Hypercycles in sparse random hypergraphs

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April 2015

Joint work with
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Outline

Outline

Introduction

Create an $m \times n$ matrix $M := M(n, m)$ over GF[2] by generating m i.i.d. rows each with n entries of 0s and 1s.

Each row has weight (number of 1s) independently distributed as some random variable $W_n \in \{1, 2, 3, \ldots, n\}$.

Given its weight, the row is chosen uniformly over all possibilities in $\{0, 1\}^n$ with that many 1s.

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Given its weight, the row is chosen uniformly over all possibilities in $\{0, 1\}^n$ with that many 1s.

A left null vector for M over GF[2] is a row vector $a \in \{0, 1\}^m$ such that $aM \equiv \mathbf{0}_n \bmod 2$, where $\mathbf{0}_n$ is the row vector of n zeros. Trivially, $\mathbf{0}_m$ is always a null vector.

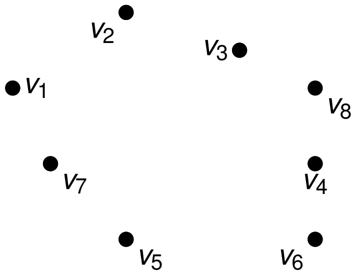
A non-trivial null vector corresponds to a non-empty subset of the row labels $\{1, \ldots, m\}$ such that the sum over the corresponding rows has no odd entries.

Introduction

Matrix

$$
\begin{matrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{matrix}
$$

Hypergraph

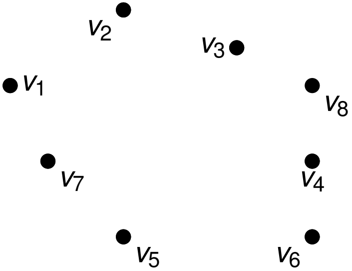


Introduction

Matrix

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
1	1	1	0	0	0	0	0
0	1	1	0	0	0	0	0
0	0	1	0	1	0	1	0
1	0	0	0	0	0	1	0
0	1	0	0	1	0	0	0
0	1	0	0	0	0	1	0
1	0	0	0	1	0	1	0

Hypergraph

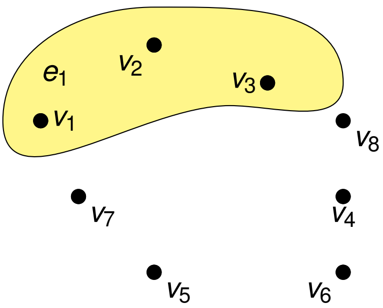


Introduction

Matrix

$$
\begin{array}{c|cccccccc}
 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 \\
\hline
e_1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{array}
$$

Hypergraph

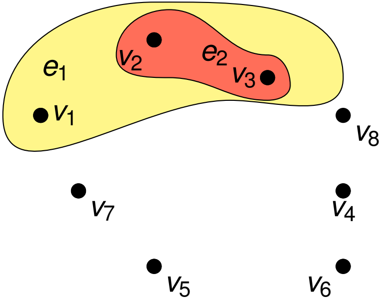


Introduction

Matrix

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
e_1	1	1	1	0	0	0	0	0
e_2	0	1	1	0	0	0	0	0
	0	0	1	0	1	0	1	0
	1	0	0	0	0	0	1	0
	0	1	0	0	1	0	0	0
	0	1	0	0	0	0	1	0
	1	0	0	0	1	0	1	0

Hypergraph



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\hline
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e_2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
e_3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
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\end{array}
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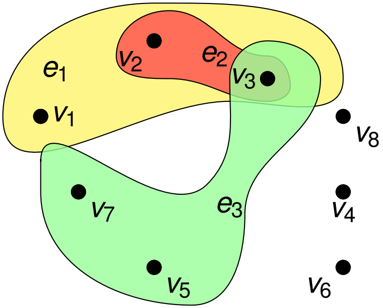

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	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
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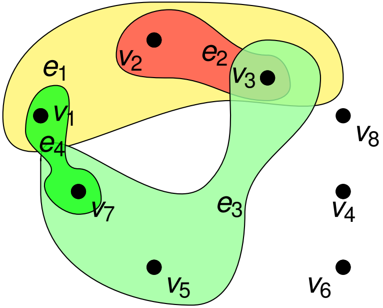
e_2	0	1	1	0	0	0	0	0
e_3	0	0	1	0	1	0	1	0
e_4	1	0	0	0	0	0	1	0
	0	1	0	0	1	0	0	0
	0	1	0	0	0	0	1	0
	1	0	0	0	1	0	1	0

Hypergraph



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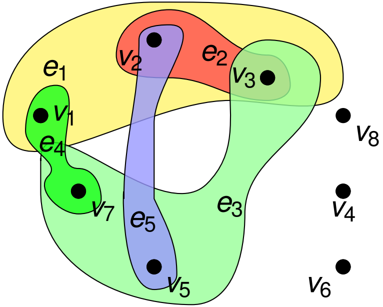
e_2	0	1	1	0	0	0	0	0
e_3	0	0	1	0	1	0	1	0
e_4	1	0	0	0	0	0	1	0
e_5	0	1	0	0	1	0	0	0
	0	1	0	0	0	0	1	0
	1	0	0	0	1	0	1	0

Hypergraph



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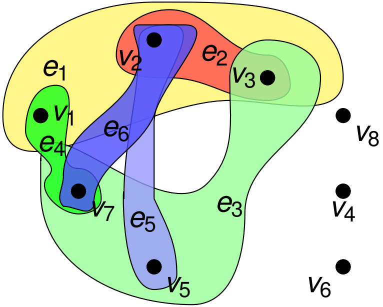
e_3	0	0	1	0	1	0	1	0
e_4	1	0	0	0	0	0	1	0
e_5	0	1	0	0	1	0	0	0
e_6	0	1	0	0	0	0	1	0
	1	0	0	0	1	0	1	0

Hypergraph



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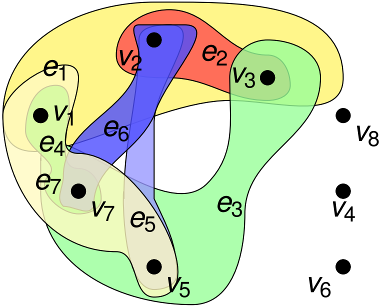
e_3	0	0	1	0	1	0	1	0
e_4	1	0	0	0	0	0	1	0
e_5	0	1	0	0	1	0	0	0
e_6	0	1	0	0	0	0	1	0
e_7	1	0	0	0	1	0	1	0

Hypergraph



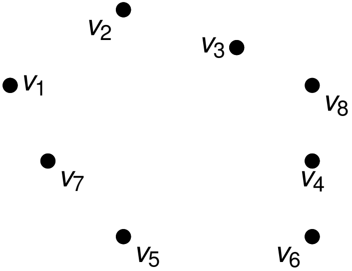
Introduction

Nullvector

$a_1 = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0)$

0	1	1	1	0	0	0	0	0
0	0	1	1	0	0	0	0	0
0	0	0	1	0	1	0	1	0
0	1	0	0	0	0	0	1	0
0	0	1	0	0	1	0	0	0
0	0	1	0	0	0	0	1	0
0	1	0	0	0	1	0	1	0
	0	0	0	0	0	0	0	0

Hypercycle



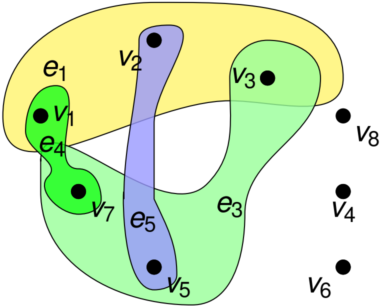
Introduction

Matrix

$a_2 = (1 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 0)$

1	1	1	1	0	0	0	0	0
0	0	1	1	0	0	0	0	0
1	0	0	1	0	1	0	1	0
1	1	0	0	0	0	0	1	0
1	0	1	0	0	1	0	0	0
0	0	1	0	0	0	0	1	0
0	1	0	0	0	1	0	1	0
	2	2	2	0	2	0	2	0

Hypercycle



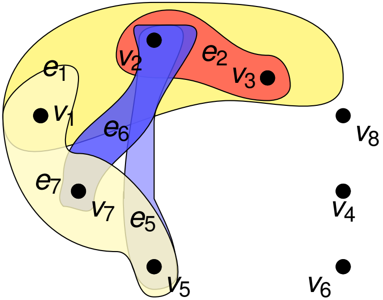
Introduction

Matrix

$a_3 = (1 \;\; 1 \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1)$

1	1	1	1	0	0	0	0	0
1	0	1	1	0	0	0	0	0
0	0	0	1	0	1	0	1	0
0	1	0	0	0	0	0	1	0
1	0	1	0	0	1	0	0	0
1	0	1	0	0	0	0	1	0
1	1	0	0	0	1	0	1	0
	2	4	2	0	2	0	2	0

Hypercycle



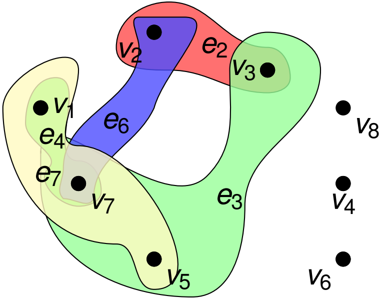
Introduction

Matrix

$$a_4 = (0 \;\; 1 \;\; 1 \;\; 1 \;\; 0 \;\; 1 \;\; 1) = a_2 + a_3$$

0	1	1	1	0	0	0	0	0
1	0	1	1	0	0	0	0	0
1	0	0	1	0	1	0	1	0
1	1	0	0	0	0	0	1	0
0	0	1	0	0	1	0	0	0
1	0	1	0	0	0	0	1	0
1	1	0	0	0	1	0	1	0
	2	2	2	0	2	0	4	0

Hypercycle



Outline

Number of null vectors

We are interested in

$$\boxed{N(n,m) = \text{number of left null vectors for } M(n,m)}$$

in the regime where $n \to \infty$, $m = m_n$ with $m_n/n \to \alpha$, and $W_n \xrightarrow{d} W$ for some limiting weight distribution W.

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Note:

$$\begin{array}{c}\text{no non-trivial}\\ \text{null vectors}\end{array} \Leftrightarrow N(n, m) = 1 \Leftrightarrow \begin{array}{c}\text{system } Mx \equiv y \bmod 2\\ \text{has a solution } x \in \{0, 1\}^n\\ \text{for every } y \in \{0, 1\}^m\end{array}$$

Also related to XORSAT, spin-glass models, Ehrenfest urn, random walk on hypercube, switch-setting problems, . . .

Expected number of null vectors

Let $\mathbf{1}_m$ denote the vector of m ones, and consider the event

$$\begin{aligned} A(n,m) &= \{\mathbf{1}_m \text{ is null for } M(n,m)\} \\ &= \{\text{all column sums of } M(n,m) \text{ even}\}. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\, N(n,m) &= \sum_{k=0}^{m} \mathbb{E}\, \text{number of null vectors of weight } k \\ &= \sum_{k=0}^{m} \binom{m}{k} \mathbb{P}\,(A(n,k)). \end{aligned}$$

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We study asymptotics of $\mathbb{E}\,(n, \alpha n)$ in terms of

$$\rho(s) := \mathbb{E}\, s^W$$

the generating function of the limiting weight distribution.

Technical device

We can show that any model in our class $W_n \xrightarrow{d} W$ is sufficiently well-approximated by the binomial model in which

$$W_n = W_n^{\text{bin}} = \begin{array}{c}\text{number of odd components in}\\ \text{multinomial}\,(W; \frac{1}{n}, \ldots, \frac{1}{n})\end{array}$$

The binomial model is equivalent to:

> *Generate each row by distributing W units uniformly and independently among the n positions and then reducing occupancies mod 2.*

Lemma

In the binomial model,

$$\mathbb{P}\left(A(n,m)\right) = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} \left(\rho(1 - \tfrac{2j}{n})\right)^m.$$

Exact formula

Proof.

Let Y_j = sum of column j and, for $J \subseteq \{1, 2, \ldots, n\}$,

$$S_{i,J} = \text{sum of entries in column } i \text{ and row subset } J.$$

Then

$$\mathbb{P}(A(n,m)) = \mathbb{P}(\text{all } Y_j \text{ even}) = \mathbb{E}\prod_{j=1}^{n}\left(\frac{1+(-1)^{Y_j}}{2}\right)$$
$$= 2^{-n}\sum_{J}\mathbb{E}\left[(-1)^{\sum_{j\in J} Y_j}\right].$$

But $\sum_{j\in J} Y_j = \sum_{i=1}^{m} S_{i,J}$ (i.i.d. summands) so

$$\mathbb{P}(A(n,m)) = 2^{-n}\sum_{J}\left(\mathbb{E}\left[(-1)^{S_{1,J}}\right]\right)^{m}.$$

But $S_{1,J}$ is given by the sum of $|J|$ components of a multinomial $(W; \frac{1}{n}, \ldots, \frac{1}{n})$ vector reduced mod 2, so $(-1)^{S_{1,J}} \stackrel{d}{=} (-1)^{Z}$ where $Z \sim \text{Bin}(W, \frac{|J|}{n})$. □

Asymptotics for $\mathbb{P}\left(A(n, \alpha n)\right)$

Lemma

Suppose that $m_n/n \to \alpha \in (0, \infty)$ *and that either (i)* m_n *is even for all* n*, or (ii)* $\mathbb{P}\left[W \text{ is even}\right] > 0$*. Then,*

$$\lim_{n \to \infty} n^{-1} \log \mathbb{P}\left(A(n, m_n)\right) = -R_\rho(\alpha),$$

for a continuous and non-decreasing function R_ρ *given by*

$$R_\rho(\alpha) = -\log \sup_{\gamma \in [0,1/2]} \left(\frac{(\rho(1-2\gamma))^\alpha}{2\gamma^\gamma (1-\gamma)^{1-\gamma}} \right).$$

Asymptotics for $\mathbb{P}(A(n, \alpha n))$

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Corollary (cf. Kolchin '94)

Let π_n *denote the probability that all the n components of a multinomial* $(m_n; \frac{1}{n}, \dots, \frac{1}{n})$ *vector are even. Suppose that* m_n *is even for all n and* $m_n/n \to \alpha = \lambda \tanh\lambda \in (0, \infty)$*. Then*

$$\lim_{n\to\infty} n^{-1} \log \pi_n = \log\cosh\lambda - (\lambda\tanh\lambda)(1 - \log\tanh\lambda).$$

Asymptotics for $\mathbb{E}\, N(n, \alpha n)$

Define the continuous non-decreasing function F_ρ by

$$F_\rho(\alpha) = \log \sup_{\gamma \in [0,1/2]} \left(\frac{(1 + \rho(1-2\gamma))^\alpha}{2\gamma^\gamma (1-\gamma)^{1-\gamma}} \right), \quad \alpha \geq 0.$$

Set $\alpha_\rho^* := \inf\{\alpha \geq 0 : F_\rho(\alpha) > 0\}$.

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Theorem

Suppose that $m_n/n \to \alpha \in (0, \infty)$. Then,

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E}\,N(n, m_n) = \begin{cases} 0 & \text{if } \alpha \leq \alpha_\rho^*; \\ F_\rho(\alpha) > 0 & \text{if } \alpha > \alpha_\rho^*. \end{cases}$$

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Here, if $\mathbb{P}(W \geq 2) = 1$ and $\mathbb{E}\, W < \infty$, then $1/2 \leq \alpha_\rho^* < 1$.

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Here, if $\mathbb{P}(W \geq 2) = 1$ and $\mathbb{E}\, W < \infty$, then $1/2 \leq \alpha_\rho^* < 1$.

In the fixed-weight case $\mathbb{P}(W = r) = 1$ versions of this result were obtained by Balakin *et al.* '92, Kolchin '94, Calkin '97, Cooper '99.

First non-trivial null vector

Consider building $M(n, m)$ one row at a time.

Let T_n be the first m for which a non-zero null vector appears, i.e., the first m for which row m is in the linear span of rows $1, \ldots, m-1$.

Theorem

Suppose $3 \leq W_n \leq B$ for some $B < \infty$. Then, with probability tending to 1, $\alpha_\rho^* < T_n/n < \underline{\alpha}_\rho$.

Here $\underline{\alpha}_\rho \leq 1$ is another threshold defined in terms of the 2-core of the hypergraph.

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Conjecture

Under the above conditions, $n^{-1} T_n \to \underline{\alpha}_\rho$ *in probability.*

An equivalent version of this conjecture in the fixed weight case has been proved by DUBOIS & MANDLER '02, DIETZFELBINGER *et al.* '10.

Fixed weight case

When $\mathbb{P}(W = r) = 1$, i.e., $\rho(s) = s^r$.

Write α_r and $\underline{\alpha}_r$ for α_ρ and $\underline{\alpha}_\rho$ in this case.

Following BALAKIN *et al.* '92, KOLCHIN '94, CALKIN '97, COOPER '99, '04 . . . .

r	1	2	3	4	5	6	7
α_r^*	0	0.5	0.889493	0.967147	0.989162	0.996228	0.998650
$\underline{\alpha}_r$	—	—	0.917935	0.976770	0.992438	0.997380	0.999064

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Lemma

- $1 - \alpha_r^* \sim \frac{e^{-r}}{\log 2}$ *(CALKIN '97);*
- $1 - \underline{\alpha}_r \sim e^{-r}$.
- *So* $\alpha_r^* < \underline{\alpha}_r < 1$ *for large enough* r.

Outline

Hypergraph 2-core

Starting from the hypergraph represented by $M(m, n)$, iterate the following:

> *Identify a column (vertex) with a single non-zero.*
> *Delete the corresponding row (hyperedge).*

When no column (vertex) with a single non-zero remains, the resulting hypergraph is the 2-core of the original hypergraph.

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Lemma (Molloy, Cooper)

If the 2-core is empty, the hypergraph has no hypercycle.

If the 2-core has more rows (hyperedges) than columns (vertices), the hypergraph contains a hypercycle.

Hypergraph 2-core

By a (minor extension of a) theorem of DARLING & NORRIS '08 one can characterize

- those α for which the 2-core asymptotically has more rows than columns,
- or vice versa,
- and hence $\underline{\alpha}_\rho$,

depending on whether

$$\psi(g^*(\alpha)) < 0 \quad \text{or} \quad \psi(g^*(\alpha)) > 0.$$

Here

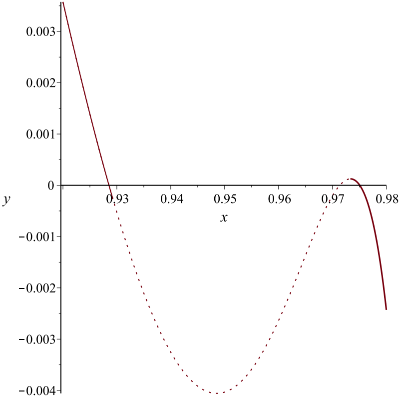
$$g^*(\alpha) = \sup\left\{x \in (0,1) : -\frac{\log(1-x)}{\rho'(x)} \leq \alpha\right\};$$

$$\psi(x) = x + \left(1 + \frac{\rho(x)}{\rho'(x)} - x\right)\log(1-x).$$

Hypergraph 2-core: non-monotonicity

In the fixed weight case, COOPER '04 showed that the core aspect ratio transition is monotone. In the random setting, more can occur.

$\rho(s) = 0.9183s^3 + 0.04s^{19} + 0.0417s^{41}$



Plot shows part of $y = \psi(x)$ (all the line) and locus of $(g^*(\alpha), \psi(g^*(\alpha)))$ (solid line).

$\psi(x)$ has 3 $+$ve zeros.

The 2 of these zeros achieved by $\psi(g^*(\alpha))$ are at $x \approx 0.928538$ and $x \approx 0.975069$.

The first corresponds to $\alpha = \underline{\alpha}_\rho \approx 0.990686$ and the second to $\alpha \approx 0.991185$.

Hence as α ranges in $(0, 1)$, $\psi(g^*(\alpha))$ changes sign from $+$ve, to $-$ve, to $+$ve, and finally to $-$ve again.

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