Hints 11-12

11.1 One way is a direct computation. Another possibility is to use the correspondence between $f(z) = \frac{az+b}{cz+d}$ and a multiplication by matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

11.3 (a) Is almost evident, but still try to find some argument.

(b) Map some intersection points to where they need to be, then use (a).

11.4 Draw two lines through P intersecting the circle, and find similar triangles.

11.5 Similar to 10.1.

12.1 Divide by $AC \cdot BC$, find (almost) two cross-ratios on the left. Then show that in case of the inscribed quadrilateral these things are exactly the cross-ratios. Finally, use results of Problem 7.8 to get the statement.

12.2 (b) Use the altitude in a right-angled triangle (E23).

(c) Use the result of Question 11.4 to find the distance PQ (where $Q \in \gamma$ is the point of contact).

(d) Use the tangent line constructed in (c) and consider similar right triangles.

(e) Invert the construction in (d).

(f) Use (b).

(g) If the circles are of different sizes, then there is a homothety with positive coefficient mapping one circle to another (why?). Find the centre of the homothety and use (c).

(h) Use (g) to find the centre and (b) to find the radius.

(i) Two equal circles may be swapped by reflection r in a line. Two different ones may be swapped by an inversion I_0 found in (h). If I is an inversion which takes two different circles to two equal ones, then $r = I \circ I_0 \circ I$. Use this both for guessing the candidate for I and for proving that $I(\gamma_1)$ and $I(\gamma_2)$ are of the same size.

12.3 By definition (look at the fixpoints).

12.4 Strictly speaking, it should be "reflections and inversions". You can present each of these transformations as a composition of two reflections/inversions: first specify the lines/circles of the inversions then write the transformations.

12.5 Look at special nice points (and the angles between curves).

12.6 Use cross-ratio.

12.7 Go through all other types: they will be easily excluded by different reasons.

12.8 Take a parabolic transformation preserving ∞ and conjugate it by a transformation mapping 1 to ∞.

12.9 Map three boundary points of the disc to three boundary points of the new domain. Don't forget to check that you obtained the right half-plane, but not the other one!

12.10 Look at the image of ∞ under the multiple application of the composition.