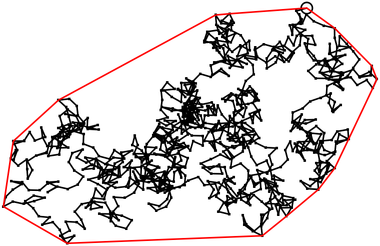
Convex hulls of planar random walks



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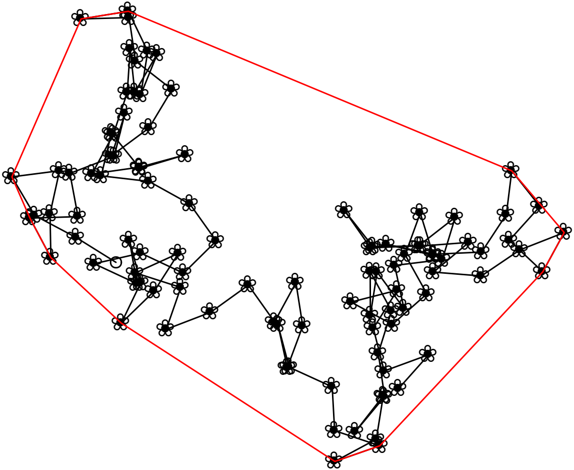
Joint work with Chang Xu, University of Strathclyde

Introduction

On each of n unsteady steps, a drunken gardener drops a seed. Once the flowers have bloomed, what is the minimum length of fencing needed to enclose the garden? What is its area?



Introduction

Let $Z_1, Z_2, \ldots$ be independent, identically distributed random vectors in $\mathbb{R}^2$.

The Z_k will be the increments of the planar random walk S_n, $n \geq 0$, started at the origin in $\mathbb{R}^2$, defined by

$$S_0 = 0, \quad \text{and} \quad S_n = \sum_{k=1}^{n} Z_k \quad \text{for } n \geq 1.$$

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We are interested in the convex hull $\operatorname{hull}(S_0, \ldots, S_n)$, i.e., the smallest convex set that contains $\{S_0, \ldots, S_n\}$.

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In particular, the $n \to \infty$ limit behaviour of the random variables

- $L_n =$ the perimeter length of $\operatorname{hull}(S_0, \ldots, S_n)$;
- $A_n =$ the area of $\operatorname{hull}(S_0, \ldots, S_n)$.

Introduction

Standing assumption: $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$ (sometimes more).

For the mean drift vector of the walk we write $\mu = \mathbb{E}\, Z_1$.

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For example, if

$$r_n := \inf\{\|x\| : x \in \mathbb{R}^2 \setminus \text{hull}(S_0, \ldots, S_n)\},$$

then, under mild conditions:

- the zero-drift walk visits all angles at arbitrary distances, so $r_n \to \infty$, i.e., the convex hull tends to the whole of $\mathbb{R}^2$;
- the walk with non-zero drift is transient in a limiting direction, so $\lim_{n\to\infty} r_n < \infty$.

Introduction

Some heuristics allow us to guess the typical scalings with n of our quantities of interest:

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We then seek distributional limit theorems. That is, for quantity Q_n is there a distributional limit for

$$n^{-\alpha} Q_n \quad \text{or} \quad n^{-\beta} \left(Q_n - \mathbb{E}\, Q_n \right) ?$$

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It turns out that each of the four cases in the table do satisfy a distributional limit theorem: one limit distribution is Gaussian; the other three are not.

Outline

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Some history

Spitzer & Widom (1961) and Baxter (1961) showed that

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So, under mild conditions:

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Snyder & Steele (1993) showed that

$$\frac{1}{n} \mathbb{V}\mathrm{ar}\,(L_n) \leq \frac{\pi^2}{2} \left(\mathbb{E}\, \|Z_1\|^2 - \|\mu\|^2 \right). \tag{1}$$

Snyder & Steele deduced from (1) the strong law

$$\lim_{n \to \infty} n^{-1} L_n = 2\|\mu\|, \text{ a.s.}$$

Some questions

The work of Snyder & Steele raised some natural questions.

- Is n the correct order for $\mathbb{V}\mathrm{ar}\,(L_n)$?
- Is there a distributional limit theorem for L_n?
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First, we introduce an important tool from convex geometry: Cauchy's formula.

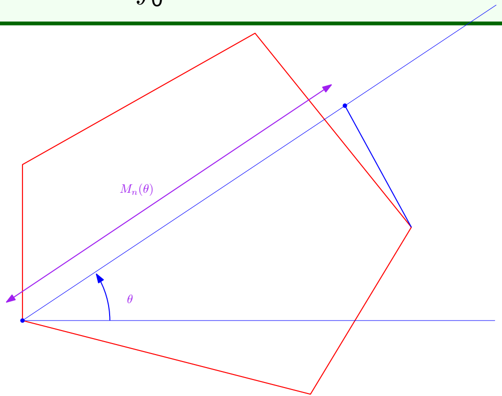
Cauchy's formula

Let $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$, unit vector in direction θ. Set

$$M_n(\theta) = \max_{0\le k\le n} (S_k \cdot \mathbf{e}_\theta), \quad m_n(\theta) = \min_{0\le k\le n} (S_k \cdot \mathbf{e}_\theta).$$

Cauchy's perimeter formula from convex geometry:

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\, \mathrm{d}\theta.$$



Cauchy's formula

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\, \mathrm{d}\theta.$$

A first consequence: classical fluctuation theory for random walk on $\mathbb{R}$ gives

$$\mathbb{E}\, M_n(\theta) = \sum_{k=1}^{n} k^{-1} \mathbb{E}\, [(S_k \cdot \mathbf{e}_\theta)^+],$$

a formula attributed variously to Kac, Hunt, Dyson, and Chung, and which can be proved combinatorially, or analytically as a consequence of the Spitzer–Baxter–Pollaczek fluctuation theory identities. Then

$$\mathbb{E}\, L_n = \sum_{k=1}^{n} k^{-1} \mathbb{E} \int_0^{2\pi} |S_k \cdot \mathbf{e}_\theta| \mathrm{d}\theta = 2 \sum_{k=1}^{n} k^{-1} \mathbb{E}\, \|S_k\|,$$

which is the Spitzer–Widom formula.

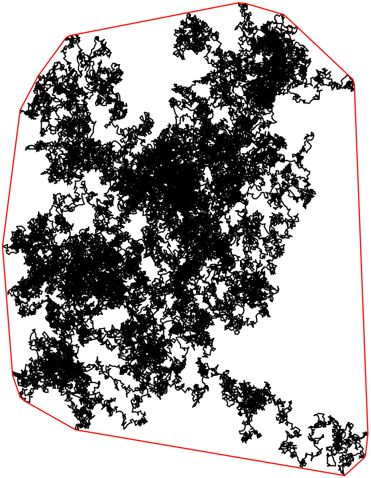
Outline

The zero drift case

Suppose $\mu = 0$. The random walk has Brownian motion as its scaling limit.

So one would expect that the convex hull of the random walk is described in the limit by the convex hull of Brownian motion. The latter was studied by Lévy; more recently by El Bachir (1983) and others.

We need to know a little about convex hulls of continuous paths, and need to set things up on the right space(s).



Paths and hulls

Consider continuous $f : [0, T] \to \mathbb{R}^d$ with $f(0) = 0$; say $f \in \mathcal{C}_d^0$.
(T is not very important—enough to take $T \equiv 1$.)

With the supremum norm $\rho_\infty(f, g) = \sup_x \|f(x) - g(x)\|$ we get a metric space $(\mathcal{C}_d^0, \rho_\infty)$.

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$\Longrightarrow$ hull $f[0, t]$ is compact (by a theorem of Carathéodory).

That is, hull $f[0, t]$ is an element of the metric space $(\mathcal{K}_d^0, \rho_H)$ of compact convex subsets of $\mathbb{R}^d$ containing 0, with the Hausdorff metric.

Paths and hulls

Metric space $(\mathcal{K}_d^0, \rho_H)$ of compact convex subsets of $\mathbb{R}^d$ containing 0, with the Hausdorff metric.

Given $K \in \mathcal{K}_d^0$ and $r > 0$, let $K^r := \{x \in \mathbb{R}^d : \rho(x, K) \leq r\}$.

For $A, B \in \mathcal{K}_d^0$,

$$\rho_H(A, B) \leq r \quad \Leftrightarrow \quad A \subseteq B^r \text{ and } B \subseteq A^r.$$

Lemma 1
For each t, the map $f \mapsto \mathrm{hull}\, f[0, t]$ is a continuous function from $(\mathcal{C}_d^0, \rho_\infty)$ to $(\mathcal{K}_d^0, \rho_H)$.

Scaling limit

Given random walk $S_n = \sum_{k=1}^{n} Z_k$ on $\mathbb{R}^d$, define

$$X_n(t) := n^{-1/2}\left(S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}\right).$$

So for each n, $X_n \in \mathcal{C}_d^0$. Let b_t, $t \geq 0$ denote standard Brownian motion on $\mathbb{R}^d$.

Donsker's Theorem

Suppose $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mu = 0$, *and* $\mathbb{E}\left(Z_1 Z_1^\top\right) = I$. *Then* $X_n \Rightarrow b$ *in the sense of weak convergence on* $(\mathcal{C}_d^0, \rho_\infty)$.

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Note hull $X_n[0,1] = n^{-1/2}$ hull$(S_0, \ldots, S_n)$. Then with Lemma 1 and the continuous mapping theorem, we get:

Theorem 2

Under the same conditions,

$n^{-1/2}$ *hull*$(S_0, \ldots, S_n) \Rightarrow$ *hull* $b[0,1]$

in the sense of weak convergence on $(\mathcal{K}_d^0, \rho_H)$.

Functionals

Now take $d = 2$.

The neatest way to define area $\mathcal{A}$ and perimeter length $\mathcal{L}$ of a set $K \in \mathcal{K}_2^0$ is:

$$\mathcal{A}(K) := |K|, \;\text{ and }\; \mathcal{L}(K) := \lim_{r \downarrow 0} \left(\frac{|K^r| - |K|}{r} \right),$$

where $| \cdot |$ is Lebesgue measure; the limit exists by the Steiner formula of integral geometry.

In particular,

$$\mathcal{L}(K) = \begin{cases} \mathcal{H}_1(\partial K) & \text{if } \operatorname{int}(K) \neq \emptyset \\ 2\mathcal{H}_1(\partial K) & \text{if } \operatorname{int}(K) = \emptyset \end{cases}$$

where $\mathcal{H}_1$ is one-dimensional Hausdorff measure.

Functionals

Lemma 3

The maps $K \mapsto \mathcal{A}(K)$ and $K \mapsto \mathcal{L}(K)$ are continuous functions from $(\mathcal{K}_2^0, \rho_H)$ to $(\mathbb{R}_+, \rho)$.

Note
$$\mathcal{L}(\text{hull}\, X_n[0,1]) = \mathcal{L}(n^{-1/2}\,\text{hull}(S_0, \ldots, S_n)) = n^{-1/2} L_n;$$
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Theorem 4

Suppose $\mathbb{E}(\|Z_1\|^2) < \infty$, $\mu = 0$, and $\mathbb{E}(Z_1 Z_1^\top) = I$. Then

$$n^{-1/2} L_n \xrightarrow{d} \ell_1, \quad \text{and} \quad n^{-1} A_n \xrightarrow{d} a_1,$$

where $\ell_1 = \mathcal{L}(\operatorname{hull} b[0,1])$ and $a_1 = \mathcal{A}(\operatorname{hull} b[0,1])$ are the perimeter length and area, respectively, of the convex hull of planar Brownian motion run for unit time.

Functionals

Some comments.

- The variables ℓ_1 and a_1 are positive and non-degenerate, and so non-Gaussian.

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- Under mild conditions, we also have that

$$n^{-1}\mathbb{V}\mathrm{ar}\,L_n \to \mathbb{V}\mathrm{ar}\,\ell_1, \quad \text{and} \quad n^{-2}\mathbb{V}\mathrm{ar}\,A_n \to \mathbb{V}\mathrm{ar}\,a_1.$$

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- We'd like an exact formula for these variances.
  Goldman (1996) manages to do an explicit computation for the planar Brownian bridge, but it is tricky.
  Using Cauchy's formula and a formula of Rogers & Shepp (2006) on the correlation of the maxima of correlated one-dimensional Brownian motions, one can show

$$\mathbb{E}\left[\ell_1^2\right] = 4\pi \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta \int_0^\infty \mathrm{d}u \cos\theta \frac{\cosh(u\theta)}{\sinh(u\pi/2)} \tanh\left(\frac{(2\theta+\pi)u}{4}\right),$$

which gives $\mathbb{V}\mathrm{ar}\,\ell_1 \approx 1.0350$.

Outline

Scaling limit in the case with drift

To get a non-degenerate scaling limit, we now must scale space by factor $1/n$ in the direction of the drift and by factor $1/\sqrt{n}$ in the orthogonal direction.

Specifically, let $\psi_n^\mu(x) = \left(\frac{x \cdot \hat{\mu}}{n\|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n\sigma^2_{\mu_\perp}}} \right)$;

Here $\sigma^2_{\mu_\perp} = \mathbb{E}\left[(Z \cdot \hat{\mu}_\perp)^2 \right]$.

Let $\tilde{b}$ denote the process on $\mathbb{R}^2$ given by $\tilde{b}(t) = (t, w(t))$, where w is standard Brownian motion on $\mathbb{R}$.

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The analogue of Donsker's theorem is as follows.

Suppose $\mathbb{E}(\|Z_1\|^2) < \infty$, $\mu \neq 0$, *and* $\sigma^2_{\mu_\perp} > 0$. *Then* $\psi_n^\mu(X_n) \Rightarrow \tilde{b}$ *weakly on* $(\mathcal{C}_2^0, \rho_\infty)$.

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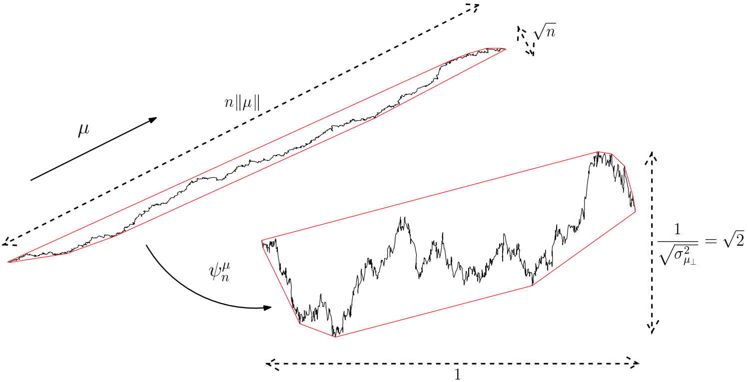

The affine map ψ_n^μ preserves the convex hull, so:

Theorem 5

Suppose $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mu \neq 0$, *and* $\sigma^2_{\mu_\perp} > 0$. *Then* $\psi_n^\mu(\mathit{hull}(S_0, \ldots, S_n)) \Rightarrow \mathit{hull}\,\tilde{b}[0,1]$ *weakly on* $(\mathcal{K}_2^0, \rho_H)$.

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Now note $\mathcal{A}(\psi^\mu_n(\mathrm{hull}(S_0, \ldots, S_n))) = n^{-3/2}\|\mu\|^{-1}(\sigma^2_{\mu_\perp})^{-1/2} A_n$.

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$$n^{-3/2}\|\mu\|^{-1}(\sigma^2_{\mu_\perp})^{-1/2}A_n \xrightarrow{d} \tilde{a}_1,$$

where $\tilde{a}_1 = \mathcal{A}(\text{hull}\,\tilde{b}[0,1])$.

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- We can show that $\mathbb{E}\,\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$, using an analogue of the Spitzer–Widom formula for areas due to Barndorff-Nielsen & Baxter (1963).
- Under mild conditions, we also have that

$$n^{-3}\mathbb{V}\mathrm{ar}\, A_n \to \|\mu\|^2 \sigma^2_{\mu_\perp} \mathbb{V}\mathrm{ar}\, \tilde{a}_1 .$$

- We can show $\mathbb{V}\mathrm{ar}\, \tilde{a}_1 > 0$.
- This scaling limit strategy is no use for the perimeter length L_n in the case $\mu \neq 0$, because ψ_n^μ does not act in a sensible way on lengths...

Outline

Perimeter length in the non-zero drift case

Theorem 7

Suppose $\mathbb{E}(\|Z_1\|^2) < \infty$ *and* $\mu \neq 0$. *Then*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{V}ar(L_n) = 4\sigma_\mu^2, \quad \text{where} \;\; \sigma_\mu^2 := \mathbb{E}\left[((Z_1 - \mu)\cdot \hat{\mu})^2\right].$$

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Theorem 8

Suppose $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mu \neq 0$, *and* $\sigma_\mu^2 > 0$. *Then*

$$\frac{L_n - \mathbb{E}\, L_n}{\sqrt{\mathbb{V}ar\, L_n}} \xrightarrow{d} \mathcal{N}(0,1), \quad \textit{and} \quad \frac{L_n - \mathbb{E}\, L_n}{\sqrt{4n\sigma_\mu^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

Remarks

(i) A little algebra shows $4\sigma_\mu^2 \leq 4(\mathbb{E}\,\|Z_1\|^2 - \|\mu\|^2)$.

Compare to the Snyder–Steele upper bound
$n^{-1}\mathbb{V}\mathrm{ar}\,(L_n) \leq \frac{\pi^2}{2}(\mathbb{E}\,\|Z_1\|^2 - \|\mu\|^2)$.

I.e., the constant in the Snyder–Steele upper bound is not sharp ($4 < \pi^2/2$).

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I.e., the constant in the Snyder–Steele upper bound is not sharp ($4 < \pi^2/2$).

(ii) $\sigma_\mu^2 = 0$ if and only if $Z_1 - \mu$ is a.s. orthogonal to μ.

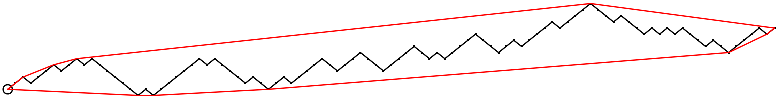
This is the case, for instance, if Z_1 takes values $(1, 1)$ or $(1, -1)$, each with probability $1/2$.

In this case Theorem 7 says that $\mathbb{V}\mathrm{ar}\,(L_n) = o(n)$.
The Snyder–Steele bound says only that $\mathbb{V}\mathrm{ar}\,(L_n) \leq \pi^2 n/2$.
Simulations suggest that actually $\mathbb{V}\mathrm{ar}\,(L_n) = O(\log n)$.

Degenerate example

Z_1 takes values $(1, 1)$ or $(1, -1)$, each with probability $1/2$.

This 2-dimensional walk can be viewed as a space-time diagram of a 1-dimensional simple symmetric random walk:



Interesting combinatorics here, related to the Bohnenblust–Spitzer algorithm; see Steele (2002).

Behaviour of L_n for this case is an open problem.

Proof idea: Martingale differences

Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Define $D_{n,i} = \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathcal{F}_i\right]$.

Lemma 9

(i) $L_n - \mathbb{E}\, L_n = \sum_{i=1}^{n} D_{n,i}$.
(ii) $\mathbb{V}ar(L_n) = \sum_{i=1}^{n} \mathbb{E}\left(D_{n,i}^2\right)$.

Sketch proof.

As Z_i' is independent of $\mathcal{F}_i$, $\mathbb{E}\left[L_n^{(i)} \mid \mathcal{F}_i\right] = \mathbb{E}\left[L_n^{(i)} \mid \mathcal{F}_{i-1}\right] = \mathbb{E}\left[L_n \mid \mathcal{F}_{i-1}\right]$.

So $D_{n,i} = \mathbb{E}\left[L_n \mid \mathcal{F}_i\right] - \mathbb{E}\left[L_n \mid \mathcal{F}_{i-1}\right]$; a standard construction of a martingale difference sequence.

$$\sum_{i=1}^{n} D_{n,i} = \mathbb{E}\left[L_n \mid \mathcal{F}_n\right] - \mathbb{E}\left[L_n \mid \mathcal{F}_0\right] = L_n - \mathbb{E}\, L_n.$$

Now use orthogonality of martingale differences. $\square$

Aside: Upper bounds

Lemma 9 with the conditional Jensen inequality gives:

$$\mathbb{V}\mathrm{ar}\,(L_n) \le \sum_{i=1}^{n} \mathbb{E}\,[(L_n - L_n^{(i)})^2].$$

A related result, a version due to Steele of the Efron–Stein inequality, says

$$\mathbb{V}\mathrm{ar}\,(L_n) \le \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\,[(L_n - L_n^{(i)})^2].$$

It is this latter result that Snyder & Steele used to obtain their upper bound.

Cauchy formula revisited

We need to study $D_{n,i} = \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathcal{F}_i\right]$.

We have the Cauchy formula for L_n, and similarly for $L_n^{(i)}$, so that

$$L_n - L_n^{(i)} = \int_0^\pi \Delta_{n,i}(\theta)\mathrm{d}\theta,$$

where

$$\Delta_{n,i}(\theta) = \left(M_n(\theta) - M_n^{(i)}(\theta)\right) - \left(m_n(\theta) - m_n^{(i)}(\theta)\right),$$

where, similarly to before,

$$M_n^{(i)}(\theta) = \max_{0\le k\le n}(S_k^{(i)} \cdot \mathbf{e}_\theta), \quad m_n^{(i)}(\theta) = \min_{0\le k\le n}(S_k^{(i)} \cdot \mathbf{e}_\theta).$$

Proof idea: Control of extrema

We want to understand the relationship between $M_n(\theta)$, $m_n(\theta)$ and $M_n^{(i)}(\theta)$, $m_n^{(i)}(\theta)$ (resampled versions).

WLOG suppose $\mathbb{E}\, Z_1 = \mu \mathbf{e}_{\pi/2} = (0, \mu)$, where $\mu > 0$.

Then for each fixed θ, $S_n \cdot \mathbf{e}_\theta$ is a one-dimensional random walk.

Indeed, $S_n \cdot \mathbf{e}_\theta = \sum_{k=1}^n Z_k \cdot \mathbf{e}_\theta$, with mean increment $\mathbb{E}\,[Z_1 \cdot \mathbf{e}_\theta] = \mathbb{E}\,[Z_1] \cdot \mathbf{e}_\theta = \mu \sin \theta$, which is positive for $\theta \in (0, \pi)$.

So, with high probability, the max $M_n(\theta)$ will be achieved nearby step n while the min $m_n(\theta)$ will be achieved nearby step 0.

To formalize this needs the strong law of large numbers, plus some care (need some uniformity in θ).

Proof idea: Control of extrema

Lemma 10

With high probability, $\Delta_{n,i}(\theta) \approx (Z_i - Z_i') \cdot \mathbf{e}_\theta$ for (almost) all i.

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So $m_n(\theta) = m_n^{(i)}(\theta)$, and

$$M_n^{(i)}(\theta) = M_n(\theta) + (Z_i' - Z_i) \cdot \mathbf{e}_\theta .$$

See the picture! □

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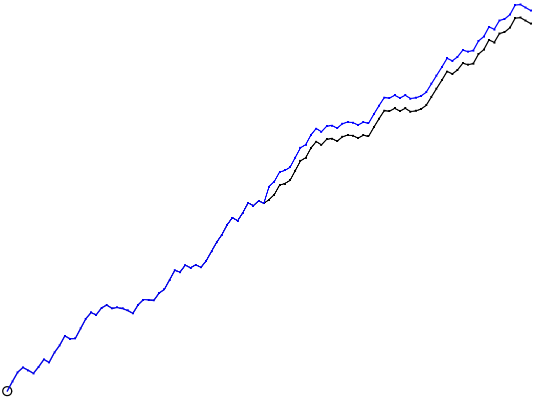


Finishing the proofs

Up to technical details, we sketched the fact that

$$D_{n,i} = \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathcal{F}_i\right] \approx \int_0^\pi \mathbb{E}\left[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i\right] \mathrm{d}\theta.$$

Here Z_i is $\mathcal{F}_i$-measurable and Z_i' is independent of $\mathcal{F}_i$, so

$$\mathbb{E}\left[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i\right] = (Z_i - \mu) \cdot \mathbf{e}_\theta.$$

Doing the integral gives

$$D_{n,i} = \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathcal{F}_i\right] \approx 2(Z_i - \mu) \cdot \hat{\mu}.$$

Finishing the proofs

Formalizing the analysis we get:

Theorem 11
Suppose $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$ *and* $\mu \neq 0$. *Then*

$$n^{-1/2}\left|L_n - \mathbb{E}\, L_n - 2\sum_{i=1}^{n}(Z_i - \mu)\cdot\hat{\mu}\right| \to 0, \text{ in } L^2.$$

So, perhaps surprisingly, $L_n - \mathbb{E}\, L_n$ is well-approximated by a sum of i.i.d. random variables.

Theorems 7 and 8 now follow from Theorem 11 easily enough.

Outline

Concluding remarks

The assumption that the Z_i are identically distributed is not essential to the main argument.

For example, let $G_n = \frac{1}{n+1} \sum_{i=0}^n S_i = \sum_{i=1}^n \frac{n+1-i}{n+1} Z_i$.

$G_0, G_1, \ldots$ is the centre-of-mass process associated with $S_0, S_1, \ldots$.

By convexity, $\text{hull}(G_0, \ldots, G_n) \subseteq \text{hull}(S_0, \ldots, S_n)$.

If L_n^{G} is the perimeter length of $\text{hull}(G_0, \ldots, G_n)$, then the statement of Theorem 11 applies to L_n^{G} in place of L_n with $\frac{n+1-i}{n+1} Z_i$ in place of Z_i.

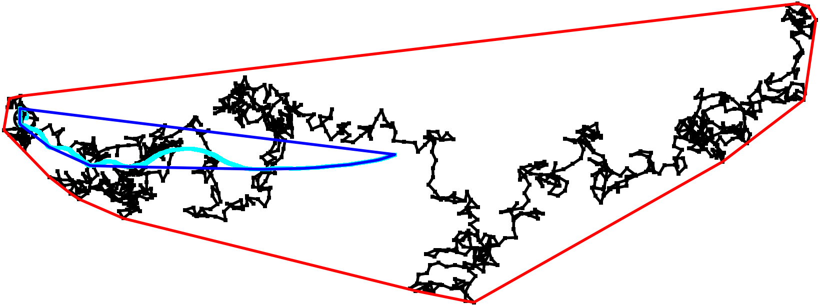
In particular, the analogue of Theorem 7 says that

$$\lim_{n \to \infty} \frac{1}{n} \mathbb{V}\mathrm{ar}\,(L_n^{\mathrm{G}}) = 4\sigma_\mu^2/3,$$

where σ_μ^2 is the same as before.

Concluding remarks

A picture:



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