeffding inequality that, for some $\rho := \rho(d, k) > 0$, $\liminf_{n\to\infty} n^{-1}\|X_n\| \geq \rho$, a.s.

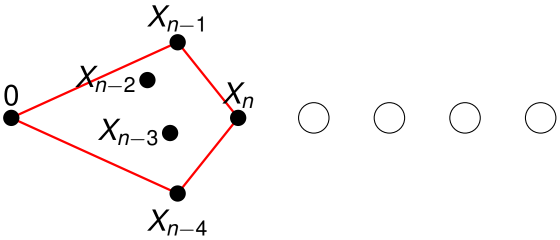
Outline

Renewal structure

The first step in establishing ballisticity is to identifty a renewal structure. Fix $\delta \in (0, 1/8)$ from now on.

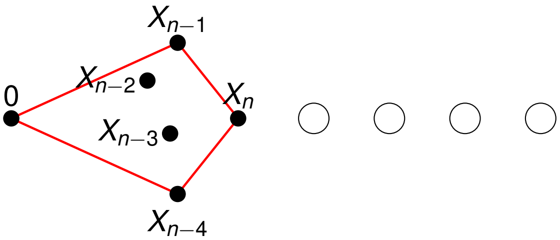
For $x \in \mathbb{R}^d$ define $\Pi(x) := \prod_{i=1}^{k} B(x + \frac{i}{2}\hat{x}; \delta) \subseteq (\mathbb{R}^d)^k$.

Say X has good geometry at time n, and that G_n occurs, if $(X_{n-k}, \ldots, X_n)$ is such that the support of $(X_{n+1}, \ldots, X_{n+k})$ contains $\Pi(X_n)$.



A configuration with good geometry for $d = 2$ and $k = 4$.

Renewal structure



A configuration with good geometry for $d = 2$ and $k = 4$.

Lemma

There exists $\alpha = \alpha(d, k, \delta) > 0$ such that for all Borel $\mathfrak{B} \subseteq (\mathbb{R}^d)^k$, on the event G_n,

$$\mathbb{P}\left((X_{n+1}, \ldots, X_{n+k}) \in \mathfrak{B} \mid \mathcal{F}_n\right) \geq \alpha \frac{|\mathfrak{B}|}{|\Pi(X_n)|} \mathbf{1}\{\mathfrak{B} \subseteq \Pi(X_n)\}.$$

Renewal structure

In other words, if the configuration has good geometry, we can extract from the law of $(X_{n+1}, \ldots, X_{n+k})$ a component that is uniform on $\Pi(X_n)$.

Thus we can introduce an i.i.d. sequence of Bernoulli-α random variables, and construct the process in blocks of k steps, as follows: At time mk,

- if G_{mk} does not occur, just construct $(X_{mk+1}, \ldots, X_{(m+1)k})$ as normal;
- if G_{mk} does occur, and the Bernoulli-α comes up heads, declare that a renewal has occurred at time mk, and construct $(X_{mk+1}, \ldots, X_{(m+1)k})$ to be uniform on $\Pi(X_{mk})$;
- if the Bernoulli-α comes up tails, use the remaining part of the law to get the correct increment distribution.

Renewal structure

These renewals occur rather frequently:

Lemma

With $\alpha > 0$ *the same constant as before,* $\mathbb{P}\left(G_{n+k} \mid \mathcal{F}_n\right) \geq \alpha$*, a.s.*

It follows that the inter-renewal times have a uniform exponential tail bound.

The renewal structure makes plain that

- the process between renewals has strictly positive radial drift; and
- the transverse fluctuations are symmetric.

This is already enough to establish a limiting direction.

Problem: the segments of the process between renewal times are not homogeneous, due to the special role played by the origin in the construction of the process, and so the radial drift is not constant. So we don't immediately get the limiting speed from this construction.

Outline

Homogeneous process

Let $\tau_1, \tau_2, \ldots$ denote the renewal times as constructed above. Our limiting speed will follow from this result:

Proposition

There are positive constants $\lambda = \lambda(d, k, \delta)$ and $u = u(d, k, \delta)$ such that, for all $\gamma \in (0, 1)$,

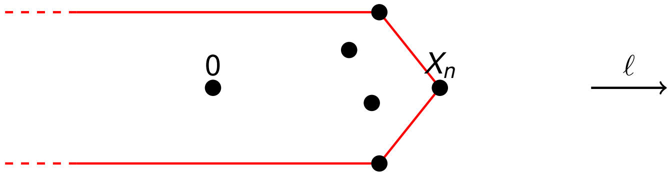
- $\mathbb{E}\left[\tau_{n+1} - \tau_n \mid \mathcal{F}_{\tau_{n+1}}\right] = \lambda + o(n^{-\gamma})$;
- $\mathbb{E}\left[X_{\tau_{n+1}} - X_{\tau_n} \mid \mathcal{F}_{\tau_{n+1}}\right] = u\hat{X}_{\tau_n} + o(n^{-\gamma})$.

We get this result by building a spatially homogeneous version of the process for which the above results hold exactly (with no $o(n^{-\gamma})$ term) and then use a coupling over the interval between renewal times.

Homogeneous process

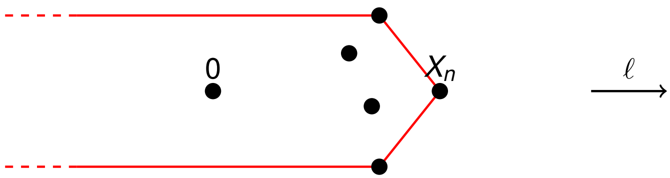
Fix a unit vector ℓ. We define the ℓ-process just like the normal process, but replace the origin by a point at infinity in the direction $-\ell$.

Here is an example with $d = 2$ and $k = 4$:



If $\ell = \hat{X}_n$ (as in the picture) then the transition law of the ℓ-process is the same as the original process.

Homogeneous process



If $\ell = \hat{X}_n$ (as in the picture) then the transition law of the ℓ-process is the same as the original process.

Idea: At time τ_n, set $\ell = \hat{X}_{\tau_n}$ and run both processes until the next renewal on the same probability space, with increments coupled in the maximal way.

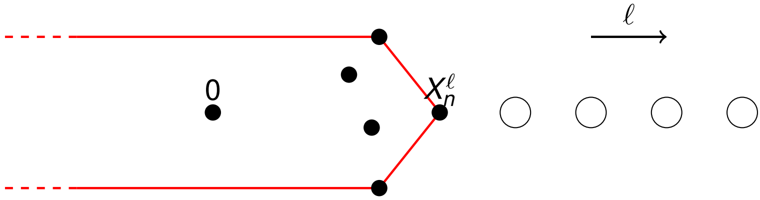
Since we know renewals are frequent, and we have linear growth of the radius, $\hat{X}$ will not deviate much from ℓ over the entire time. So coupling has a good chance of success.

We need to explain what a renewal is for the ℓ-process, and why it is homogeneous.

Homogeneous process

For $x \in \mathbb{R}^d$ define $\Pi^\ell(x) := \prod_{i=1}^k B(x + \frac{i}{2}\ell; \delta) \subseteq (\mathbb{R}^d)^k$.

Say X^ℓ has good geometry at time n, and that G_n^ℓ occurs, if $(X_{n-k}^\ell, \ldots, X_n^\ell)$ is such that the support of $(X_{n+1}^\ell, \ldots, X_{n+k}^\ell)$ contains $\Pi^\ell(X_n^\ell)$.



Note: for original process the definition of Π had $\hat{x}$ in place of ℓ.

The law of $(X_m^\ell, m \geq \tau_n)$ depends on X_{τ_n} only through $\hat{X}_n = \ell$. Moreover, the law of the process is invariant under rotations that leave ℓ fixed.

Homogeneous process and coupling

With the obvious notation, we get:

Proposition

For positive constants $\lambda = \lambda(d, k, \delta)$ and $u = u(d, k, \delta)$,

- $\mathbb{E}\left[\tau^\ell_{n+1} - \tau^\ell_n \mid \mathcal{F}_{\tau^\ell_n+1}\right] = \lambda$;
- $\mathbb{E}\left[X^\ell_{\tau^\ell_{n+1}} - X_{\tau^\ell_n} \mid \mathcal{F}^\ell_{\tau^\ell_n+1}\right] = u\hat{X}^\ell_{\tau^\ell_n}$.

The final step in the proof is the coupling:

Proposition

Starting with $\ell = \hat{X}_{\tau_n}$ we can build on one probability space copies of the processes X and X^ℓ such that their paths coincide up to time $\tau_{n+1} = \tau^\ell_{n+1}$ with probability at least

$$1 - \frac{C \log^2 n}{n}.$$

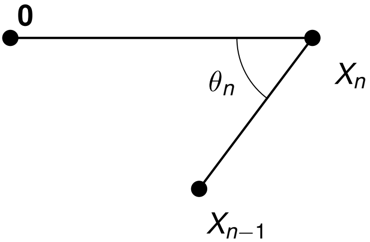
Details are technical, but exploit that $\hat{X} \approx \ell$ throughout.

Outline

Unit memory

From now on take $d = 2$ and $k = 1$.

Denote by $\theta_n \in [0, \pi]$ the magnitude of the interior angle of the convex hull of $\{0, X_{n-1}, X_n\}$ at X_n:



Unit memory

Theorem

If $d = 2$ and $k = 1$, then

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\,\|X_n\| = \frac{8}{9\pi^2} \approx 0.09006327.$$

Combined with our general result that $n^{-1}\|X_n\| \to v_{2,1}$, a.s., this gives the value of $v_{2,1}$.

The following relates local drift to global speed.

Lemma

Let $\xi_0, \xi_1, \ldots$ be a process on $\mathbb{R}^d$ with $\xi_0 = 0$ and $\|\xi_{n+1} - \xi_n\| \leq B$, a.s., for some constant $B < \infty$. Suppose that $\|\xi_n\| \to \infty$, a.s. Then

$$\lim_{n\to\infty} \left| \frac{1}{n}\mathbb{E}\,\|\xi_n\| - \frac{1}{n}\sum_{m=0}^{n-1} \mathbb{E}\left[(\xi_{m+1} - \xi_m)\cdot \hat{\xi}_m\right] \right| = 0.$$

Proof.

Note first that for any x, y with $\|y\| \leq B$,

$$\big| \|x + y\| - \|x\| - \hat{x} \cdot y \big| \leq C(1 + \|x\|)^{-1}.$$

Setting $\Delta_m := \xi_{m+1} - \xi_m$,

$$\mathbb{E}\,\|\xi_n\| = \sum_{m=0}^{n-1} \mathbb{E}\left[\|\xi_m + \Delta_m\| - \|\xi_m\right] = \sum_{m=0}^{n-1} \mathbb{E}\,[\hat{\xi}_m \cdot \Delta_m] + \sum_{m=0}^{n-1} \mathbb{E}\,\zeta_m,$$

where $|\zeta_m| \leq C(1 + \|\xi_m\|)^{-1} \to 0$, a.s. $\square$

Unit memory

A calculation shows that the local drift of the process is:

Lemma

$$\mathbb{E}\left[(X_{n+1} - X_n) \cdot \hat{X}_n\right] = \mathbb{E}\left[\frac{2\sin\theta_n}{6\pi - 3\theta_n}\right].$$

Idea: We show that θ_n converges in distribution, so that the limiting speed $v_{2,1}$ is given by

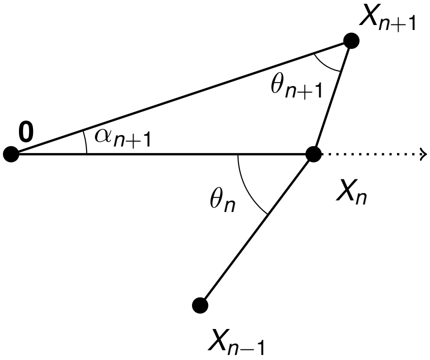
$$v_{2,1} = \lim_{n\to\infty} \mathbb{E}\left[\frac{2\sin\theta_n}{6\pi - 3\theta_n}\right].$$

Unit memory

Note that θ_n is not Markov, but some geometry shows that:

Lemma

$\theta_{n+1} = |(2\pi - \theta_n)U_{n+1} - \pi| - \alpha_{n+1}$, where $U_1, U_2, \dots$ are i.i.d. $U[0,1]$ and $\alpha_n \to 0$ a.s.



Unit memory

Lemma

As $n \to \infty$, $\theta_n \xrightarrow{d} \theta$ where $\theta \in [0, \pi]$ has the distribution uniquely determined by the fixed-point equation

$$\theta \stackrel{d}{=} |(2\pi - \theta)U - \pi|, \; \theta \in \mathbb{R}, \tag{1}$$

where U is $U[0,1]$ and independent of θ. Moreover, the random variable θ has probability density function f given by

$$f(t) = \frac{2}{3\pi^2}(2\pi - t), \; \text{for } t \in [0, \pi].$$

It follows that

$$v_{2,1} = \mathbb{E}\left[\frac{2\sin\theta}{6\pi - 3\theta}\right] = \frac{4}{9\pi^2}\int_0^\pi \sin t \, \mathrm{d}t = \frac{8}{9\pi^2}.$$

Unit memory

Proof.

Let $T(u,x) := |(2\pi - x)u - \pi|$. Then $\theta_{n+1} = T(\theta_n, U_{n+1}) - \alpha_{n+1}$.

Define a Markov operator Q on $[0,\pi]$ by $Q(x,A) := \mathbb{P}\,(T(x,U) \in A)$. Then the fixed-point equation (1) reads $\mathbb{E}\,Q(\theta, A) = \mathbb{P}\,(\theta \in A)$ for all A.

So solutions to (1) are the invariant measures of Q.

But Q satisfies a Doeblin condition, so there is a unique invariant measure μ and $\sup_\nu \rho(\nu Q^m, \mu) \to 0$, where $\sup_\nu$ is over all probability measures ν on $[0,\pi]$. (Here ρ is a suitable metric on distributions.)

Let ν_n denote the law of θ_n. Then $\sup_k \rho(\nu_k Q^m, \mu) \to 0$ as $m \to \infty$.

But also, since $|T(x,u) - T(y,u)| \le |x - y|$ and $\alpha_n \to 0$ we get $\sup_m \rho(\nu_{k+m}, \nu_k Q^m) \to 0$ as $k \to \infty$.

Combining these gives $\nu_{k+m} \to \mu$.

Then one can check that the density f solves the fixed-point equation (1). $\square$

Final remarks

The renewal structure depends crucially on the finite memory.

Explicit computation of $v_{d,k}$ seems hard beyond the case $v_{2,1}$.

One expects a central limit theorem for $n^{-1/2}(X_n - v_{d,k} n \hat{X}_n)$.

There is an obvious coupling between the process with memory k and the process with memory $k+1$, but it is not clear whether it is useful.

Including the origin in the convex hull to exclude is crucial; otherwise the process is diffusive (cf. the correlated random walk).

Thank you!

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