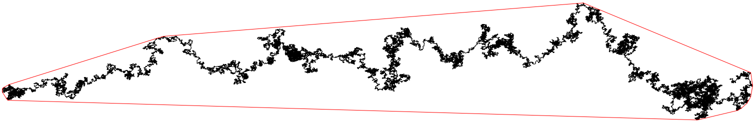
Iterated-logarithm laws for convex hulls of random walks with drift



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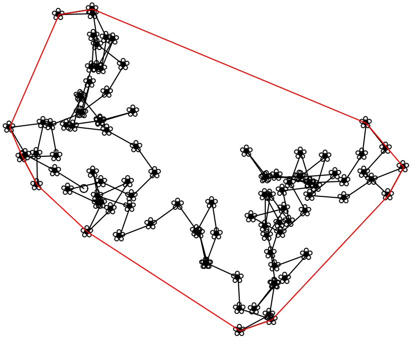
Introduction

On each of n unsteady steps, a drunken gardener drops a seed. Once the flowers have bloomed, what is the area of the garden enclosed by the minimal-length fence?



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Introduction

Let $Z, Z_1, Z_2, \ldots \in \mathbb{R}^d$ ($d \geq 2$) be independent and identically distributed.

The Z_k will be the increments of the random walk S_n, $n \geq 0$, started at the origin 0 in $\mathbb{R}^d$, defined by

$$S_0 = 0, \quad \text{and} \quad S_n = \sum_{k=1}^{n} Z_k \quad \text{for } n \geq 1.$$

We are interested in the convex hull

$$\mathcal{H}_n := \text{hull}\{S_0, \ldots, S_n\},$$

i.e., the smallest convex set that contains $\{S_0, \ldots, S_n\}$.

In particular, the $n \to \infty$ limit behaviour of the random variables

- $V_d(\mathcal{H}_n) =$ the volume of $\mathcal{H}_n$;
- $D(\mathcal{H}_n) =$ the diameter of $\mathcal{H}_n$;
- other intrinsic volumes.

Outline

Drift: zero vs. non-zero

Standing assumption: $\mathbb{E}\,\|Z\| \in (0, \infty)$.

For the mean drift vector of the walk we write $\mu = \mathbb{E}\,Z$.

There is going to be a clear distinction between the zero drift case ($\mu = 0$) and the non-zero drift case ($\mu \neq 0$).

For a qualitative result, observe that $\mathcal{H}_\infty := \cup_{n \geq 0} \mathcal{H}_n$ exists (by monotonicity) and $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) \in \{0, 1\}$ (by Hewitt–Savage zero–one law).

Theorem (López Hernández, W., 2021).
We have $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 1$ *if* $\mu = 0$ *and* $\mathbb{P}(\mathcal{H}_\infty = \mathbb{R}^d) = 0$ *if* $\mu \neq 0$.

Law of large numbers

View $\mathcal{H}_n$ as a sequence in the metric space of convex, compact subsets of $\mathbb{R}^d$ containing 0, with Hausdorff metric. Let $\ell_\mu := \text{hull}\{0, \mu\}$, the line segment from 0 to μ.

A consequence of the strong law of large numbers plus continuity:

Proposition (cf. Lo, McRedmond, Wallace, 2018).
As $n \to \infty$, $n^{-1}\mathcal{H}_n \to \ell_\mu$, a.s.

In non-zero drift case, this tells us the first-order asymptotic shape of convex hull, and (by continuity) implies that, e.g.,

$$\lim_{n\to\infty} n^{-1} D(\mathcal{H}_n) = \|\mu\|, \text{ and } \lim_{n\to\infty} n^{-d} V_d(\mathcal{H}_n) = 0, \text{ a.s.}$$

Zero-drift case

When $\mu = 0$, the strong laws says only $n^{-1}\mathcal{H}_n \to \{0\}$, a.s.

New standing assumption: $\mathbb{E}(\|Z\|^2) \in (0, \infty)$.

Let $\Sigma := \mathbb{E}(ZZ^\top)$ denote the increment covariance matrix.

A consequence of Donsker's theorem plus continuity:

Proposition (cf. W., Xu, 2015; Lo, McRedmond, Wallace, 2018).
Suppose that $\mu = 0$. For $b : [0,1] \to \mathbb{R}^d$ the trajectory of a standard Brownian motion, $n^{-1/2}\mathcal{H}_n \xrightarrow{\mathrm{d}} \Sigma^{1/2} \operatorname{hull} b[0,1]$.

A consequence is that (for $\Sigma =$ identity, say)

$$n^{-1/2} D(\mathcal{H}_n) \xrightarrow{\mathrm{d}} \operatorname{diam} b[0,1], \text{ and } n^{-d/2} V_d(\mathcal{H}_n) \xrightarrow{\mathrm{d}} V_d(\operatorname{hull} b[0,1]).$$

For $d = 2$, the expected area of the Brownian convex hull is $\mathbb{E}\, V_2(\operatorname{hull} b[0,1]) = \pi/2$ (El Bachir, 1983). We don't know the expected diameter (cf. McRedmond, Xu, 2017).

Scaling limit in the case with drift

How to go beyond law of large numbers when $\mu \neq 0$? To get a non-degenerate scaling limit, we now must scale space by factor $1/n$ in the direction of the drift and by factor $1/\sqrt{n}$ in the orthogonal directions.

Take $d = 2$ so we can draw a picture.

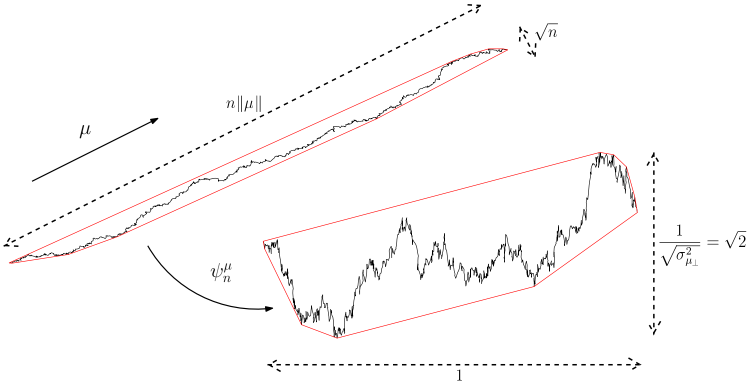
$$\text{Then, let } \ \varphi_n^\mu(x) = \left(\frac{x \cdot \hat{\mu}}{n\|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n\sigma^2_{\mu_\perp}}} \right);$$

$$\text{Here } \sigma^2_{\mu_\perp} = \mathbb{E}\left[\left(Z \cdot \hat{\mu}_\perp \right)^2 \right].$$

Let $\tilde{b}$ denote the process on $\mathbb{R}^2$ given by $\tilde{b}(t) = (t, w(t))$, where w is standard Brownian motion on $\mathbb{R}$.

The analogue of Donsker's theorem says that $\varphi_n^\mu(X_n)$ converges weakly to $\tilde{b}$ as $n \to \infty$; proof combines the functional LLN and CLT (cf. W. & Xu, 2015).

Scaling limit in the case with drift



The affine map φ_n^μ preserves the convex hull, so:

Theorem (W. & Xu, 2015).
If $\mu \neq 0$ *and* $\sigma^2_{\mu_\perp} > 0$*, then as* $n \to \infty$*,* $\varphi_n^\mu(\mathcal{H}_n)$ *converges weakly to* hull $\tilde{b}[0,1]$.

Scaling limit in the case with drift

By continuity and scaling of volumes (one coordinate by the LLN scaling n, the other $d-1$ coordinates by the CLT scaling $\sqrt{n}$) this leads to distributional limit for volumes:

Corollary (W. & Xu, 2015; McRedmond, 2019).
Suppose that $\mu \neq 0$ and $\sigma^2_{\mu_\perp} > 0$. Then, as $n \to \infty$,

$$n^{-(d+1)/2} \|\mu\|^{-1} (\sigma^2_{\mu_\perp})^{-1/2} V_d(\mathcal{H}_n) \xrightarrow{\mathrm{d}} V_d(\operatorname{hull} \tilde{b}[0,1]).$$

W. & Xu (2015) show that, when $d = 2$, $\mathbb{E}\, V_2(\operatorname{hull} \tilde{b}[0,1]) = \frac{1}{3}\sqrt{2\pi}$.

This scaling limit strategy does not work so nicely for diameter or perimeter length when $\mu \neq 0$, because φ_n^μ does not act in a sensible way on lengths. This leads to another story (and a different class of limit phenomena): W. & Xu (2015) for perimeter, McRedmond & W. (2018) for diameter.

Outline

Iterated-logarithm laws: Overview

We want to study a.s. behaviour of upper envelope of e.g. $V_d(\mathcal{H}_n)$: we seek appropriate versions of the law of the iterated logarithm (LIL) from classical fluctuation theory.

In the zero-drift case, the answer is an elegant theorem due to KHOSHNEVISAN (1992), using STRASSEN'S (1964) functional LIL. For example, when $d = 2$, $\mu = 0$, and $\Sigma = I$ (identity), Khoshnevisan shows that area satisfies

$$\limsup_{n\to\infty} \frac{V_2(\mathcal{H}_n)}{n \log \log n} = \frac{1}{\pi}, \text{ a.s.}$$

The constant $1/\pi$ arises from solving a variational problem (this is typical for a Strassen-type argument).

The analogue of this result for Brownian motion had already been obtained in a formidable paper of LÉVY (1955), who anticipated to some extent the functional LIL of STRASSEN (1964).

Iterated-logarithm laws: Overview

Khoshnevisan (1992): when $d = 2$, $\mu = 0$, and $\Sigma = I$,

$$\limsup_{n\to\infty} \frac{V_2(\mathcal{H}_n)}{n \log\log n} = \frac{1}{\pi}, \text{ a.s.}$$

In the non-zero drift case, Khoshnevisan's LIL does not apply. We obtain:

Theorem (Cygan, Sandrić, Šebek, W., 2023).
If $d = 2$, $\mu \neq 0$, and $\Sigma = I$,

$$\limsup_{n\to\infty} \frac{V_2(\mathcal{H}_n)}{n^{3/2}\sqrt{\log\log n}} = \frac{\|\mu\|}{\sqrt{6}}, \text{ a.s.}$$

Our general result covers all intrinsic volumes and (like Khoshnevisan's) is founded on Strassen's functional LIL, modified appropriately to apply to walks with non-zero drift; in our setting, as in Khoshnevisan's, limiting constants can often be characterized by variational problems, but in only a limited number of instances is the solution known.

Strassen's theorem

Let $\mathcal{C}_d$ denote the set of continuous $f : [0, 1] \to \mathbb{R}^d$, and let $\mathcal{C}_d^0$ denote the subset of those $f \in \mathcal{C}_d^0$ for which $f(0) = 0$. Define the linearly-interpolated random walk trajectory

$$Y_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}, \text{ for } t \in [0, 1].$$

Then $Y_n \in \mathcal{C}_d^0$ for every $n \in \mathbb{Z}_+$. The KHINCHIN scaling function for the classical LIL is

$$\ell(n) := \sqrt{2n \log \log n} \text{ for } n \geq 3.$$

The symmetric, non-negative definite matrix Σ has a unique symmetric, non-negative definite square-root $\Sigma^{1/2}$, which acts as a linear transformation of $\mathbb{R}^d$.

Strassen's theorem is a statement about the a.s. limit points of the sequence $Y_n/\ell(n)$ in the metric space $\mathcal{C}_d^0$ (endowed with the supremum metric).

Strassen's theorem

Theorem (Strassen's theorem for random walk).

Let $d \in \mathbb{N}$ and $\mu = 0$. With probability 1, the sequence $Y_n/\ell(n)$ in $\mathcal{C}_d^0$ is relatively compact, and its set of limit points is $\Sigma^{1/2} U_d$.

Here

$$U_d := \left\{ \text{a.c. } f : f(0) = 0, \int_0^1 \|f'(s)\|^2 \mathrm{d}s \leq 1 \right\}$$

is unit ball in Cameron–Martin space for the Wiener measure, and f' is componentwise derivative.

In words, the theorem states that, a.s., (a) every subsequence of $Y_n/\ell(n)$ contains a further subsequence that converges, its limit being some $f \in \Sigma^{1/2} U_d$, and (b) for every $f \in \Sigma^{1/2} U_d$, there is a subsequence of $Y_n/\ell(n)$ that converges to f.

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Example: among $f \in U_d$, maximum $f(1) = 1$ achieved by $f(s) \equiv s$; so corollary to Strassen's theorem is the classical LIL: for $\Sigma = I$,
$$\limsup_{n \to \infty} \frac{S_n}{\ell(n)} = 1, \text{ a.s.}$$

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Example: also yields the extension that for $\Sigma = I$,

$$\liminf_{n \to \infty} \left| \frac{S_n}{\ell(n)} - \theta \right| = 0, \text{ a.s., if and only if } \theta \in [-1, 1].$$

A Strassen theorem for non-zero drift

Idea: Use different scalings, like in the W. & Xu weak convergence result; this time LLN scaling in drift direction, LIL scaling in the rest.

WLOG, choose coordinates so that the standard orthonormal basis $(e_1, \dots e_d)$ of $\mathbb{R}^d$, $d \geq 2$, has e_1 in the direction of μ.

Let $\Sigma_{\mu^\perp}$ denote the matrix obtained from Σ by omitting the first row and column (reduced covariance matrix).

For $n \in \mathbb{N}$, define $\psi_n^\mu : \mathbb{R}^d \to \mathbb{R}^d$, acting on $x = (x_1, \dots, x_d)$, by

$$\psi_n^\mu(x_1, \dots, x_d) = \left(\frac{x_1}{n}, \frac{x_2}{\ell(n)}, \dots, \frac{x_d}{\ell(n)} \right).$$

Let $I_\mu : [0,1] \to \mathbb{R}_+$ denote the function $I_\mu(t) = \|\mu\| t$, and set

$$W_{d,\mu,\Sigma} := \{ g = (I_\mu, \Sigma_{\mu^\perp}^{1/2} f) : f \in U_{d-1} \}, \text{ for } d \geq 2.$$

A Strassen theorem for non-zero drift

Theorem (Cygan, Sandrić, Šebek, W., 2023).

Suppose that $d \geq 2$ and $\mu \neq 0$. With probability 1, the sequence $\psi_n^\mu(Y_n)$ in $\mathcal{C}_d^0$ is relatively compact, and its set of limit points is $W_{d,\mu,\Sigma}$.

Proof.

Combine the strong LLN (in functional form) for the first component, with Strassen's LIL for the remaining $d-1$ components. □

Corollary.

Suppose that $d \geq 2$ and $\mu \neq 0$. Let G be a real-valued, continuous function on compact, convex sets. Then

$$\limsup_{n \to \infty} G(\psi_n^\mu(\mathcal{H}_n)) = \sup_{g \in W_{d,\mu,\Sigma}} G(\operatorname{hull} g[0,1]), \text{ a.s.}$$

Note: Not necessarily immediate to use, because of the involved nature of the ψ_n^μ map.

Application to volumes

Theorem (Cygan, Sandrić, Šebek, W., 2023).
Suppose that $d \geq 2$ and $\mu \neq 0$. Then, a.s.,

$$\limsup_{n\to\infty} \frac{V_d(\mathcal{H}_n)}{\sqrt{2^{d-1} n^{d+1} (\log\log n)^{d-1}}} = \|\mu\| \cdot \sqrt{\det \Sigma_{\mu^\perp}} \cdot \lambda_d,$$

where

$$\lambda_d := \sup_{f \in U_{d-1}} V_d(\mathrm{hull}\{(t, f(t)); t \in [0,1]\}).$$

Theorem (Cygan, Sandrić, Šebek, W., 2023).
When $d = 2$, the constant takes value $\lambda_2 = \sqrt{3}/6$.

Together, these results give the LIL for area of the planar convex hull stated earlier.

Application to volumes

Proof of first Theorem.

By the scaling property of volumes, for $n \in \mathbb{N}$,

$$V_d(\psi_n^\mu(\mathcal{H}_n)) = \frac{V_d(\mathcal{H}_n)}{n\ell(n)^{d-1}}.$$

Applying the Corollary with $G = V_d$, we get

$$\limsup_{n\to\infty} \frac{V_d(\mathcal{H}_n)}{\sqrt{2^{d-1}n^{d+1}(\log\log n)^{d-1}}} = \sup_{g\in W_{d,\mu,\Sigma}} V_d(\operatorname{hull} g[0,1]).$$

Now $g \in W_{d,\mu,\Sigma}$ has $g = (I_\mu, \Sigma_{\mu^\perp}^{1/2} f)$ for some $f \in U_{d-1}$, and, by scaling, if $g_0 := (I_{e_1}, f)$,

$$\begin{aligned} V_d(\operatorname{hull} g[0,1]) &= \|\mu\| \cdot \det \Sigma_{\mu^\perp}^{1/2} \cdot V_d(\operatorname{hull} g_0[0,1]) \\ &= \|\mu\| \cdot \sqrt{\det \Sigma_{\mu^\perp}} \cdot V_d(\operatorname{hull}\{(t, f(t)); t \in [0,1]\}). \end{aligned}$$

□

General intrinsic volumes

For $k \in \{1, \ldots, d\}$, let $V_k(\mathcal{H}_n)$ denote the kth intrinsic volume of $\mathcal{H}_n$. (V_d = volume, $V_{d-1} \approx$ surface area, etc.)

Theorem (Cygan, Sandrić, Šebek, W., 2023).

Suppose that $d \geq 2$ and $\mu \neq 0$. Let $k \in \{1, 2, \ldots, d\}$. Then there exists a constant $\Lambda \in (0, \infty)$, depending on d, k, and the law of Z, such that, a.s.,

$$\limsup_{n \to \infty} \frac{V_k(\mathcal{H}_n)}{\sqrt{2^{k-1} n^{k+1} (\log \log n)^{k-1}}} = \Lambda.$$

- Case $k = d$ is the LIL for volumes. For other k, V_k does not scale so nicely through ψ_n^μ, so the proof is less direct, and the constant less explicit.
- Proof uses some further ingredients, including a zero–one law.

Outline

The planar constant: isoperimetric problem

We turn back to:

Theorem (Cygan, Sandrić, Šebek, W., 2023).
When $d = 2$, the constant in the LIL for area is $\lambda_2 = \sqrt{3}/6$.

Recall that λ_2 was characterized via

$$\lambda_2 = \sup_{f \in U_1} V_2(\mathrm{hull}\{(t, f(t)); t \in [0,1]\}),$$

where U_1 was the Strassen ball, i.e., a.c. $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$ and

$$\Gamma(f) := \int_0^1 f'(s)^2 \mathrm{d}s \leq 1.$$

Denoting by $\bar{f}$, $\underline{f}$ the least concave majorant and greatest convex minorant, respectively, of f, we can write

$$V_2(\mathrm{hull}\{(t, f(t)); t \in [0,1]\}) = A(f) := \int_0^1 \left(\bar{f}(s) - \underline{f}(s)\right) \mathrm{d}s.$$

The planar constant: isoperimetric problem

We can express the variational problem to identify λ_2 as

$$\text{maximize } A(f) \text{ subject to } \Gamma(f) \leq 1,$$

where $f(0) = 0$ and

$$\Gamma(f) = \int_0^1 f'(s)^2 \mathrm{d}s; \quad A(f) = \int_0^1 \left(\bar{f}(s) - \underline{f}(s) \right) \mathrm{d}s.$$

Theorem (Cygan, Sandrić, Šebek, W., 2023).

The optimal f is $f = f^\star$ given by

$$f^\star(u) = \sqrt{3}u(1-u), \text{ for } 0 \leq u \leq 1,$$

which has $\Gamma(f^\star) = 1$ and $A(f^\star) = \sqrt{3}/6$.

We sketch the proof.

The planar constant: isoperimetric problem

Three important reductions:

- Suffices to work with bridges, $f(0) = f(1) = 0$.
  Easy: a calculation shows the bridge $\hat{f}$ given by $\hat{f}(s) := f(s) - sf(1)$ has $A(\hat{f}) = A(f)$ and $\Gamma(\hat{f}) \leq \Gamma(f)$.
- Suffices to work with positive bridges, $f(s) > 0$ for $s \in (0, 1)$.
  Not so easy: proof uses symmetrization.
- Suffices to work with concave positive bridges.
  Easy: replace positive bridge by its concave majorant to decrease Γ.

Problem then reduces to

$$\text{maximize } \int_0^1 f(s)\mathrm{d}s \text{ subject to } \Gamma(f) \leq 1,$$

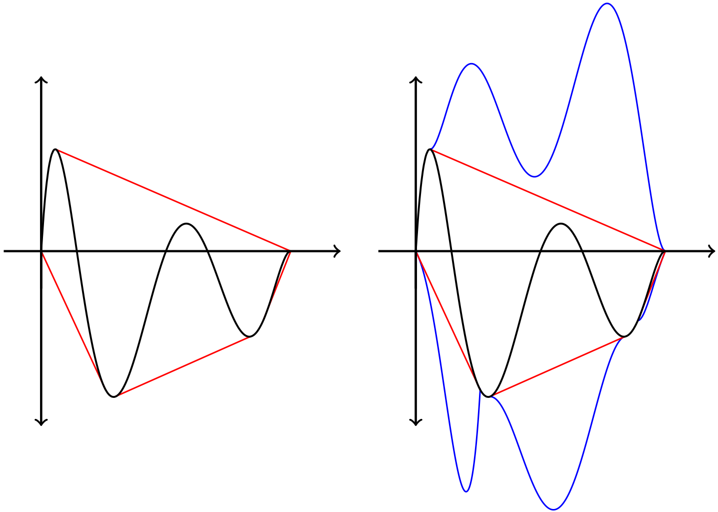
and to show that optimal f is $f = f^\star$ given above.

This is a "Cameron–Martin" or "Strassen" version of the Dido problem of antiquity to find maximal enclosed area for a curve of given arc length; here arc length is replaced by Strassen cost Γ. Adjacent results by Schmidt (1940).

The planar constant: isoperimetric problem

Proposition.

For every bridge f, there is a positive bridge f^{s} (produced by symmetrization) for which $\Gamma(f^{\mathrm{s}}) = \Gamma(f)$ and $A(f^{\mathrm{s}}) \geq A(f)$.



Outline

Concluding remarks

- The LIL is closely related to large deviations. For planar random walks with Gaussian increments, some recent results are given by Akopyan & Vysotsky (2021).

- The infinite-variance, multidimensional case (when the random walk is in the domain of attraction of a d-dimensional stable law), distributional limit theory recently studied by Cygan, Sandrić, Šebek (2022). LIL-type behaviour still open.

- As hinted earlier, some functionals fall into a different class of limit theorems, e.g. perimeter in case $\mu \neq 0$ satisfies a CLT (W., Xu, 2015) and we would expect a LIL there, too, but existing approaches do not apply.

Thank you!

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