Assignment 17-18
Starred problems due on Friday, 18 March

17.1. Prove that any pair of parallel lines can be transformed to any other pair of parallel lines by an isometry.

17.2. Let $A, B \in \gamma$ be two points on a horocycle γ. Show that the perpendicular bisector to AB is orthogonal to γ.

17.3. Let l_1, l_2, l_3 be three lines in $\mathbb{H}^2$, let r_i be the reflection with respect to l_i and let $f = r_3 \circ r_2 \circ r_1$. Show that f is either a reflection or a glide reflection, i.e. a hyperbolic translation along some line composed with a reflection with respect to the same line.

Assuming that the lines l_1, l_2, l_3 are not passing through the same point and not having a common perpendicular, show that f is a glide reflection.

17.4. (*) Given an isometry f of the hyperbolic plane such that the distance from A to $f(A)$ is the same for all points $A \in \mathbb{H}^2$, show that f is the identity map.

17.5. (*) Let a and b be two vectors in the hyperboloid model such that $\langle a, a\rangle > 0$ and $\langle b, b\rangle > 0$. Let l_a and l_b be the lines determined by equations $\langle x, a\rangle = 0$ and $\langle x, b\rangle = 0$ respectively. And let r_a and r_b be reflections with respect to l_a and l_b.

(a) For $a = (0, 1, 0)$ and $b = (1, 0, 0)$ write down r_a and r_b.
Find $r_b \circ r_a(v)$, where $v = (0, 1, 2)$.

(b) What type is the isometry $\phi = r_b \circ r_a$ for $a = (1, 1, 1)$ and $b = (1, 1, -1)$?
(*Hint:* you don't need to compute r_a and r_b).

(c) Find an example of a and b such that $\phi = r_b \circ r_a$ is a rotation by $\pi/2$.

18.1 Let l be a line on the hyperbolic plane and let E_l be the equidistant curve for l.

(a) Let C_1 and C_2 be two connected components of the same equidistant curve E_l. Show that that C_1 is also equidistant from C_2, i.e. given a point $A \in C_1$ the distance $d(A, C_2)$ from A to C_2 does not depend on the choice of A.

(b) Let $A \in E_l$ be a point on the equidistant curve, and let $A_l \in l$ be the point of l closest to A. Show that the line AA_l is orthogonal to the equidistant curve.

(c) Let $P, Q \in l$ be two points on l. Let $A \in E_l$ be a point of the equidistant curve such that the segments AP and AQ contain no point of E_l except A. Continue the rays AP and AQ till the next intersection points with E_l, denote the resulting intersection points by B and C. Let T be a curvilinear triangle ABC (with geodesic sides AB and AC, but BC being a segment of the equidistant curve). Assuming that all angles of ABC are acute show that the area of T does not depend on the choice of $A \in E_l$.

(d) With the assumptions of (c), show that the area of the geodesic triangle ABC does not depend on the choice of A.

18.2. (*)

(a) Let l and l' be ultra-parallel lines. Let γ be an equidistant curve for l intersecting l' in two points A and B. Denote by h the common perpendicular to l and l' and let $H = h \cap l'$ be the intersection point. Show that $AH = HB$.

(b) Let l be a line and γ be an equidistant curve for l. For two points A, B on one component of γ, show that the perpendicular bisector of AB is also orthogonal to l.

(c) Let ABC be a triangle in the Poincare disc model. Let γ be a Euclidean circumscribed circle (i.e. a circumscribed circle for ABC considered as a Euclidean triangle). Suppose that γ intersects the absolute at points X and Y. Show that the (hyperbolic) perpendicular bisector to AB is orthogonal to the hyperbolic line XY.

(d) Show that three perpendicular bisectors in a hyperbolic triangle are either concurrent, or parallel, of have a common perpendicular.

References:

- Material on types of isometries in hyperbolic geometry, and on horocycles and equidistant curves is based on Lecture IX of Prasolov's book.
  Alternatively, see pp.113-116 of Section 5.3 in Prasolov and Tikhomirov.