Random directed and on-line spatial graphs

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A partial survey, including some joint work with
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Introduction: Spatial networks

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- It is natural to study simple models of random spatial networks, for inference or prediction, for insight into typical behaviour, or for assessing performance of algorithms or processes that take place on networks.
- In this talk I will survey some results on some networks constructed on random points in Euclidean space with a connectivity rule that incorporates proximity and some ordering constraint.

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- More generally, α-weighted total length: $\sum_{\text{edges } (i,j)} \|U_i - U_j\|^\alpha$, for a fixed parameter $\alpha > 0$.
- Main question: What can we say about the (asymptotic) distribution of $\mathcal{L}_n$ in the large sample limit $n \to \infty$?

Outline

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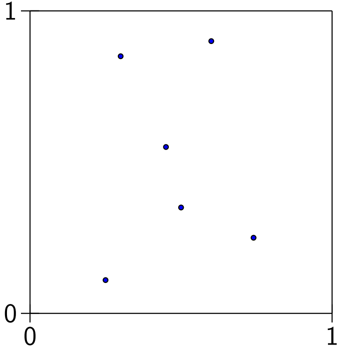
Nearest-neighbour graph

Vertices $1, \ldots, n$ are given independent, uniformly random locations $U_1, \ldots, U_n$ in the unit d-cube $[0,1]^d$. For every vertex, insert a (directed) edge from that vertex to its nearest neighbour, i.e., the closest (Euclidean distance) other vertex. (Ties have probability 0.) Here is a picture for $d = 2$, $n = 6$:



Points are distributed uniformly at random.

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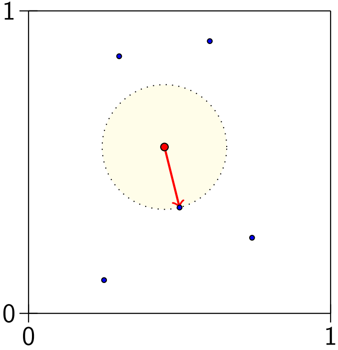

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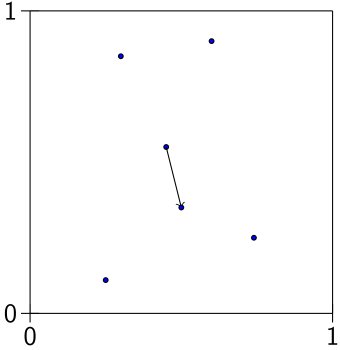

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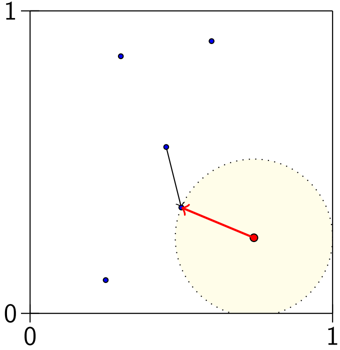

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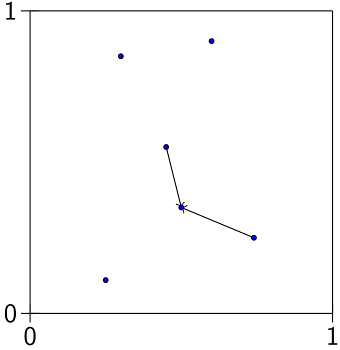

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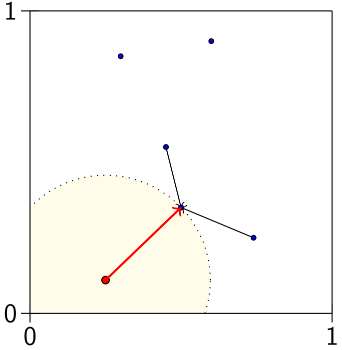

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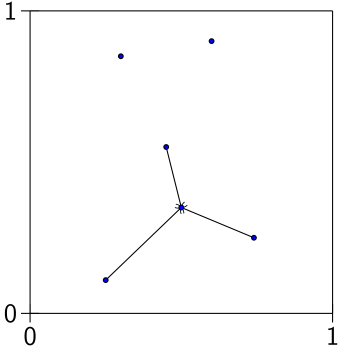

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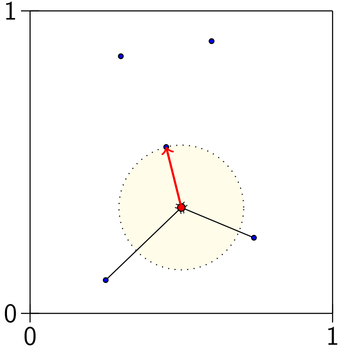

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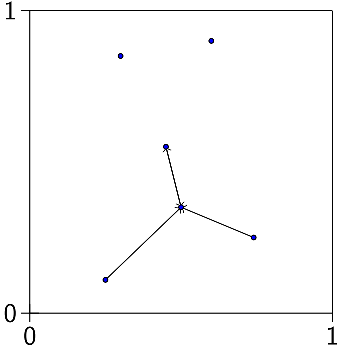

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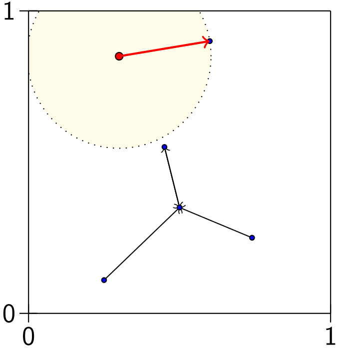

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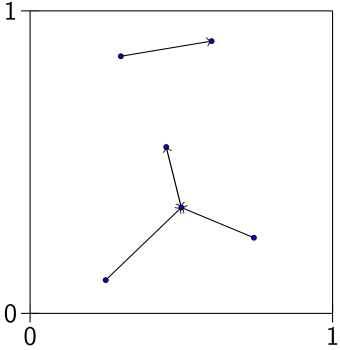

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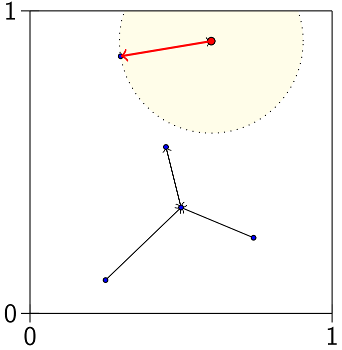

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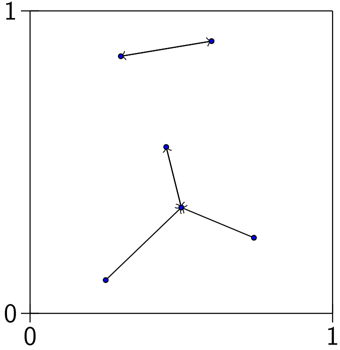

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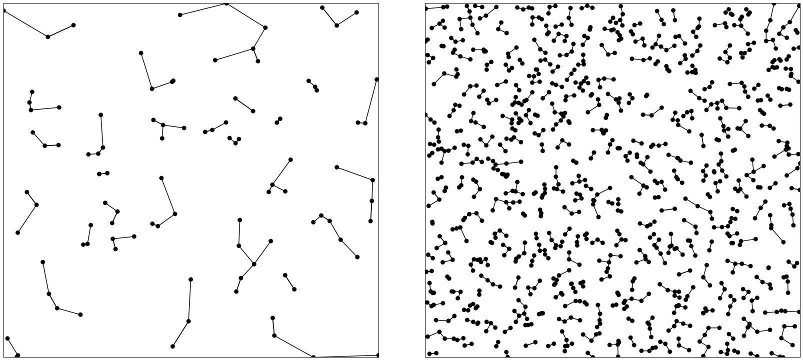

Simulations of nearest-neighbour graph (NNG) in $[0,1]^2$, on $n = 100$ points (*left*) and $n = 1000$ points (*right*).

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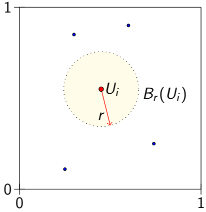
We consider the total power-weighted edge length

$$\mathcal{N}_n^{d,\alpha} := \sum_{i=1}^{n} \min_{1 \le j \le n,\, j \neq i} \|U_j - U_i\|^{\alpha},$$

where $\alpha > 0$ (e.g., $\alpha = 1$ is the total edge length).

The limit theory of $\mathcal{N}_n^{d,\alpha}$ was considered by Bickel & Breiman (1983), Avram & Bertsimas (1993), and Penrose & Yukich (2001), among others.

Nearest-neighbour graph: Heuristics



What is the n-dependence in $\mathcal{N}_n^{d,\alpha}$? Say in $\mathbb{E}\mathcal{N}_n^{d,\alpha}$?

Consider vertex $i \in \{1, \ldots, n\}$ at $U_i \in [0,1]^d$.

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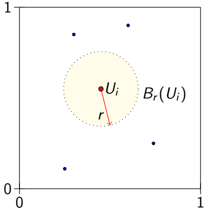

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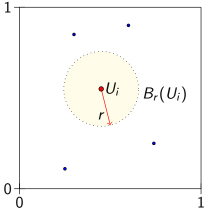

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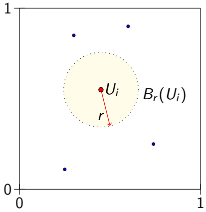

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From here it is not hard to see that $\operatorname{med}(D_{n,i}) \asymp n^{-1/d}$, and indeed

$$\mathbb{E}\, D_{n,i}^{\alpha} \asymp n^{-\alpha/d}, \text{ and } \mathbb{V}\mathrm{ar}\, D_{n,i}^{\alpha} \asymp n^{-2\alpha/d}.$$

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However, the $D_{n,i}$ are not independent, so this is not a sum of i.i.d. random variables. The dependence is in some sense only local: $D_{n,i}$ and $D_{n,j}$ are close to independent if U_i and U_j are far apart.

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An appropriate formulation of this local dependence allows one to obtain laws of large numbers and central limit theorems, involving scale factors $\mathbb{E}\,\mathcal{N}_n^{d,\alpha} \asymp n^{1-(\alpha/d)}$ and $\mathbb{V}\mathrm{ar}\,\mathcal{N}_n^{d,\alpha} \asymp n^{1-(2\alpha/d)}$.

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Now an extensive theory of stabilization to obtain such results for spatial random graphs with local dependence: Aldous & Steele (1992), Avram & Bertsimas (1993), Kesten & Lee (1996), Penrose & Yukich (2001–), many others.

Nearest-neighbour graph: Limit theory

Theorem (Bickel & Breiman 1983, Avram & Bertsimas 1993, Penrose & Yukich 2001, etc.)

For $d \in \mathbb{N}$ and $\alpha > 0$, as $n \to \infty$,

$$n^{(\alpha/d)-1} \mathcal{N}_n^{d,\alpha} \to c_{d,\alpha} := \pi^{-\alpha/2} \Gamma(1 + \tfrac{\alpha}{d}) \Gamma(1 + \tfrac{d}{2})^{\alpha/d}, \text{ in } L^1.$$

Moreover, as $n \to \infty$,

$$n^{(\alpha/d)-(1/2)} \left(\mathcal{N}^{d,\alpha} - \mathbb{E}\mathcal{N}^{d,\alpha} \right) \xrightarrow{d} Z,$$

where Z has a mean zero, finite variance normal distribution.*

* Variance depends on d and α; explicit values known only for $d = 1$ (Penrose & Wade, 2008).

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Example. When $d = 2$, $\alpha = 1$, we have that the total edge length of the nearest-neighbour graph on $[0,1]^2$ satisfies

$$\mathbb{E}\mathcal{N}^{2,1} \sim \frac{\sqrt{n}}{2}, \text{ and } \mathcal{N}^{2,1} - \mathbb{E}\mathcal{N}^{2,1} \xrightarrow{d} \text{non-degenerate normal.}$$

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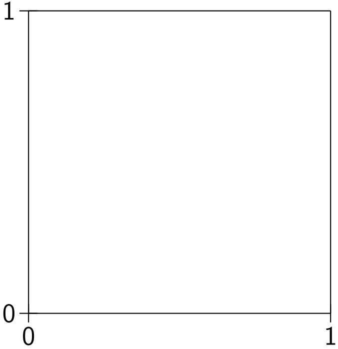

$n = 0$: empty graph!

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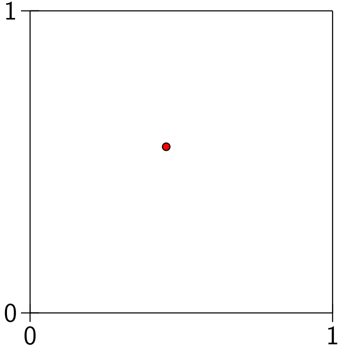

$n = 1$: first vertex arrives at a uniform random location.

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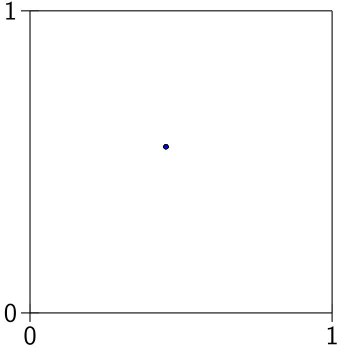

$n = 1$: first vertex has no neighbours, so no edge.

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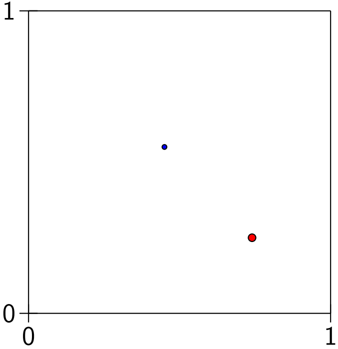

$n = 2$: next vertex arrives.

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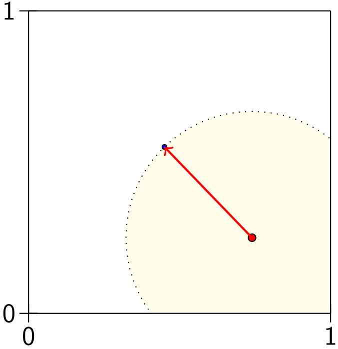

$n = 2$: new vertex joins to nearest existing vertex.

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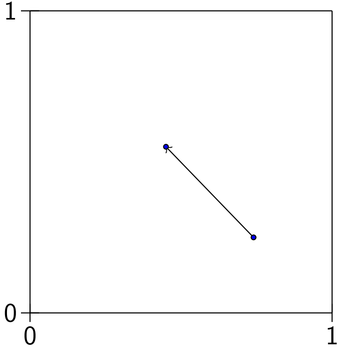

$n \geq 2$: new vertex joins to nearest existing vertex.

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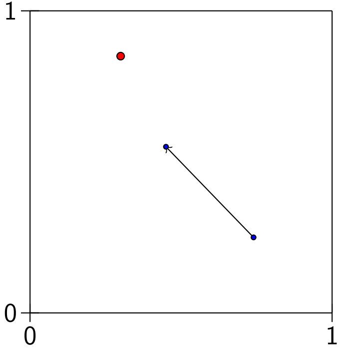

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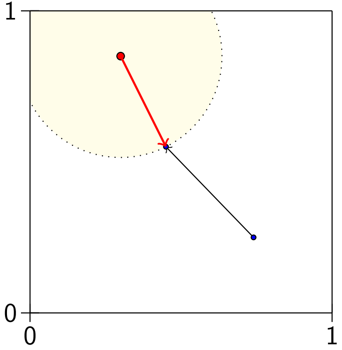

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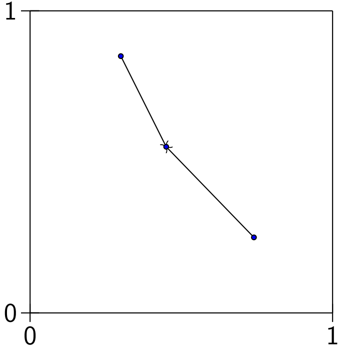

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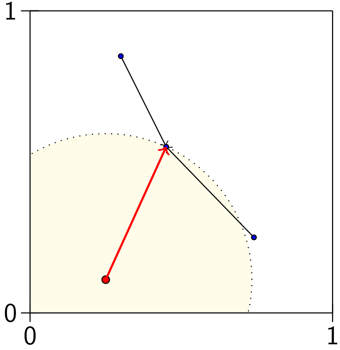

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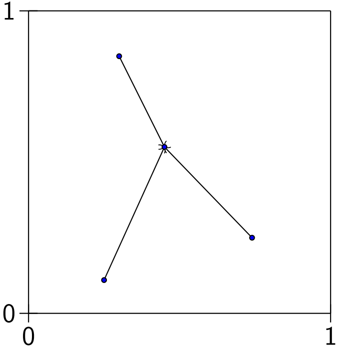

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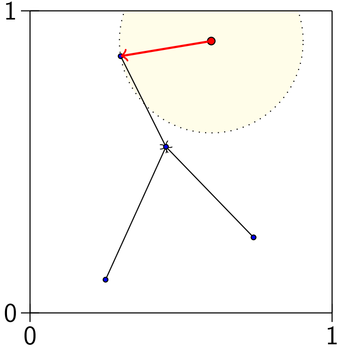

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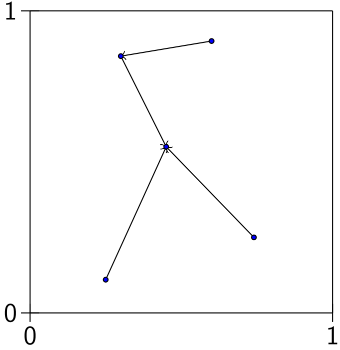

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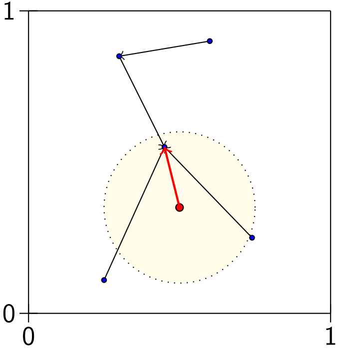

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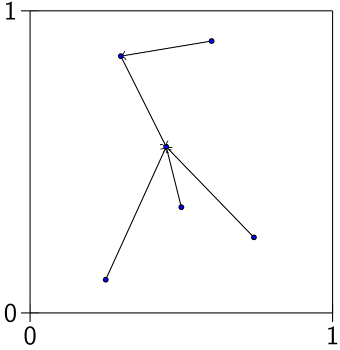
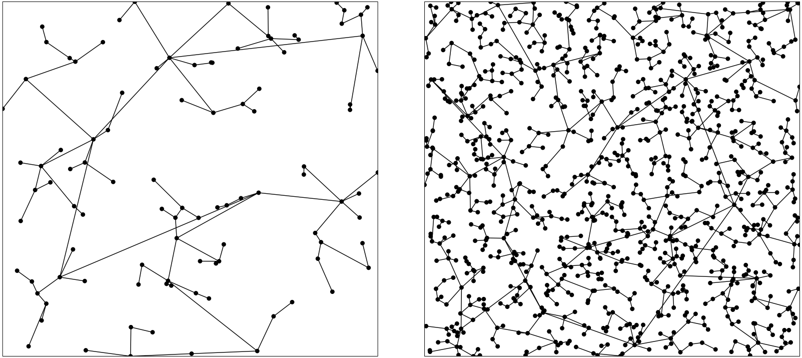


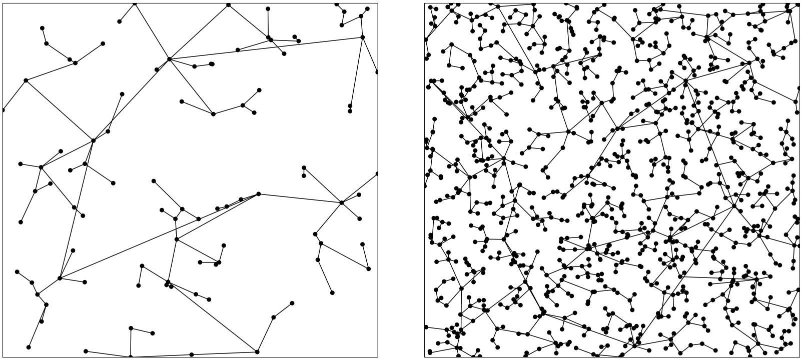
NB. Graph is a tree; different order will give different graph.

On-line nearest-neighbour graph



Simulations of on-line nearest-neighbour graph (ONG) in $[0,1]^2$, on $n = 100$ points (*left*) and $n = 1000$ points (*right*).

On-line nearest-neighbour graph



Simulations of on-line nearest-neighbour graph (ONG) in $[0,1]^2$, on $n = 100$ points (*left*) and $n = 1000$ points (*right*).

Note the ONG is more inhomogeneous than the ordinary NNG: Old vertices tend to be more highly connected; old edges tend to be longer.

On-line nearest-neighbour graph

Although it is a natural model of a time-evolving spatial network, the earliest appearance of the on-line nearest-neighbour graph (ONG) that I am aware of is in Steele (1989). It also appears as a limiting case of network models of Fabrikant *et al.* (2002), Manna & Sen (2002) and Flaxman *et al.* (2006). The name "on-line nearest-neighbour graph" is due to Penrose (2005).

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We consider the total power-weighted edge length

$$\mathcal{O}_n^{d,\alpha} := \sum_{i=2}^{n} \min_{1 \le j < i} \| U_j - U_i \|^{\alpha},$$

as studied by Penrose (2005), Penrose & Wade (2008), and Wade (2007–9).

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As in the heuristic for the nearest-neighbour graph, $\mathbb{E}\, D_{i,i}^\alpha \asymp i^{-\alpha/d}$ and $\mathbb{V}\mathrm{ar}\, D_{i,i}^\alpha \asymp i^{-2\alpha/d}$, so

$$\mathbb{E}\, \mathcal{O}_n^{d,\alpha} \begin{cases} \to \text{const.} & \text{if } \alpha > d, \\ \asymp \log n & \text{if } \alpha = d, \\ \asymp n^{1-(\alpha/d)} & \text{if } \alpha < d. \end{cases}$$

Moreover, assuming local dependence, one might guess

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On-line nearest-neighbour graph: Law of large numbers

Theorem (Wade 2007, Penrose & Wade 2008)

Let $d \in \mathbb{N}$ and $\alpha > 0$. Then, as $n \to \infty$,

$$n^{(\alpha/d)-1}\mathcal{O}_n^{d,\alpha} \to c'_{d,\alpha}, \text{ in } L^1, \text{ if } 0 < \alpha < d.$$

On the other hand for $\alpha = d$ we have $\mathbb{E}\,\mathcal{O}_n^{d,d} \sim \pi^{-d/2}\Gamma(1+(d/2))\log n$, and, for $\alpha > d$ we have $\mathcal{O}_n^{d,\alpha} \to W^{d,\alpha}$ in L^2, where $W^{d,\alpha}$ is an $\mathbb{R}_+$-valued random variable with finite variance (not Gaussian).

Here

$$c'_{d,\alpha} := \frac{d}{d-\alpha} c_{d,\alpha} \text{ for } \alpha \in (0,d).$$

For example, $c'_{2,1} = 2c_{2,1} = 1$.

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Intuition: Increasing α increases the relative importance of long ($\approx$ early) edges. For example, if $\alpha > d/2$ then the very first edge contributes a positive fraction of the total variance (order 1).

On-line nearest-neighbour graph: Large α

Theorem (Penrose & Wade 2008, Wade 2009)

Let $d \in \mathbb{N}$ and $\alpha > d/2$. Then, as $n \to \infty$,

$$\mathcal{O}_n^{d,\alpha} - \mathbb{E}\,\mathcal{O}_n^{d,\alpha} \to Q_{d,\alpha}, \text{ in } L^2,$$

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- Note if $\alpha \in (d/2, d]$ one has $\mathbb{E}\,\mathcal{O}_n^{d,\alpha} \to \infty$, so the centering is non-trivial.

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- Note if $\alpha \in (d/2, d]$ one has $\mathbb{E}\,\mathcal{O}_n^{d,\alpha} \to \infty$, so the centering is non-trivial.
- We know $Q_{d,\alpha}$ is non-Gaussian for $d = 1$, $\alpha > 1/2$ (when it is characterized by a distributional fixed-point equation), and when $d \geq 2$, $\alpha > d$ (when $Q_{d,\alpha} = W_{d,\alpha} - \mathbb{E}\,W_{d,\alpha}$ from above). Conjecture is that $Q_{d,\alpha}$ is non-Gaussian for all $\alpha > d/2$.

On-line nearest-neighbour graph: Small α

Conjecture (Penrose 2005, Wade 2009)

Let $d \in \mathbb{N}$ *and* $\alpha \in (0, d/2)$. *Then, as* $n \to \infty$,

$$n^{(\alpha/d)-(1/2)} \left(\mathcal{O}_n^{d,\alpha} - \mathbb{E}\, \mathcal{O}_n^{d,\alpha} \right) \xrightarrow{d} \text{non-degenerate normal.}$$

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This conjecture holds for $\alpha \in (0, d/4)$.

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Example. For the case of the total length ($\alpha = 1$) we have that the central limit theorem holds for $d \geq 5$; it is conjectured for $d \geq 3$. When $d = 1$, we have $\mathcal{O}_n^{1,1} - \mathbb{E}\, \mathcal{O}_n^{1,1} \to Q_{1,1}$, non-Gaussian distribution characterized by a fixed point.

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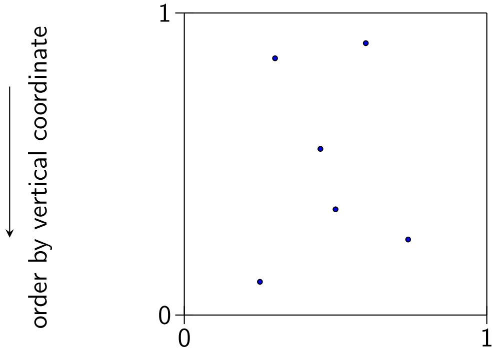
Outline

Minimal directed spanning tree

Vertices $1, \ldots, n$ have uniform random locations $U_1, \ldots, U_n$ in $[0, 1]^d$. For every vertex, insert a (directed) edge from that vertex to its directed nearest neighbour, i.e., the closest vertex with strictly smaller dth coordinate. (The vertex with minimal dth coordinate does not emit an edge.) Here is a picture for $d = 2$, $n = 6$:



Points are distributed uniformly at random.

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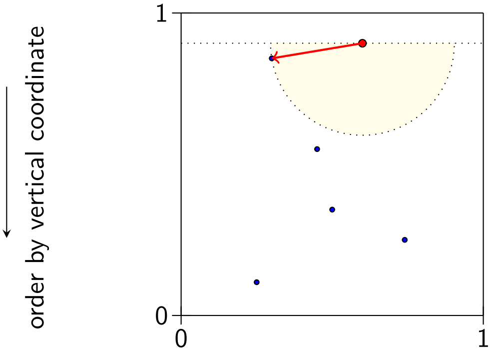

Join each point to its nearest neighbour below.

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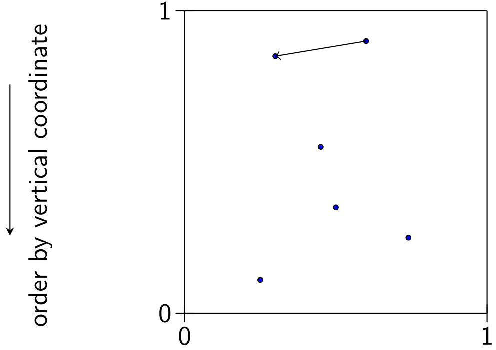

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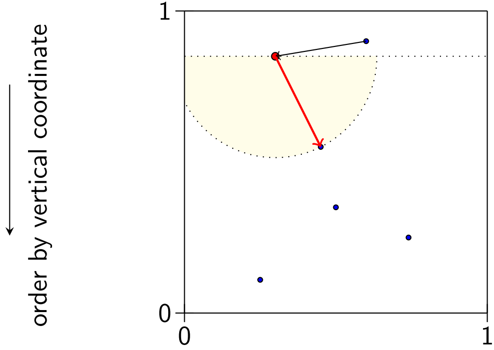

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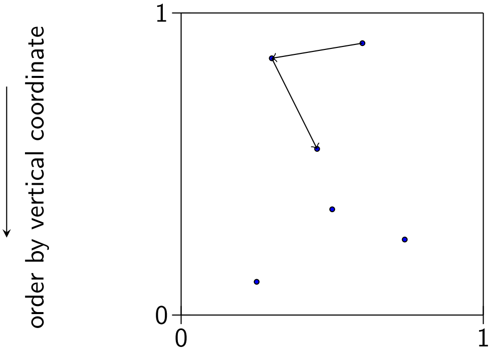

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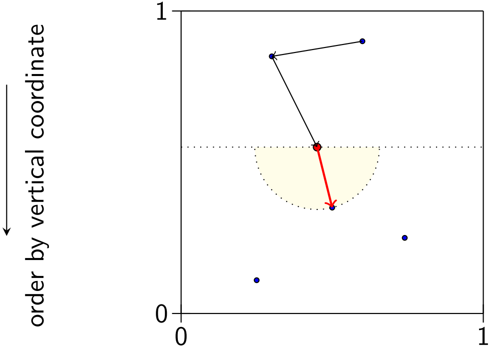

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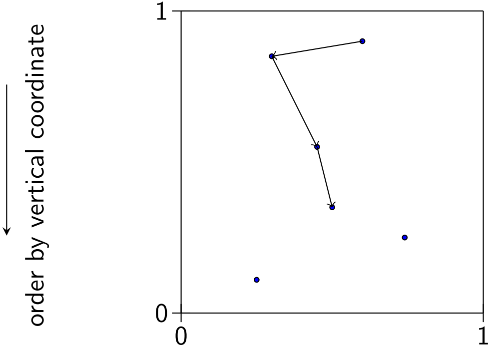

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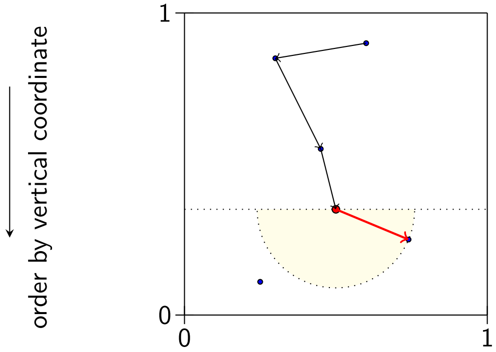

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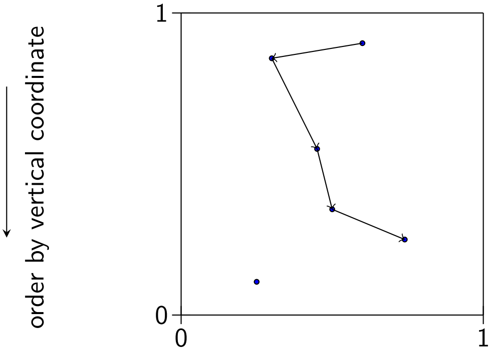

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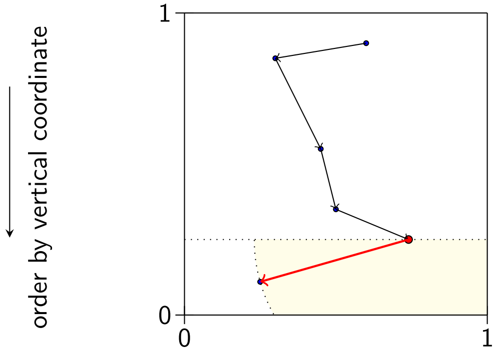

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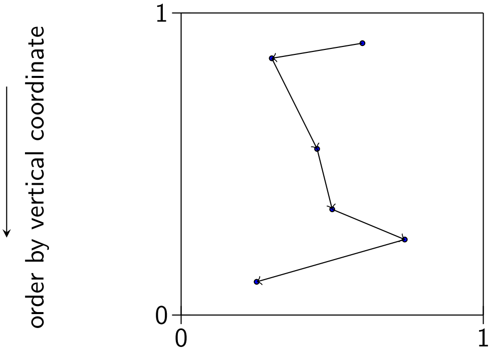
Vertices $1, \ldots, n$ have uniform random locations $U_1, \ldots, U_n$ in $[0,1]^d$. For every vertex, insert a (directed) edge from that vertex to its directed nearest neighbour, i.e., the closest vertex with strictly smaller dth coordinate. (The vertex with minimal dth coordinate does not emit an edge.) Here is a picture for $d = 2$, $n = 6$:



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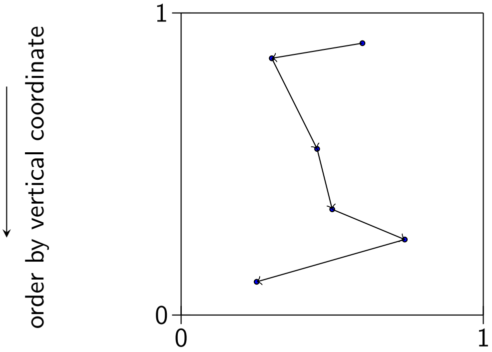

NB. The graph is tree.

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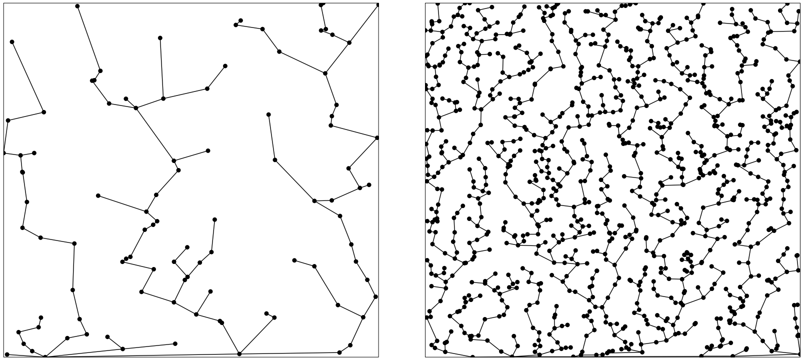

Simulations of minimal directed spanning tree (MDST) in $[0,1]^2$, on $n = 100$ points (*left*) and $n = 1000$ points (*right*).

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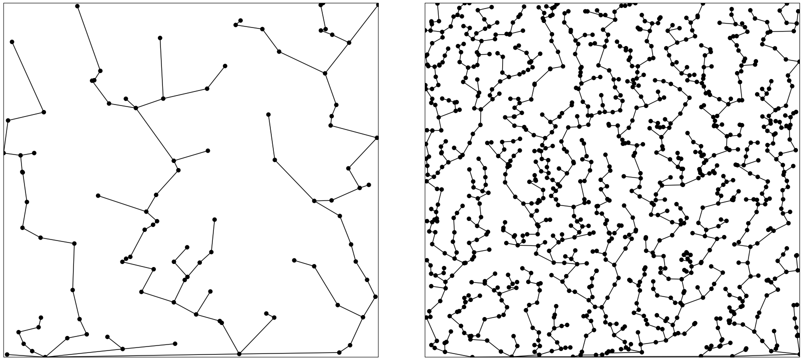

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Note again some inhomogeneity, this time due to spatial boundary effects near the base of the d-cube, where edges are typically much longer than the edges in the bulk.

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$$\mathcal{M}_n^{d,\alpha} := \sum_{i=1}^{n} \min_{1 \leq j \leq n,\, j \neq i} \| U_j - U_i \|^{\alpha},$$

where $\alpha > 0$ (e.g., $\alpha = 1$ is the total edge length).

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Increasing α increases relative importance of long edges, and leads to a Gaussian/non-Gaussian phase transition. Small α: only bulk contributes (Gaussian), large α only boundary contributes (non-Gaussian).

Minimal directed spanning tree: Limit theory

Theorem (Penrose & Wade* 2010)

- *If* $0 < \alpha < d/2$, *then, as* $n \to \infty$,

$$n^{(\alpha/d)-(1/2)} \left(\mathcal{M}_n^{d,\alpha} - \mathbb{E}\,\mathcal{M}_n^{d,\alpha} \right) \xrightarrow{d} Z,$$

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- *If* $\alpha = d/2$, *then, as* $n \to \infty$,

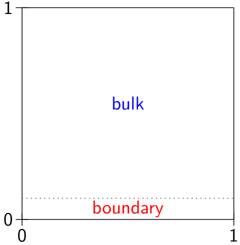
$$\mathcal{M}_n^{d,d/2} - \mathbb{E}\, \mathcal{M}_n^{d,d/2} \xrightarrow{d} Z + Q_{d-1,d/2},$$

  where Z *is non-degenerate normal, independent of* $Q_{d-1,d/2}$.

* Caveat: Penrose & Wade actually prove this for the Poissonized version.

Minimal directed spanning tree: Heuristics

Define $\varepsilon_n = n^{-\beta}$ for $\beta \approx 1/2$.



The normal component (when present) in the limit theorem arises from the bulk of the domain, vertices in $[0,1]^{d-1} \times [\varepsilon_n, 1]$.

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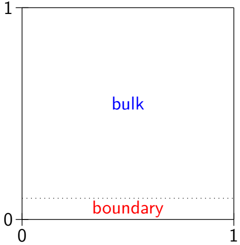

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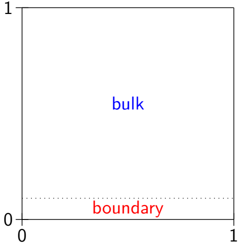

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Similarly to the ordinary nearest-neighbour graph, local dependence and roughly homogeneous edge lengths imply

$$n^{(\alpha/d)-(1/2)} \left(\mathcal{M}_n^{d,\alpha:\text{bulk}} - \mathbb{E}\,\mathcal{M}_n^{d,\alpha:\text{bulk}} \right) \xrightarrow{d} Z.$$

In particular, if $\alpha = d/2$, there's no scaling, while if $\alpha > d/2$, then $\mathcal{M}_n^{d,\alpha:\text{bulk}} - \mathbb{E}\,\mathcal{M}_n^{d,\alpha:\text{bulk}} \to 0$ in probability.

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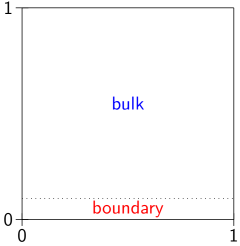

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$$\mathcal{M}_n^{d,\alpha} = \mathcal{M}_n^{d,\alpha:\text{bulk}} + \mathcal{M}_n^{d,\alpha:\text{bndry}}.$$

On the other hand, we claim $\mathcal{M}_n^{d,\alpha:\text{bndry}} \approx \mathcal{O}_{N_n}^{d-1,\alpha}$ for $N_n \to \infty$. If $\alpha \geq d/2$, then $\alpha > (d-1)/2$, and, by limit theorem for the ONG, $\mathcal{M}_n^{d,\alpha:\text{bndry}} - \mathbb{E}\,\mathcal{M}^{d,\alpha:\text{bndry}} \xrightarrow{d} Q_{d-1,\alpha}$. If $\alpha < d/2$, then

$$n^{(\alpha/d)-(1/2)} \left(\mathcal{M}_n^{d,\alpha:\text{bndry}} - \mathbb{E}\,\mathcal{M}_n^{d,\alpha:\text{bndry}}\right) \to 0,$$

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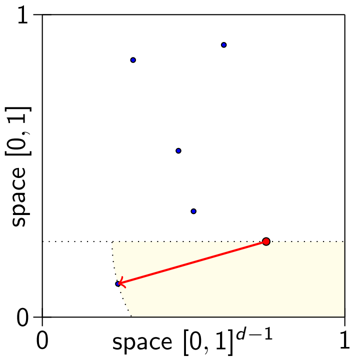
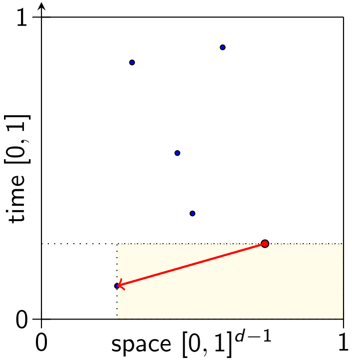
Why the approximation by the ONG near the boundary?

Minimal directed spanning tree: Heuristics

Why the approximation by the ONG near the boundary? Couple MDST on $[0,1]^d$ to an ONG on $[0,1]^{d-1}$ by projecting points onto the first $d-1$ coordinates, and use dth coordinate order for the time in the ONG.



MDST on $[0,1]^d$



ONG on $[0,1]^{d-1}$

Distance on $[0,1]^d$ in MDST but distance on $[0,1]^{d-1}$ in ONG.

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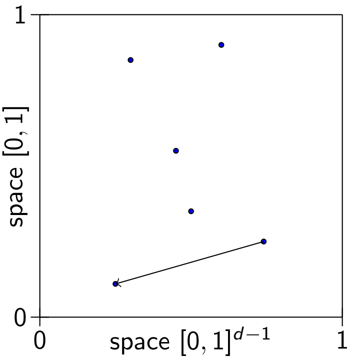

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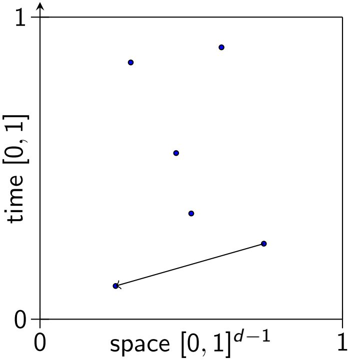

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Edge is the “same” but length will be different.

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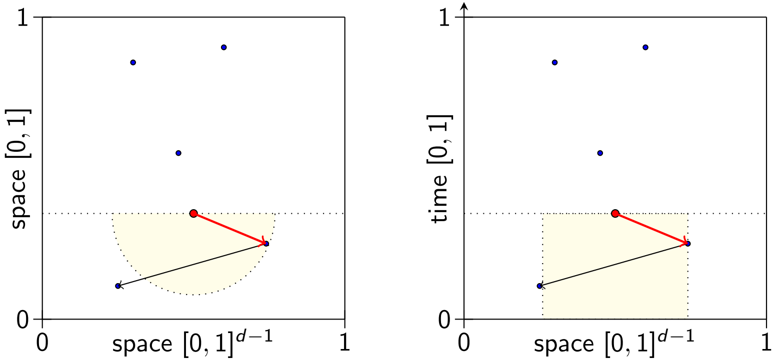

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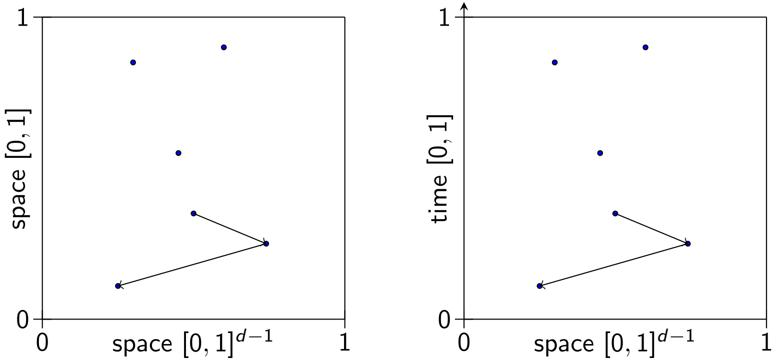

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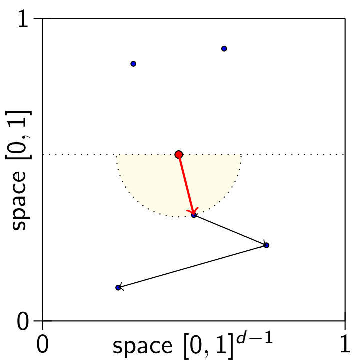

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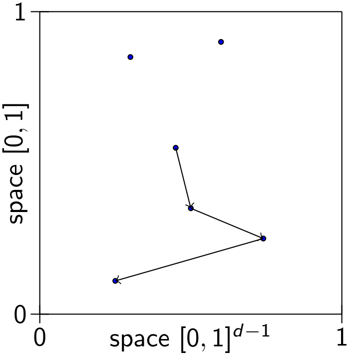

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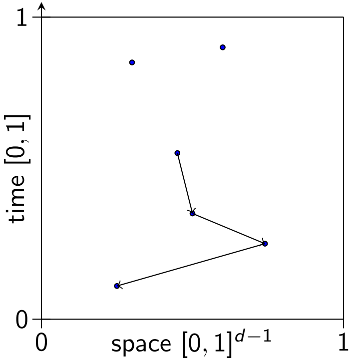

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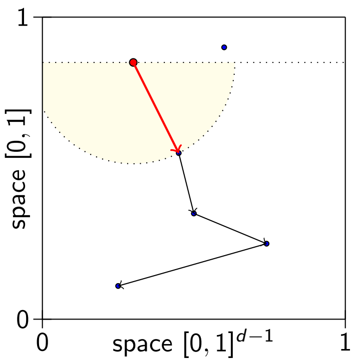

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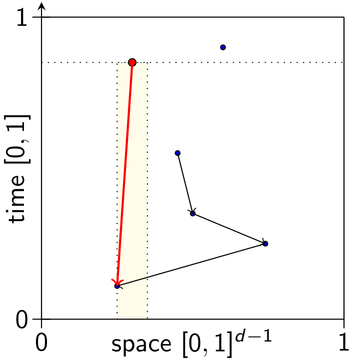

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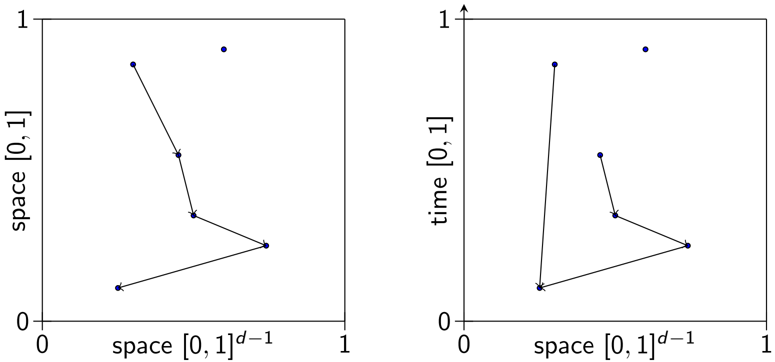

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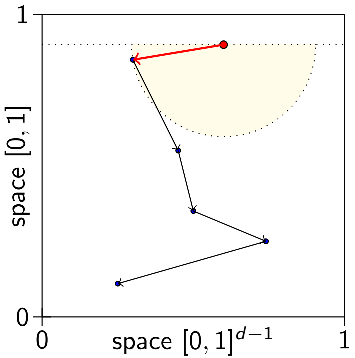

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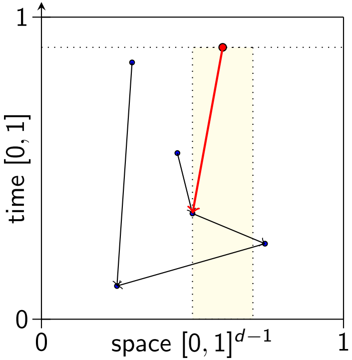

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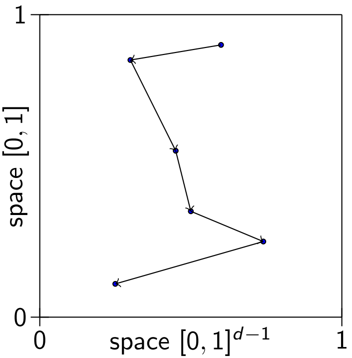

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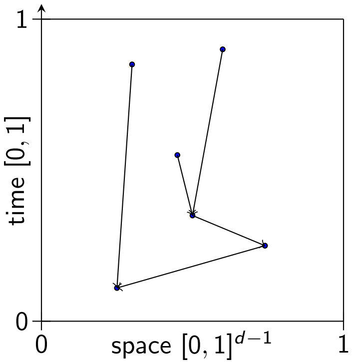

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Thus one obtains that the MDST length in the boundary zone is well approximated by the ONG in $(d-1)$ dimensions.

Outline

Closing remarks

Some other directions:

- Rates of convergence: Avram, Bertsimas, Penrose, Yukich, Schulte, Eichelsbacher, Thäle, Lachièze-Rey, Peccati, Yang, Bhattacharjee (2022), ...

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Thank you for listening!

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