Assignment 11-12

Starred problems due on Friday, 4 February

11.1 Show that Möbius transformations form a group.

11.2 Find a Möbius transformation which takes $1, 2, 3$ to $0, 1, \infty$.

11.3 (*)

(a) Let l be a line and γ be a circle. Show that γ is orthogonal to l if and only if l contains the centre of γ.

(b) Let γ_1, γ_2, γ_3 be three mutually orthogonal circles on the plane. Show that there exists a Möbius transformation which takes them to the curves $\{x = 0\}$, $\{y = 0\}$ and $x^2 + y^2 = 1$.

11.4 (*) Let γ be a circle and P be a point lying outside of γ. Let l be a line through P and A, B be the intersection points of l with γ. Prove that the product $|PA| \cdot |PB|$ does not depend on the choice of l. (This product is also called *power of P with respect to γ*).

11.5 The same question as 11.4, but P lies inside γ.

12.1 Prove the theorem of Ptolemy: for a cyclic quadrilateral $ABCD$, the following equality holds:

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

12.2 (*) (Inversion with ruler and compass).
To do this question, you DO NOT need to use/have actual ruler and compass! You need rather to find and describe algorithms for certain constructions as well as to justify them. You can use without proofs and further descriptions the following two constructions:

- midpoint of a given segment;
- the line perpendicular to a given line through a given point.

The above constructions (together with examples how to write down such questions) can be found here, see p. 2, Question 1.3, parts (b), (c) (you do not need to go through all details/parts of that question, just check how to write if you are unsure!).

This is a long question with an easy start and more complicated parts at the end. Please, try to submit parts (b)-(e) (may be using hints if needed). If you want to present solutions/sketches for later parts, I would be happy to read also that (but formally it is not a part of the written assignment).

(a) Given a circle γ, construct its centre.

(b) Given segments of length a and b construct a segment of length h satisfying $h^2 = a \cdot b$.

(c) Given a circle γ and a point P outside the circle, construct a line PQ tangent to γ.

(d) Given a circle γ and a point A outside the circle, construct the inversion image of A

(e) Construct the inversion image for the point A' lying inside the circle γ.

(f) Let O, A' and A be three points lying on a line (A' lies between O and A).
Construct a circle γ centred at O such that the inversion with respect to γ takes A to A'.

(g) Given two circles γ_1 and γ_2, construct a line tangent to both of them.

(h) Given two circles γ_1 and γ_2 of different sizes, construct an inversion which takes γ_1 to γ_2 and takes γ_2 to γ_1.
(You need to construct the centre and the radius of the circle of inversion).

(i) Given two circles γ_1 and γ_2 of different sizes,
find an inversion which takes them to a pair of equal circles.
(You need to construct the centre and the radius of the circle of inversion).

12.3 What type is the transformation $1/z$?
(Hint: parabolic or not? if not, then is it elliptic, or hyperbolic, or loxodromic?)

12.4 Write the following transformations as compositions of inversions and/or reflections:
(a) $2z$ (b) $-z$ (c) $z+1$ (d) $\frac{1}{z}$

12.5 Let I be an inversion with respect to the unit circle $|z| = 1$. Find the image $I(l)$ of the line l given by the equation $\mathrm{Re}(z) = 2$.

12.6 Do the points $-1-2i$, $-1+2i$, $3+i$, $3-i$ lie on one line or circle?

12.7 Show that a finite order Möbius transformation is elliptic.
(g is called of finite order if $g^n = id$ for some integer n).

12.8 Find a parabolic Möbius transformation preserving the point $z = 1$.

12.9 Find a Möbius transformation mapping the disc $|z| < 1$ to the half-plane $Rez > 2$.

12.10 I_0 is the inversion with respect to the circle $|z| = 1$. I_1 is the inversion with respect to the circle $|z-1| = 1$. The composition $I_1 \circ I_0$ is a Möbius transformation. What type is the composition $I_1 \circ I_0$?
(Hint: try to find a geometric solution, without writing the formulae).

References:

1. Lectures 21-24 (Möbius transformations, Inversion, Stereographic projection) are expansions of Lecture V in Prasolov's book. Alternatively, see pp. 93–95 in Prasolov, Tikhomirov.

2. A good exposition about Möbius transformations is contained it Section 1.1 of *Hyperbolic Geometry* by Caroline Series.

3. For introduction and discussion of inversion you can check the following sources (both linked from the course's on-line resources page):

   - Malin Christersson, *Circle inversion.* (Nice illustrated introduction with proofs).
   - Tom Davis, *Inversion in a circle.* Beautiful article showing how to use inversions (but not giving the proofs of basic properties of inversion).