Heavy-tailed random walk on complexes of half-lines

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Joint work with
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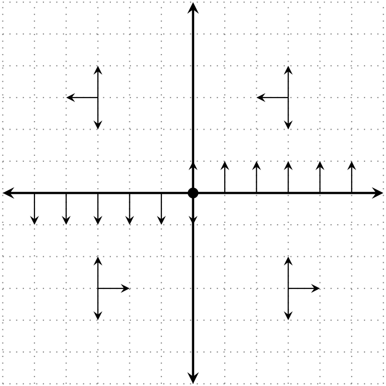
Introduction



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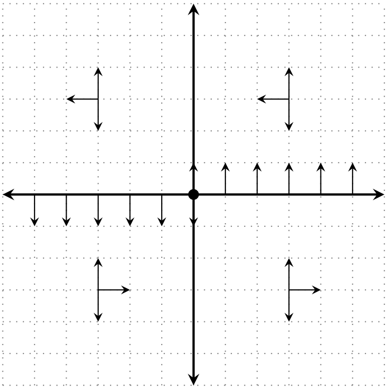

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Is this walk transient or recurrent?

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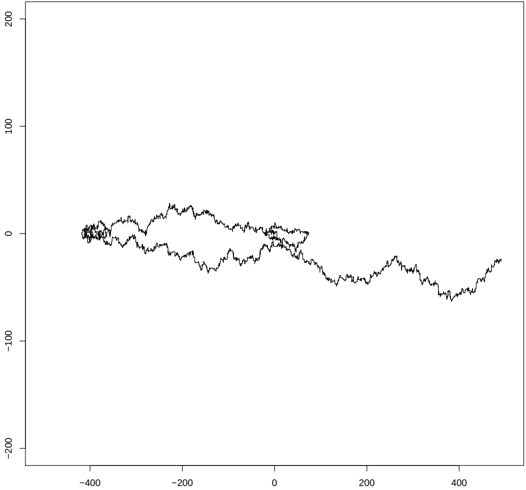
Here is a simulated trajectory of the random walk on $\mathbb{Z}^2$.



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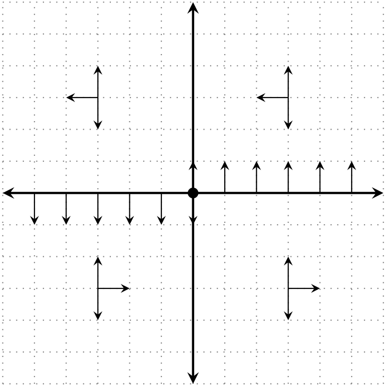

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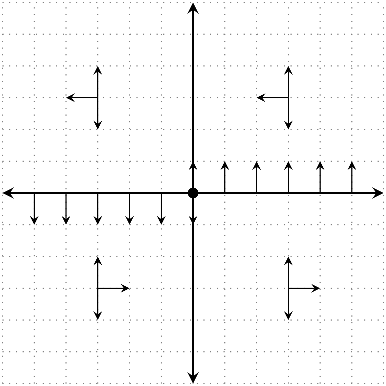

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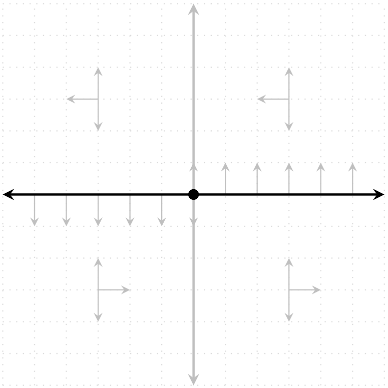

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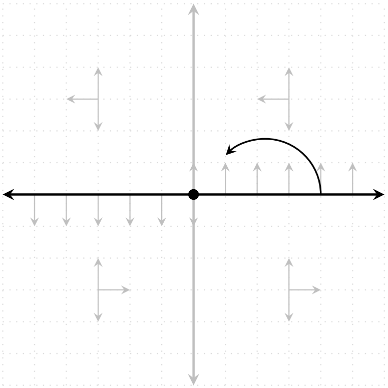

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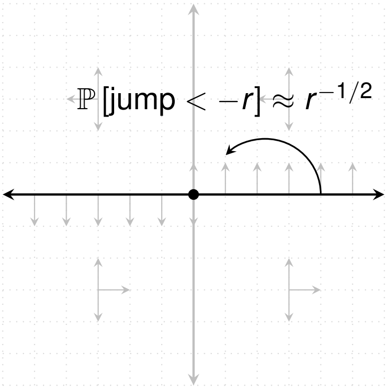

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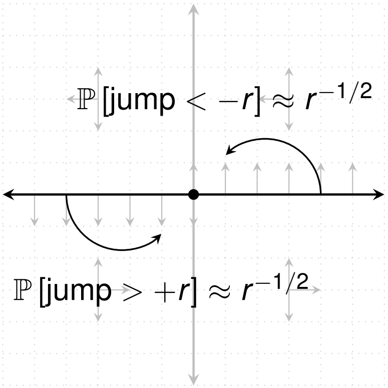

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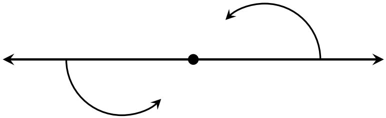

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This random walk on $\mathbb{Z}$ is an example of the oscillating random walk studied by Kemperman (1974).

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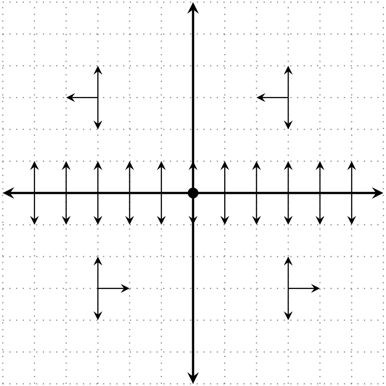

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This walk was studied by Campanino & Petritis (2003). Is it transient or recurrent?

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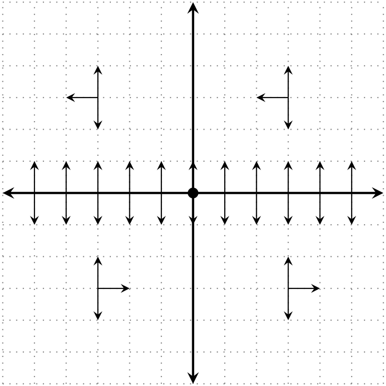

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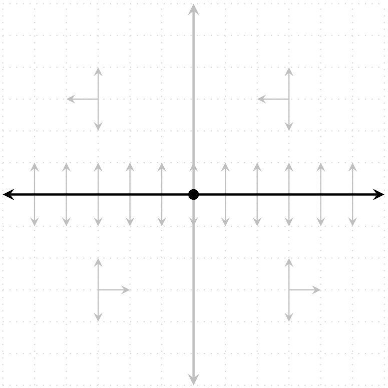

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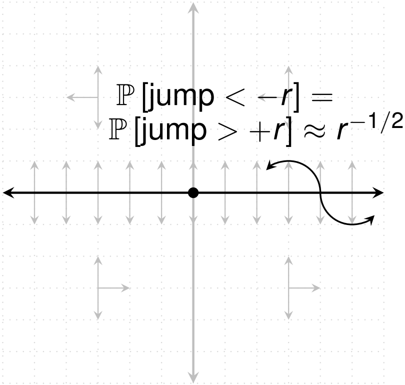

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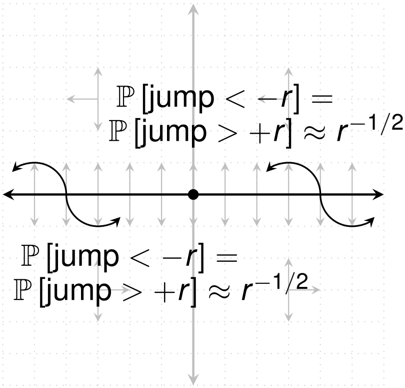

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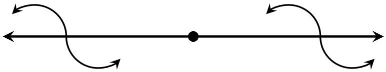

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This random walk on $\mathbb{Z}$ is homogeneous and symmetric; see Shepp (1962).

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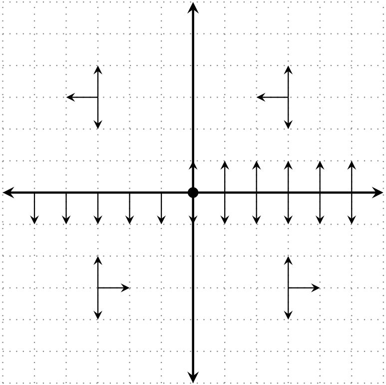

One more example, with a third rule for transitions on the horizontal axis.

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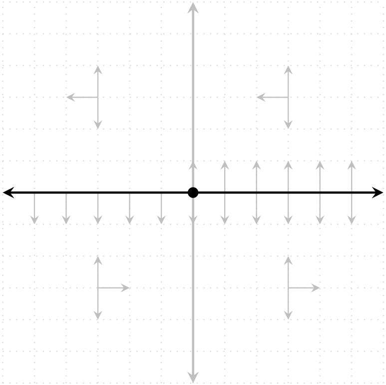

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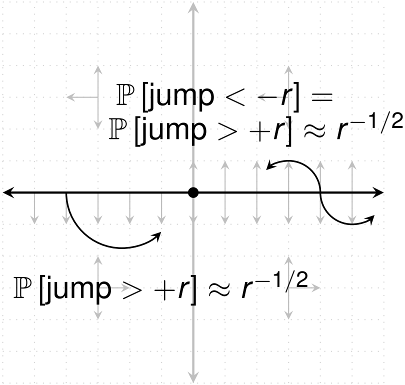

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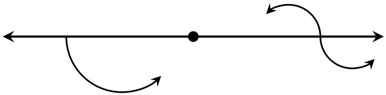

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This is another example of the oscillating random walk; this one was studied by Rogozin & Foss (1978).

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2. Oscillating random walk
3. Complexes of half-lines
4. Ideas of the proofs

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I.e., Z_n is a time-homogeneous Markov chain on $\mathbb{R}$ with transition kernel given by

$$\mathbb{P}\left[Z_{n+1} \in B \mid Z_n = x\right] = \begin{cases} \int_B w_+(z-x)\mathrm{d}z & \text{if } x \geq 0, \\ \int_B w_-(z-x)\mathrm{d}z & \text{if } x < 0. \end{cases}$$

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We consider certain heavy-tailed jump densities.

Write $v \in \mathfrak{D}_\alpha$ to mean

$$v(y) = \begin{cases} c(y)y^{-1-\alpha} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0, \end{cases}$$

where $c(y) \to c$ a positive constant.

Oscillating random walk: Example 1

For $\alpha \in (0, \infty)$, suppose $v \in \mathfrak{D}_\alpha$ and $w_+(y) = w_-(y) = \frac{1}{2} v(|y|)$.

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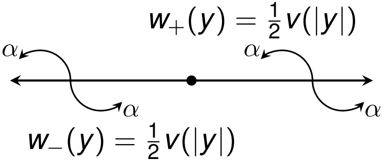

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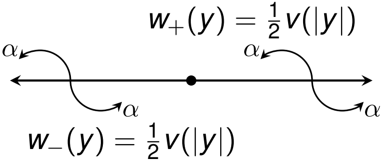

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The random walk is transient if $\alpha < 1$ and recurrent if $\alpha > 1$.

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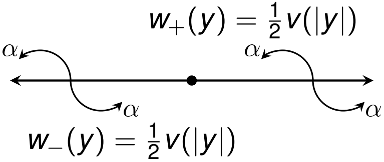

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> **Theorem (Shepp 1962)**
>
> *The random walk is transient if $\alpha < 1$ and recurrent if $\alpha > 1$.*

Under a slightly stronger assumption on $c(y)$, the critical case $\alpha = 1$ is recurrent.

Oscillating random walk: Example 1

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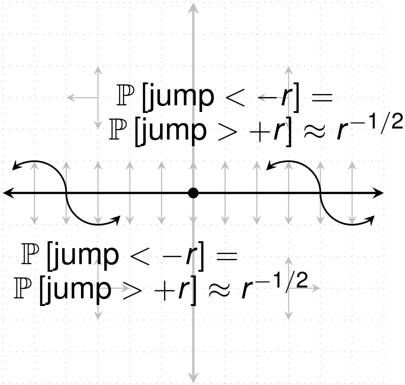

The embedded random walk in this example is Shepp's symmetric walk with $\alpha = 1/2$.

So this random walk is (comfortably) transient.

This result was obtained by other methods by Campanino & Petritis (2003).

Oscillating random walk: Example 2

For $\alpha, \beta \in (0, \infty)$, suppose $v_+ \in \mathfrak{D}_\alpha$, $v_- \in \mathfrak{D}_\beta$, and take $w_+(y) = \frac{1}{2} v_+(|y|)$ and $w_-(y) = \frac{1}{2} v_-(|y|)$.

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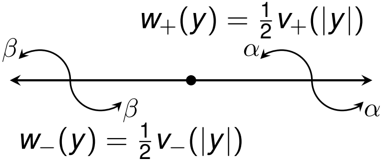

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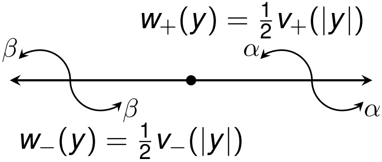

Generalizes the symmetric random walk to a two-sided oscillating random walk in the vein of Kemperman (1974).

Theorem (Kemperman 1974, Rogozin & Foss 1978, Sandrić 2014)

The random walk is transient if $\alpha + \beta < 2$ and recurrent if $\alpha + \beta > 2$.

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For $\alpha, \beta \in (0, \infty)$, suppose $v_+ \in \mathfrak{D}_\alpha$, $v_- \in \mathfrak{D}_\beta$, and take $w_+(y) = v_+(-y)$ and $w_-(y) = v_-(y)$.

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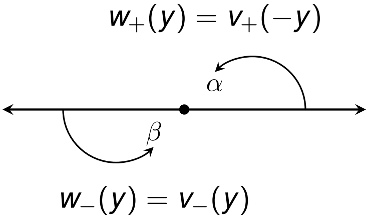

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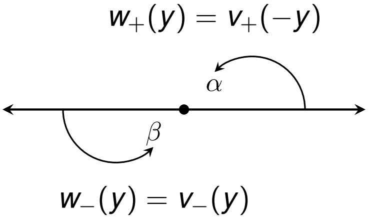

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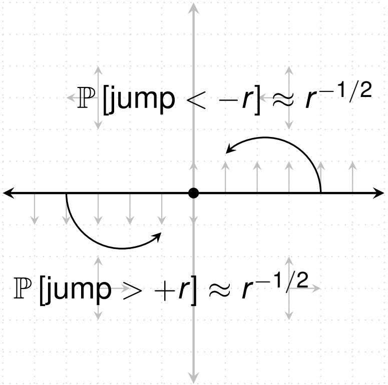

The embedded random walk in this example is the one-sided oscillating random walk with $\alpha = \beta = 1/2$.

This is exactly the critical case, and needs some more delicate analysis.

We conjecture that this walk is recurrent.

Oscillating random walk: Example 4

For $\alpha, \beta \in (0, \infty)$, suppose $v_+ \in \mathfrak{D}_\alpha$, $v_- \in \mathfrak{D}_\beta$, and take $w_+(y) = \frac{1}{2} v_+(|y|)$ and $w_-(y) = v_-(y)$.

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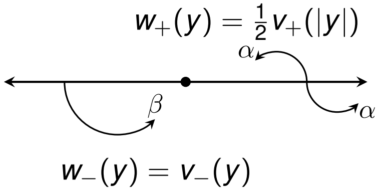

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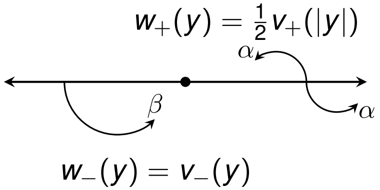

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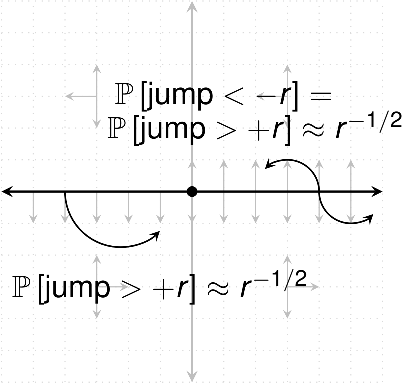

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So $\alpha + 2\beta = 3/2 < 2$ and this random walk is transient.

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Complexes of half-lines

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We view $\mathbb{R}_+ \times \mathcal{S}$ as a complex of half-lines $\mathbb{R}_+ \times \{k\}$ connected at a central origin $\mathcal{O} = \{0\} \times \mathcal{S}$; at time n, coordinate ξ_n describes which branch the process is on, and X_n describes the distance along the branch at which the process sits.

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The transition law of the process will be described by

- an irreducible stochastic matrix $P = (p(i,j); i,j \in \mathcal{S})$;
- a collection $(w_i; i \in \mathcal{S})$ of probability density functions.

Complexes of half-lines

The transition rule is as follows. Given $(X_n, \xi_n) = (x, i) \in \mathbb{R}_+ \times \mathcal{S}$, generate (independently) a spatial increment φ_{n+1} from w_i and a random $\eta_{n+1} \in \mathcal{S}$ according to $p(i, \cdot)$. Then

- if $x + \varphi_{n+1} \geq 0$, set $(X_{n+1}, \xi_{n+1}) = (x + \varphi_{n+1}, i)$; else
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The transition kernel of the process is given for $(x, i) \in \mathbb{R}_+ \times \mathcal{S}$ by

$$\begin{aligned}
&\mathbb{P}\left[(X_{n+1}, \xi_{n+1}) \in B \times \{j\} \mid (X_n, \xi_n) = (x, i)\right] \\
&\quad = p(i, j) \int_B w_i(-z - x) \mathrm{d}z + \mathbf{1}\{i = j\} \int_B w_i(z - x) \mathrm{d}z.
\end{aligned}$$

Complexes of half-lines

We assume that the densities w_i are either one-sided or symmetric: partition $\mathcal{S}$ as $\mathcal{S}^{\text{one}} \cup \mathcal{S}^{\text{sym}}$.

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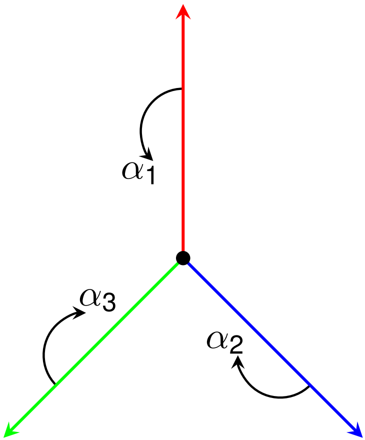
In the case where $\mathcal{S}$ has two elements, mapping $\mathbb{R}_+ \times \mathcal{S}$ naturally into $\mathbb{R}$ we recover the oscillating random walk.

Complexes of half-lines: Example



Here is an example with $\mathcal{S} = \mathcal{S}^{\text{one}} = \{1, 2, 3\}$.

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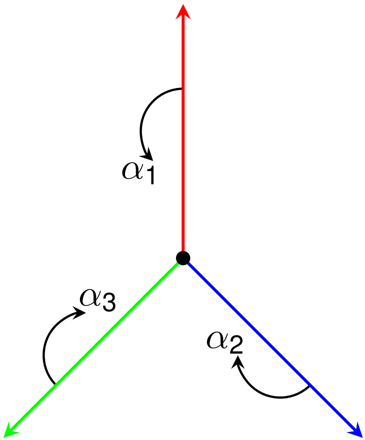

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Question: how does recurrence and transience of this walk depend on $\alpha_1, \alpha_2, \alpha_3$ and P?

Complexes of half-lines: Theorem

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Theorem

(a) Suppose that $\max_k \chi_k \alpha_k \geq 1$. *Then the walk is recurrent.*

(b) Suppose that $\max_k \chi_k \alpha_k < 1$.

- *If* $\sum_k \mu_k \cot(\chi_k \pi \alpha_k) < 0$, *then the walk is recurrent.*
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In the critical case $\sum_k \mu_k \cot(\chi_k \pi \alpha_k) = 0$ we have recurrence under slightly stronger conditions.

Complexes of half-lines: Theorem

This result includes as special cases the results on oscillating random walk described earlier.

For example, in the two-sided oscillating random walk, $\mu = (1/2, 1/2)$ and

$$\sum_k \mu_k \cot(\chi_k \pi \alpha_k) = \frac{1}{2}\cot(\pi\alpha/2) + \frac{1}{2}\cot(\pi\beta/2)$$
$$= \frac{\sin(\pi(\alpha+\beta)/2)}{2\sin(\pi\alpha/2)\sin(\pi\beta/2)},$$

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While in these special cases the critical surface in α_k is linear, in the general setting our cotangent criterion shows that the critical surface is in general non-linear.

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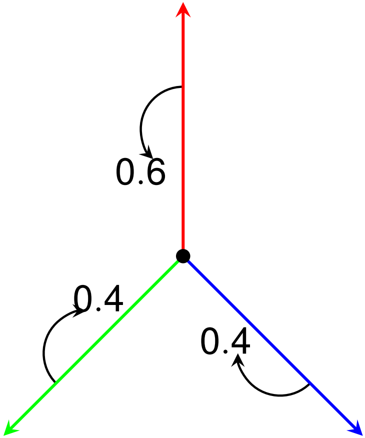

Suppose

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}.$$

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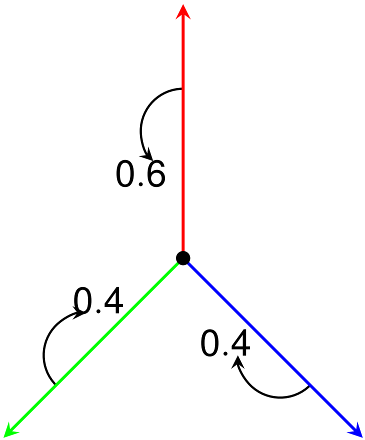

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$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\mu = \frac{1}{10}(4, 3, 3)$.

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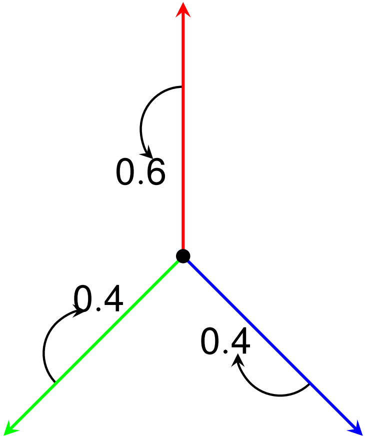

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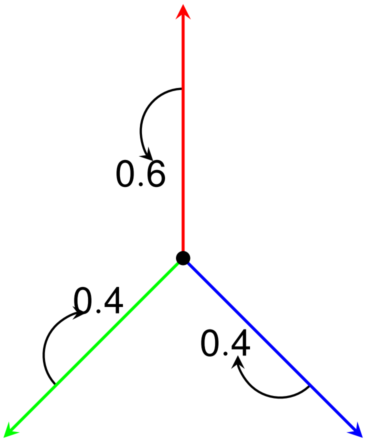

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$$\sum_k \mu_k \cot(\chi_k \pi \alpha_k) = 0.06,$$

so the random walk is transient.

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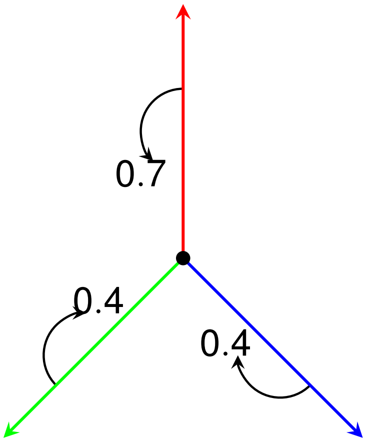

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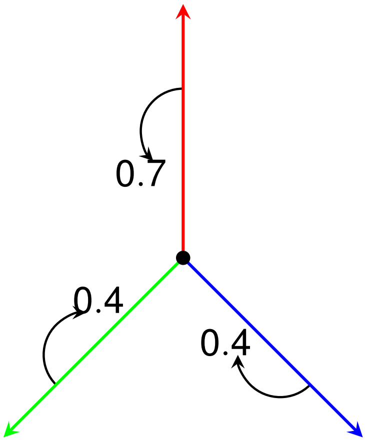

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Suppose $(\alpha_1, \alpha_2, \alpha_3) = (0.7, 0.4, 0.4)$.

Then

$$\sum_k \mu_k \cot(\chi_k \pi \alpha_k) = -0.10,$$

so the random walk is recurrent.

Outline

1 Introduction

2 Oscillating random walk

3 Complexes of half-lines

4 Ideas of the proofs

Ideas of the proofs

We use a Lyapunov function $f(x, i) = \lambda_i x^\nu$ where $\lambda_i \in \mathbb{R}$ are carefully chosen constants.

We show that for suitable choices of the λ_i $f(X_n, \xi_n)$ satisfies a local supermartingale condition

Ideas of the proofs

We use a Lyapunov function $f(x,i) = \lambda_i x^\nu$ where $\lambda_i \in \mathbb{R}$ are carefully chosen constants.

We show that for suitable choices of the λ_i $f(X_n, \xi_n)$ satisfies a local supermartingale condition, which if $\nu > 0$ establishes recurrence and if $\nu < 0$ establishes transience.

To obtain our sharp phase transition one takes $\nu \to 0$ and $\lambda \to 1$; the choice of λ_i depends on the μ_i and the α_i in a way captured by the cotangent criterion.

Why does cot appear?

When computing the expected increment of $f(X_n, \xi_n)$, one ends up with several integrals, such as

$$\int_x^\infty (y-x)^\nu y^{-1-\alpha} \mathrm{d}y = x^{\nu-\alpha} \int_1^\infty (u-1)^\nu u^{-1-\alpha} \mathrm{d}u.$$

By Taylor's theorem, as $\nu \to 0$,

$$\int_1^\infty (u-1)^\nu u^{-1-\alpha} \mathrm{d}u = \frac{1}{\alpha} + \nu \int_1^\infty \log(u-1) u^{-1-\alpha} \mathrm{d}u + o(\nu).$$

Here

$$\int_1^\infty \log(u-1) u^{-1-\alpha} \mathrm{d}u = -\frac{1}{\alpha}(\gamma + \psi(\alpha)),$$

where γ is Euler's constant and ψ is the digamma function. Together with similar expressions from other terms, the cotangent arises from the digamma reflection formula

$$\psi(1-z) - \psi(z) = \pi \cot \pi z.$$

References

- M. Campanino & D. Petritis, Random walks on randomly oriented lattices, *Markov Process. Related Fields* **9** (2003) 391–412.
- J.H.B. Kemperman, The oscillating random walk, *Stoch. Process. Appl.* **2** (1974) 1–29.
- D. Petritis, M.V. Menshikov & A.R. Wade, Heavy tailed random walks on complexes of half-lines, Preprint (2016).
- B.A. Rogozin & S.G. Foss, The recurrence of an oscillating random walk, *Theor. Probab. Appl.* **23** (1978) 155–162.
- N. Sandrić, Recurrence and transience property for a class of Markov chains, *Bernoulli* **19** (2013) 2167–2199.
- N. Sandrić, Recurrence and transience criteria for two cases of stable-like Markov chains, *J. Theoret. Probab.* **27** (2014) 754–788.
- N. Sandrić, Long-time behavior of stable-like processes, *Stoch. Process. Appl.* **123** (2013) 1276–1300.
- L.A. Shepp, Symmetric random walk, *Trans. Amer. Math. Soc.* **104** (1962) 144–153.