Phase transitions for random geometric preferential attachment graphs

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Introduction

Construct a sequence of random graphs $G_1, G_2, \dots$ with $G_n = (V_n, E_n)$, where $V_n = \{0, 1, \dots, n\}$.

Vertices will be associated with sites located in $S \subset \mathbb{R}^d$, for S convex, compact, and of positive measure.

Let $X_0, X_1, X_2, \dots$ be i.i.d. random variables on S with a common density f, bounded away from 0 and ∞.

Vertex i is associated with site $X_i \in S$.

Vertices arrive one at a time. Each new vertex after the first is joined by an edge to an existing vertex according to a probabilistic rule that mixes preferential attachment by degree and spatial attachment by proximity.

The model originates with MANNA & SEN (2002), FLAXMAN *et al.* (2006), and JORDAN (2010).

Introduction

Start with $G_1 = (\{0,1\},\{(1,0)\})$: a single edge from 1 to 0.
Given G_n, $n \geq 1$, and $\{X_0,\ldots,X_n\}$, construct G_{n+1} as follows.

Vertex $n+1$ arrives at site $X_{n+1} \in S$.

An edge is added from vertex $n+1$ to a random vertex among $\{0,1,\ldots,n\}$, where vertex k is chosen with probability proportional to

$$\deg_n(k) F(\rho(X_{n+1}, X_k)).$$

Here

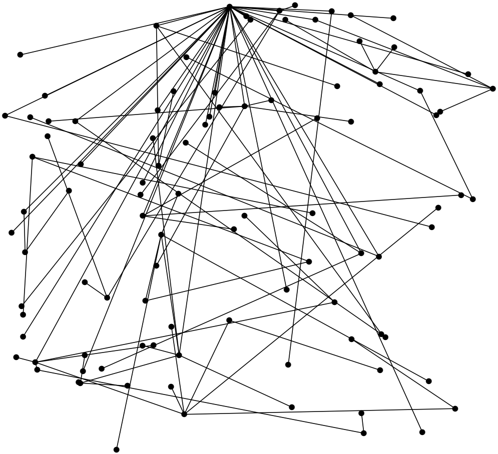
- $\deg_n(k)$ is the (total) degree of vertex k in G_n;
- ρ is the Euclidean distance on $\mathbb{R}^d$;
- $F : (0,\infty) \to (0,\infty)$ is an attractiveness function.

Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0,1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-1}$, $n = 100$.

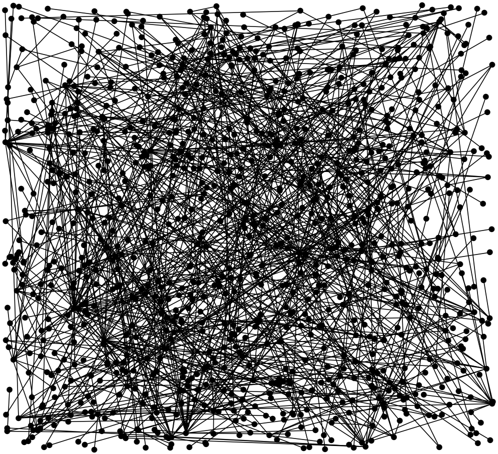


Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0,1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-1}$, $n = 1000$.

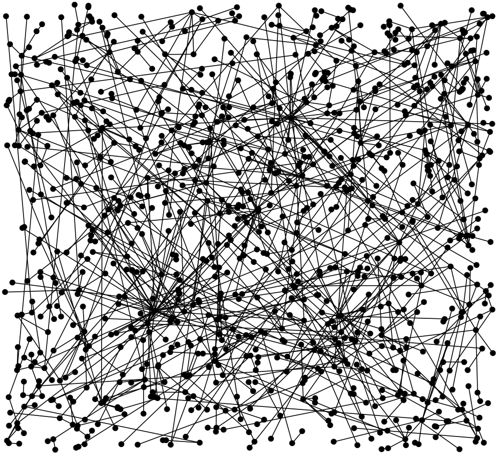


Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0, 1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-2}$, $n = 1000$.

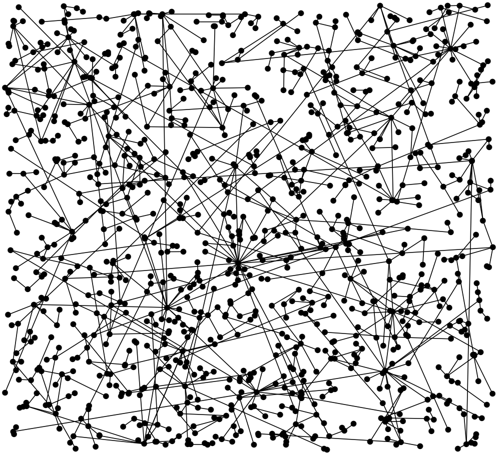


Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0,1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-3}$, $n = 1000$.

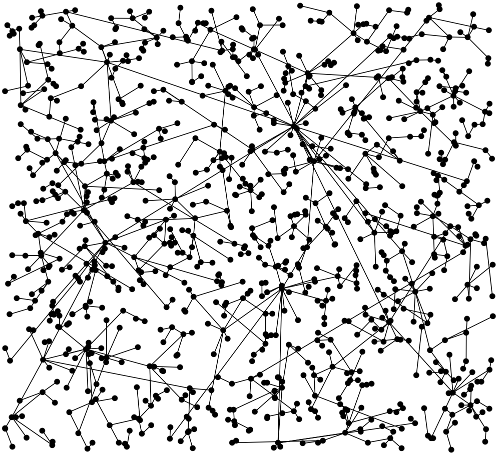


Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0, 1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-4}$, $n = 1000$.

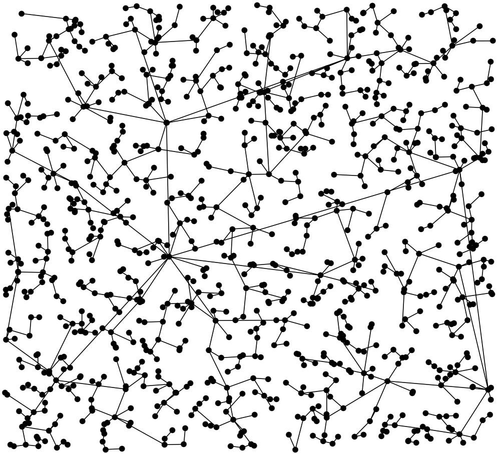


Simulations

Typically $F(r) \uparrow \infty$ as $r \downarrow 0$. The rate at which F blows up at 0 regulates the influence of the geometry.

Example: Consider $S = [0,1]^2$ and $f = \mathbf{1}_S$; uniform points on the unit square.

Simulation shown for $F(r) = r^{-10}$, $n = 1000$.



Power law degrees

We are interested in the degree distribution, for example

$$\pi_n(k) := \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{1}\{\deg_n(i) \geq k\},$$

the proportion of vertices in G_n whose degree is at least k.

For F relatively flat, we expect preferential attachment to dominate and power-law degrees.

Formalized by JORDAN (2010). Assume (S, f) homogeneous in the sense that $\int_{S \cap B(x;r)} f(y) \mathrm{d}y$ depends only on r:

Theorem (Jordan)

If $\int_S F(\rho(x,y)) f(y) dy < \infty$ then, for any k, $\lim_{n\to\infty} \pi_n(k) = \lambda_{\mathrm{BA}}(k)$ in L^2, where $\lambda_{\mathrm{BA}}(k) \asymp k^{-2}$.

For example, this applies for $F(r) = r^{-s}$, $s \in (0, d)$.

Phase transitions

We are interested in other regimes.

Our main results are:

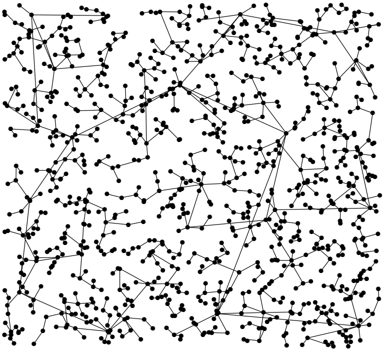
- If F blows up fast enough, π_n has exponential tails. Here 'fast enough' is faster than any polynomial.
- There is an intermediate regime in which π_n has stretched exponential tails.

In the first case, we show that the geometric component is so dominant that G_n is well approximated by the on-line nearest-neighbour graph (ONG), in which each new vertex is joined to its nearest neighbour among the existing vertices.

On-line nearest-neighbour graph

ONG studied by MANNA & SEN (2002) and PENROSE (2005).

Simulation shown for $S = [0,1]^2$, $f = \mathbf{1}_S$, and $n = 1000$.



Degree distribution for ONG in the case of $S = [0,1]^2$ and $f = \mathbf{1}_S$ studied by BERGER *et al.* (2007), who showed, for some constants A, A', C, C',

$$A' \mathrm{e}^{-C'k} \leq \liminf_{n\to\infty} \mathbb{E}\pi_n^{\mathrm{ONG}}(k) \leq \limsup_{n\to\infty} \mathbb{E}\pi_n^{\mathrm{ONG}}(k) \leq A\mathrm{e}^{-Ck}.$$

We extend this.

On-line nearest-neighbour graph

Theorem

For any d and f, for any k,

$$\lim_{n\to\infty} \pi_n^{\mathrm{ONG}}(k) \stackrel{a.s.}{=} \lim_{n\to\infty} \mathbb{E}\pi_n^{\mathrm{ONG}}(k) = \lambda_{\mathrm{ONG}}(k),$$

where λ_{ONG} depends on d but not on f and satisfies exponential bounds as a function of k.

Existence of the limit *in probability* follows from stabilization, e.g. PENROSE (2005,07); then use a concentration argument.

Conjecture

We expect that $\lim_{k\to\infty}\left(-k^{-1}\log\lambda_{\mathrm{ONG}}(k)\right) = \mu(d)$ exists in $(0,1]$ for each d, and we conjecture that $\lim_{d\to\infty}\mu(d) = \log 2$, the exponent in the simpler (non-spatial) uniform attachment model.

Exponential tail regime

Back to G_n. Take $F = F_\gamma$ given by

$$F_\gamma(r) := \exp\left\{(\log(1/r))^\gamma\right\}.$$

For $\gamma > 1$, this blows up faster than any r^{-s} as $r \downarrow 0$.

Theorem

Let $F = F_\gamma$, $\gamma > 3/2$. Then a.s. in G_n the proportion of vertices joined to a vertex other than their on-line nearest neighbour is $o(1)$, and, for any k,

$$\lim_{n\to\infty} \pi_n(k) \stackrel{L^1}{=} \lim_{n\to\infty} \mathbb{E}\pi_n(k) = \lambda_{\mathrm{ONG}}(k).$$

Conjecture

The conclusion holds for all $\gamma > 1$.

Intermediate regime

Take $F(r) = r^{-s}$ where now $s > d$. (Recall $s < d$ is the power-law case studied by Jordan.)

Theorem

Let $F(r) = r^{-s}$ for $s > d$. Then for any $\theta \in (0, \frac{s-d}{2s-d})$ there exists a constant $C < \infty$ such that, for any k,

$$\limsup_{n\to\infty} \pi_n(k) \leq C \exp\{-k^{\theta}\}, \ \textit{a.s.}$$

As $s \to \infty$, this gives a bound of almost $e^{-\sqrt{k}}$.

Question: is this sharp?

Intermediate regime

Outline of the proof:

- The denominator in the probability of the new edge being joined to a given vertex can be estimated in terms of a quantity of the form $n^{-s/d} \sum_{j=0}^{n-1} \rho(X_j, x)^{-s}$, which is dominated by a random variable in the domain of attraction of a stable law of index $d/s \in (0, 1)$.
- Hence we get an estimate for the probability of the new edge being joined to a given vertex in terms of the current degrees and a random quantity we can understand.
- Actually, we need an estimate conditional on the locations of the existing vertices, but this functional has nice concentration properties.
- We use these estimates and an inductive stochastic approximation argument to obtain the result.

References

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