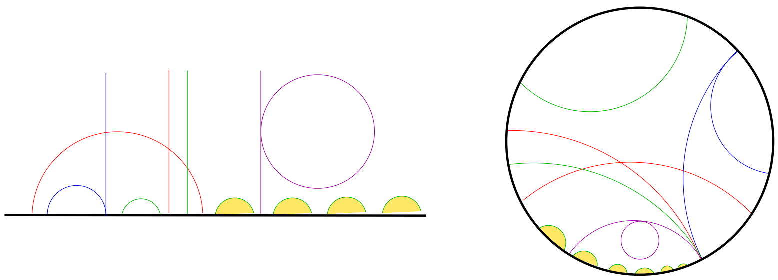
# Solutions 13-14

13.1. Draw in each of the two conformal models (Poincare disc and upper half-plane):

(a) two intersecting lines;
(b) two parallel lines;
(c) two ultra-parallel lines;
(d) infinitely many disjoint (hyperbolic) half-planes;
(e) a circle tangent to a line.

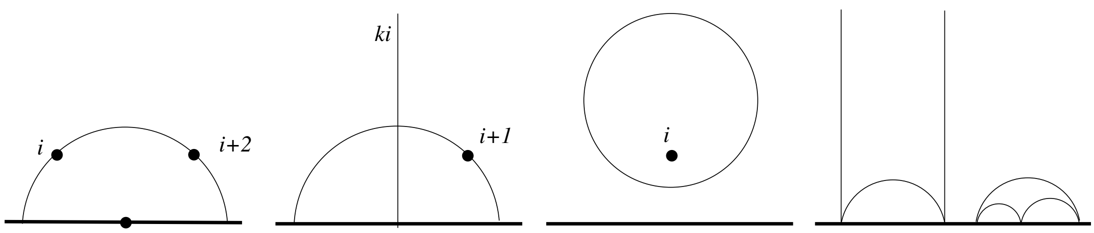
**Solution:** One solution is indicated by colors: (a), (b), (c), (d), (e).



13.2. In the upper half-plane model draw

(a) a (hyperbolic) line through the points $i$ and $i+2$;
(b) a (hyp.) line through $i+1$ orthogonal to the (hyp.) line represented by the ray $\{ki \mid k>0\}$;
(c) a (hyperbolic) circle centred at $i$ (just sketch it, no formula needed!);
(d) a triangle with all three vertices at the absolute (such a triangle is called *ideal*).

**Solution:**




13.3. Prove SSS, ASA and SAS theorems of congruence of triangles on hyperbolic plane.

**Solution: 1. SSS:** Let $A_1B_1C_1$ and $A_2B_2C_2$ be two hyperbolic triangles satisfying SSS (i.e. having the same side lengths). First we apply an isometry which takes $A_2$ to $A_1$ and $B_2$ to $B_1$. Then the points $C_1$ and $C_2$ lie on the intersection of (hyperbolic) circles $\gamma_A$ (centred at $A_1$ of radius $A_1C_1$) and $\gamma_B$ (centred at $B_1$ of radius $B_1C_1$). Consider these circles in Poincaré disc or half-plane model. Since hyperbolic circles are represented by Euclidean circles, two circles have at most two intersection points. Moreover, the two intersection points are symmetric with respect to the (hyperbolic) line $A_1B_1$ Hence, there is an isometry which takes $A_2B_2C_2$ to $A_1B_1C_1$.

**2. ASA:** Suppose that $A_1B_1 = A_2B_2$ $\angle A_1 = \angle A_2$ and $\angle B_1 = \angle B_2$. Apply an isometry which takes s $A_2$ to $A_1$ and $B_2$ to $B_1$. Then $C_1$ and $C_2$ lie on a (hyperbolic) ray starting from $A_1$

and making angle $\angle A_1$ with $A_1B_1$. There exists exactly one such a ray in each half-plane with respect to the line $A_1B_1$ (this is especially clear if $A_1$ is the centre of the Poincaré disc model). Similarly, $C_1$ and $C_2$ lie on a (hyperbolic) ray starting from $B_1$ and making angle $\angle B_1$ with $A_1B_1$. As two rays have at most one intersection, we get at most one possibility for the point $C_1$ and $C_2$ in each of two half-planes, also these two possibilities are symmetric with respect to $A_1B_1$. Hence, $A_2B_2C_2$ may be transformed to $A_1B_1C_1$ by an isometry.

**3. SAS:** Suppose that $A_1B_1 = A_2B_2$, $\angle A_1 = \angle A_2$ and $A_1C_1 = A_2C_2$. First, map the angle $\angle A_2$ to $\angle A_1$, then the points $C_1$ and $C_2$ lie on the given distances on the given lines.

**Remark.** This proofs concern triangles with finite sides. For the discussion concerning triangles with one or more vertices at the absolute, see Problems Class 7.

13.4. Let $ABC$ be a triangle. Let $B_1 \in AB$ and $C_1 \in AC$ be two points such that $\angle AB_1C_1 = \angle ABC$. Show that $\angle AC_1B_1 > \angle ACB$.

**Solution:** Consider the quadrilateral $B_1BCC_1$. If $\angle AC_1B_1 \leq \angle ACB$ then the sum of angles of $B_1BCC_1$ is greater or equal to $2\pi$. On the other hand, we can divide $B_1BCC_1$ by a diagonal into two triangles, each having a sum of angles less than $\pi$. The contradiction proves $\angle AC_1B_1 > \angle ACB$.

13.5. Show that there is no "rectangle" in hyperbolic geometry (i.e. no quadrilateral has four right angles).

**Solution:** Suppose there is a rectangle $ABCD$. Then its sum of angles is $2\pi$. Decompose it into two triangles by a diagonal $AC$. The sum of angles of $ABC$ is smaller than $\pi$, the sum of angles of $ACD$ is smaller than $\pi$ but the sum of these two sums of angles equal to the sum of angles of $ABCD$. Contradiction.

13.6. (*) Given an acute-angled polygon $P$ (i.e. a polygon with all angles smaller or equal to $\pi/2$) and lines $m$ and $l$ containing two disjoint sides of $P$, show that $l$ and $m$ are ultra-parallel.

**Solution:** Let $A_1A_2\dots A_n$ be an acute-angled $n$-gon $P$.

First, let as prove that the lines containing two "almost adjacent" rays $A_1A_2$ and $A_4A_3$ are disjoint: i.e. suppose $B = A_1A_2 \cap A_4A_3$. Then the triangle $A_2A_3B$ has at least two non-acute angles, which contradicts to the fact that the angle sum of a hyperbolic triangle is less than $\pi$.

Similarly, assuming that the rays $A_1A_2$ and $A_5A_4$ do intersect, we see a (non-convex) quadrilateral, with two non-acute angles and one angle bigger than $\pi$, which again contradicts the angle sum of hyperbolic quadrilateral (which should be less than $2\pi$, as a quadrilateral can be decomposed to two triangles).

More generally, an intersection of the rays $A_1A_2$ and $A_kA_{k-1}$, $k \geq 4$, will result in an $(k-1)$-gon breaking the angle sum theorem. In more details, if $B = A_1A_2 \cap A_kA_{k-1}$ then the polygon $P' = BA_2A_3\dots A_{k-1}$ has $k-1$ vertices (so, should have area at most $(k-3)\pi$), with obtuse angles $\angle A_2$ and $\angle A_{k-1}$ and $k-4$ angles $(\angle A_3, \angle A_4, \dots, \angle A_{k-2})$ each of the size $3\pi/2$ at least. Which implies that the sum of angles of $P'$ satisfies

$$\sum \alpha_i \geq 2 \cdot \frac{\pi}{2} + (k-4) \cdot \frac{3\pi}{2} = \pi + (k-3)\pi + (k-3)\frac{\pi}{2} - \frac{3\pi}{2} = (k-3)\pi + (k-4)\frac{\pi}{2} > (k-3)\pi,$$

in contradiction to the previous conclusion that the sum of angles of $P'$ does not exceed $(k-3)\pi$.

14.7. Given $\alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma < \pi$, show that there exists a hyperbolic triangle with angles $\alpha, \beta, \gamma$.

**Solution:** Put a vertex $A$ of angle $\alpha$ in the centre of the Poincaré disc model. Let $AX$ and $AY$ be the rays forming the angle. Let $T$ be a point on $AX$. Let $TZ$ be a ray emanating from $T$ and such that $\angle ZTA = \beta$. Consider the point $C(T) = TZ \cap AY$. When $T$ is very close to $A$ the hyperbolic line $TZ$ is very close to a Euclidean line through the same points, so the hyperbolic sum of angles of triangle $ATC(T)$ is very close to $\pi$. As $T$ runs away from $A$ to $Y$, angle $AC(T)T$ became smaller and smaller (see Question 13.4), then it tends to zero (when $TZ$

is parallel to $AY$) and disappears. By the continuity we see that for some intermediate point $T_0$ the sum of angles of $AT_0C(T_0)$ coincides with $\alpha+\beta+\gamma$ which implies that $\angle AC(T_0)T_0=\gamma$ and $AT_0C(T_0)$ is the required triangle.

14.8. Show that there exists a hyperbolic pentagon with five right angles.

**Solution:** Consider a Euclidean regular pentagon $P_{Eucl}$. Draw $P_{Eucl}$ so that the centre of $P_{Eucl}$ coincides with the centre $O$ of Poincaré disc model. Consider the hyperbolic pentagon $P$ spanned by the vertices of $P_{Eucl}$. When $P_{Eucl}$ is very small (but still centred at $O$) the angle of $P$ are almost the same as the angles of $P_{Eucl}$. When $P_{Eucl}$ is inscribed into the absolute the angles of $P$ are zero (as the adjacent sides of $P$ are tangent). Notice that the angles of $P_{Eucl}$ are obtuse (more precisely, they are equal to $3\pi/5$). So, by continuity we see that there exist some intermediate size of $P_{Eucl}$ such that the corresponding hyperbolic regular pentagon has right angles.

14.9. (*) An *ideal* triangle is a hyperbolic triangle with all three vertices on the absolute.

(a) Show that all ideal triangles are congruent.
(b) Show that the altitudes of an ideal triangle are concurrent.
(c) Show that an ideal triangle has an inscribed circle.

**Solution:**

(a) There exists a hyperbolic isometry which takes any given triple of points on the absolute to any other triple. So, it takes a triangle spanned by the given three points to the triangle spanned by the other three points.
(b) Part (a) implies that any ideal triangle may be represented by a "regular" ideal triangle in the Poincaré disc model (i.e. by a triangle with vertices $1, e^{i\pi/3}, e^{2i\pi/3}$). The symmetry shows that all altitudes of this triangle pass through the centre $O$ of the model, so, are concurrent.
(c) Similarly to part (b), looking at the "regular" representative, we can see that there is an inscribed circle centred at $O$ (we can take a very small circle and start to blow it up till it will touch one of the sides; by symmetry reasons it will touch all other sides at the same time).

14.10 (*) We have proved that an isometry fixing 3 points of the absolute is identity map. How many isometries fix two points of the absolute?

**Solution:** We will work in the upper half-plane model. Let $f$ be an isometry fixing two points of the absolute. First, we can conjugate $f$ by an isometry $h$ which takes the fixpoints of $f$ to 0 and $\infty$. Then $h^{-1}\circ f\circ h$ fixes 0 and $\infty$. Moreover, for every isometry $f'$ fixing the same two points as $f$, the isometry $h^{-1}\circ f'\circ h$ fixes 0 and $\infty$. This implies that for answering the question it is sufficient to consider isometries fixing 0 and $\infty$.

Now, any orientation-preserving isometry of the upper half-plane may be written as $\frac{az+b}{cz+d}$ with real $a,b,c,d$. Preserving 0 and $\infty$ means $b=0$ and $c=0$, so any orientation-preserving isometry fixing 0 and $\infty$ writes as $az$, $a\in\mathbb{R}^+$. Hence, we get 1-parametric family of orientation-preserving isometries (all hyperbolic).

Similarly, we obtain a one-parametric family of orientation-reversing isometries $-a\bar{z}$, $a\in\mathbb{R}^+$.

So, we obtained two one-parametric (i.e. infinite) families of isometries fixing 0 and $\infty$, which also implies that there are two one-perametric families of isometries fixing any other pair of points.

**Remark.** Aleternatively, one can show that the general pair of points is fixed by infinitely many isometries in the following way. Let $A, B$ be two points of the absolute. And let $X\in\partial H^2$ be a third point. Then by triple transitivity of isometries on the points of the absolute, there exists an isometry which preserves $A$ and $B$ and takes $X$ to $Y$ for any given $Y\in\partial H^2$. Also, any tow such isometries are different (as mapping $X$ to two different points). So there are infinitely many isometries preserving $A$ and $B$.

14.11 (a) Show that the group of isometries of hyperbolic plane is generated by reflections.

(b) How many reflections do you need to map a triangle $ABC$ to a congruent triangle $A'B'C'$?

**Solution:** We can do (a) and (b) simultaneously, using the same procedure as in $\mathbb{E}^2$ or $S^2$:
first apply reflection $r_1$ to take $A$ to $A'$ (with respect to the perpendicular bisector);
then use the reflection in $A'M$ (where $M$ is a midpoint of $B'r(B)$);
last, if needed, the reflection in $A'B'$.
So, we need at most 3 reflections.

14.12 (*)

(a) Does there exist a regular triangle in hyperbolic plane?

(b) Does there exist a right-angled regular polygon in hyperbolic plane? How many edges does it have (if exists)?

**Solution:**

(a) In the Poincaré disc model, consider a triangle whose vertices are represented by vertices of a regular Euclidean triangle with centre at $O$. By symmetry, this triangle is also a regular hyperbolic triangle (in other words, all Euclidean isometries we would use to check that the Euclidean triangle is regular are also isometries of the hyperbolic plane).

(b) The same construction as in (a) shows that there is a regular $n$-gon for all integer $n \geq 3$ in hyperbolic plane. When we make this $n$-gon very small, its sides are almost Euclidean lines, so its angles are almost the same as the angles of a regular Euclidean $n$-gon, i.e. $(n-2)\pi/n$. When the regular Euclidean $n$-gon grows, the angles of the corresponding hyperbolic $n$-gon decrease monotonically (to see this use Question 13.4), when all vertices of the $n$-gon are on the absolute, the angles are 0. So, the angles take every intermediate value between $(n-2)\pi/n$ and 0. In particular, if $n > 4$ then $(n-2)\pi/n > \pi/2$, which implies that there is a right-angled $n$-gon for every $n > 4$. We also know (from the sum of angles) that there are no right angled triangles and quadrilaterals.

14.13 (a) Show that the angle bisectors in a hyperbolic triangle are concurrent.

(b) Show that every hyperbolic triangle has an inscribed circle.

(c) Does every hyperbolic triangle have a circumscribed circle?

**Solution:**

(a) Similarly to Euclidean/spherical cases, an angle bisector is a locus of points on the same distance from the rays forming the angle (this is most clear if you put the vertex of the angle at the centre of the Poincare disc model). So, the intersection point of two angle bisectors lies on the same distance from all three sides of the triangle, which implies that it actually lies on the third angle bisector.

(The intersection point does exists since the two ends of one angle bisector -say $AA_1$ -lie on two different sides of the other angle: $A \in BA$, $A_1 \in BC$, so they are separated by the angle bisector $BB_1$).

(b) Blowing the small circle centred at the point of intersection of angle bisectors, we obtain at some point a circle tangent to all three sides.

(c) The vertices of the hyperbolic triangle (in any of two Poincaré models) are represented by vertices of Euclidean triangle. Consider a Euclidean circle $\gamma$ passing through the vertices of this Euclidean triangle (it does exist by E15). If $\gamma$ lies entirely inside the hyperbolic plane (i.e. in the disc or in the upper half-plane) then it represents some hyperbolic circle passing through the given points. However, the circle $\gamma$ may intersect the boundary of hyperbolic plane. Then it does not represent any hyperbolic circle. Moreover, in the latter case no hyperbolic circle passes through the given points, as each hyperbolic circle is represented by some Euclidean circle.