Solutions 15-16

15.1. (*)

(a) Let P and Q be feet of the altitudes in an ideal hyperbolic triangle. Find PQ.

(b) Find the radius of a circle inscribed into an ideal hyperbolic triangle.

(c) Show that a radius of a circle inscribed into a hyperbolic triangle does not exceed $arcosh(2/\sqrt{3})$.

Solution.

(a) We will compute in the upper half-plane model. Let $X = 0$, $Y = 1$, $Z = \infty$ (we can assume that as all ideal triangles are congruent). By symmetry reasons, we can also assume that P and Q lie on XZ and YZ respectively. The (hyperbolic) line through Y orthogonal to XZ is represented by an arc of the unit circle, so $P = i$. Similarly, $Q = i + 1$. Hence,

$$\cosh d(P,Q) = 1 + \frac{1}{2 \cdot 1 \cdot 1} = \frac{3}{2}$$

and $d(P,Q) = arcosh(\frac{3}{2})$.

(b) The incentre I of the ideal triangle is the intersection of three altitudes (this is especially clear if we place the ideal triangle in the Poincaré disc so that the vertices form a regular Euclidean triangle). One of the altitudes is the unit circle, another is the line $x = 1/2$. So, $I = e^{i\pi/3}$. The required radius r is the distance from I to (any) foot of an altitude, say to $R = (1+i)/2$. Hence,

$$\cosh r = 1 + \frac{(\frac{\sqrt{3}}{2} - \frac{1}{2})^2}{2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}} = 1 + \frac{3 + -2\sqrt{3} + 1}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

and $r = arcosh(\frac{2}{\sqrt{3}})$.

(c) We will show that any triangle ABC may be enclosed into some ideal triangle. Notice that the incircle is the largest circle sitting inside the given triangle. So, the radius of the incircle of ABC does not exceed the radius of the incircle of the ideal triangle (which is $arcosh(2/\sqrt{3})$, as computed in (b)).

Let $X, Y \in \partial H^2$ be the endpoints of the line AB, and let $Z \in \partial H^2$ be the second endpoint of the line XC. Then ABC lies inside the ideal triangle XYZ.

15.2. For a right hyperbolic triangle ($\gamma = \frac{\pi}{2}$) show:

$$\text{(a)} \tanh b = \tanh c \cos\alpha, \qquad \text{(b)} \sinh a = \sinh c \sin\alpha.$$

Solution. We will use the same notation as in the proof of Theorem 6.21 (Pythagorean Theorem), see Fig. 1. Also, we will use the values $\cosh b = \frac{1+k^2}{2k}$ and $\cosh c = \frac{1+k^2}{2k \sin\varphi}$ computed in the proof of Theorem 6.21.

First, we show

$$\sin^2\alpha = \frac{4k^2\cos^2\varphi}{(k+1)^2 - 4k^2\sin^2\varphi} = \frac{4k^2\cos^2\varphi}{(k^2-1)^2 + 4k^2\cos^2\varphi}. \tag{1}$$

Let $X = (x_0, 0)$ be the (Euclidean) centre of the (Euclidean) circle representing the hyperbolic line AB. Then $\alpha = \angle AXO$ (as XA is a radius, so is orthogonal to the circle and the horizontal line XO is orthogonal to the vertical line AC). So,

$$\sin^2\alpha = \sin^2\angle AXO = \frac{k^2}{k^2 + x_0^2}.$$

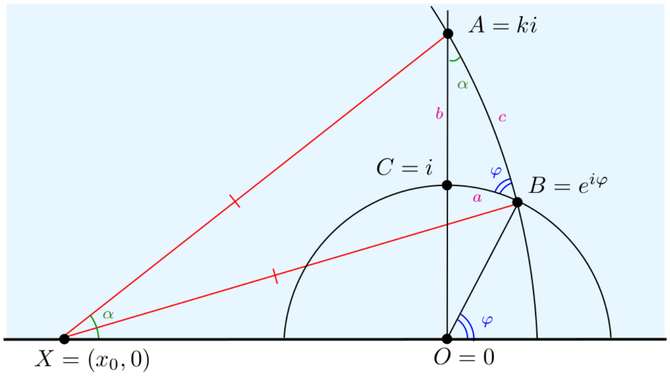


Figure 1: Notation for Quesion 15.2.

To find x_0, notice that $XA = XB$ (as X is the centre of the circle), which implies

$$x_0^2 + k^2 = (\cos^2\varphi - x_0)^2 + \sin^2\varphi \qquad \Leftrightarrow \qquad k^2 = 1 - 2x_0\cos\varphi,$$

i.e.

$$x_0 = \frac{1-k^2}{2\cos\varphi}.$$

Hence,

$$\sin^2\alpha = \sin^2\angle AXO = \frac{k^2}{k^2+x_0^2} = \frac{k^2}{k^2 - (\frac{k^2-1}{2\cos\varphi})^2} = \frac{4k^2\cos^2\varphi}{(k^2-1)^2 + 4k^2\cos^2\varphi} = \frac{4k^2\cos^2\varphi}{(k^2+1)^2 - 4k^2 + 4k^2\cos^2\varphi} = \frac{4k^2\cos^2\varphi}{(k^2+1)^2 - 4k^2\sin^2\varphi}$$

(a) Using the expressions for $\cosh b$ and $\cosh c$ we get respectively

$$\tanh^2 b = \frac{\sinh^2 b}{\cosh^2 b} = \frac{\cosh^2 b - 1}{\cosh^2 b} = 1 - \frac{1}{\cosh^2 b} = 1 - \frac{4k^2}{(1+k^2)} = \left(\frac{1-k^2}{1+k^2}\right)^2$$

and

$$\tanh^2 c = 1 - \frac{1}{\cosh^2 c} = 1 - \frac{4k^2\sin^2\varphi}{(1+k^2)^2} = \frac{(1-k^2)^2 + 4k^2\cos^2\varphi}{(1+k^2)^2}.$$

On the other hand,

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{4k^2\cos^2\varphi}{(k^2-1)^2 + 4k^2\cos^2\varphi} = \frac{(k^2-1)^2}{(k^2-1)^2 + 4k^2\cos^2\varphi},$$

which clearly satisfies the required identity $\tanh^2 b = \tanh^2 c \cos^2\alpha$.

(b) Similarly, using the expressions for $\cosh a$ and $\cosh c$ we get respectively

$$\sinh^2 a = \cosh^2 a - 1 = \frac{1}{\sin^2\varphi} - 1 = \frac{\cos^2\varphi}{\sin^2\varphi}$$

and

$$\sinh^2 c = \cosh^2 c - 1 = (\frac{1+k^2}{2k\sin\varphi})^2 - 1 = \frac{(k^2+1)^2 - 4k^2\sin^2\varphi}{4k^2\sin^2\varphi}.$$

Hence, comparing to (**??**), we get $\sinh a = \sinh c \sin\alpha$.

15.3. Show that in the upper half-plane model the following distance formula holds:

$$2\sinh^2\frac{d}{2}=\frac{|z-w|^2}{2Im(z)Im(w)}$$

Solution. $\sinh^2\frac{d}{2}=(\frac{e^{d/2}+e^{-d/2}}{2})^2=\frac{e^d+e^{-d}-2}{4}=\frac{1}{2}(\cosh d-1)=\frac{1}{2}\frac{|z-w|^2}{2Im(z)Im(w)}$.

15.4. Find an area of a right-angled hyperbolic pentagon.

Solution. Subdividing the pentagon into 3 triangles, we see that $S=3\pi-5\frac{\pi}{2}=\frac{\pi}{2}$.

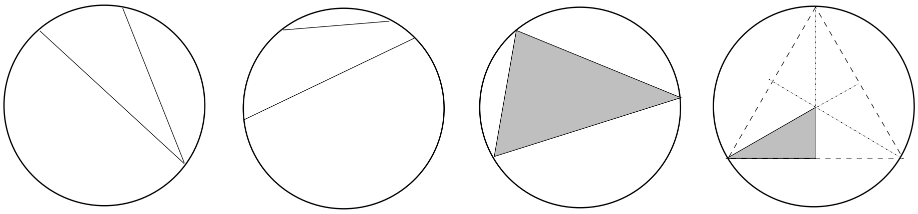
15.5. In the upper half-plane model, find the locus of points z lying on distance d from the line 0∞.

Solution. Consider the isometry $z\to kz$, for $k>0$ of the upper half-plane model. Let z_0 be a point on distance d from 0∞. Then every point kz_0 lies on the same distance from 0∞. So, we get a (Euclidean) ray lying in the locus. Now, applying reflection with respect to the imaginary axis $z\to-\bar{z}$, we see that the locus contains also all points on another Euclidean ray $-k\bar{z}_0$.

Let us prove now that the locus contained no other points except the two rays described above. The distance from a point A to a line l is the length of the segment AH perpendicular to l, $H\in l$. Clearly, each line perpendicular to l contains exactly two points on the given distance d from l (one point in each half-plane). All lines perpendicular to 0∞ are represented by circles centred in 0, and each of them intersects each of the two rays. So, there are no other points in the locus.

16.1. In the Klein disc model draw two parallel lines, two ultra-parallel lines, an ideal triangle, a triangle with angles $(0,\frac{\pi}{2},\frac{\pi}{3})$.

Solution.



16.2. (*) Show that three altitudes of a hyperbolic triangle either have a common point or are pairwise parallel or there is a unique line orthogonal to all three altitudes.

Solution. Without loss of generality we may assume that A is the centre of the Klein disc and B and C are any two other points in $\overline{\mathbb{H}}^2$ Let AH_a, BH_b and CH_c be the (Euclidean) altitudes of the Euclidean triangle with vertices A,B,C. Then AH_a, BH_b and CH_c are also (hyperbolic) altitudes of hyperbolic triangle ABC. Indeed, $AH_a\perp BC$ since AH_a lie on a diameter of the disc, $BH_b\perp AC$ and $CH_c\perp AB$ since AC and AB lie on the diameters of the disc.

Being altitudes of a Euclidean triangle, the lines AH_a, BH_b and CH_c have a common point T, however, T does not necessarily belongs to the disc. If T lies in the disc, the altitudes of ABC have a common point. If T lies on the boundary of the disc then the altitudes of ABC are pairwise parallel. If T lies outside the disc then there exists a unique (hyperbolic) line l orthogonal to all three altitudes (to find this line l consider the (Euclidean) lines t_1 and t_2 passing through T and tangent to the boundary of the disc, then l is the line through the points $t_1\cap\partial\mathbb{H}^2$ and $t_2\cap\partial\mathbb{H}^2$).

16.3. Let u,v be two vectors in $\mathbb{R}^{2,1}$. Denote $Q=|\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}|$, where $\langle x,y\rangle=x_1y_1+x_2y_2-x_3y_3$. Show the following distance formulae:

(a) if $\langle u,u\rangle<0$, $\langle v,v\rangle<0$, then u and v give two points in $\mathbb{H}^2$, and $\cosh^2(u,v)=Q$.

(b) if $\langle u,u\rangle<0$, $\langle v,v\rangle>0$, then u gives a point and v give a line l_v on $\mathbb{H}^2$, and $\sinh^2 d(u,l_v)=Q$.

(c) if $\langle u,u\rangle > 0$, $\langle v,v\rangle > 0$ then u and v define two lines l_u and l_v on $\mathbb{H}^2$ and
  - if $Q<1$, then l_u intersects l_v forming angle φ satisfying $\quad Q=\cos^2\varphi$;
  - if $Q=1$, then l_u is parallel to l_v;
  - if $Q>1$, then l_u and l_v are ultra-parallel lines satisfying $\quad Q=\cosh^2 d(l_u,l_v)$.

Solution. We will compute in the hyperboloid model. Moreover, we will use isometry group to reduce the question to a 2-dimensional one.

(a) By transitivity of isometry group on $\mathbb{H}^2$ we may assume $u=(0,0,1)$. Applying a rotation around this point (in 3-dimensional space it is represented by a rotation around the third coordinate axis) we may assume that $v=(v_1,0,v_3)$, $v_1^2-v_3^2=-1$. We will also assume $v_1>0$.

We find $d(u,v)$ by definition, as a cross-ratio of four lines.

The line (plane in the model) through u and v has the equation $x_2=0$, i.e. it is the line $\langle x,a\rangle=0$ the vector $a=(0,1,0)$. This line intersects the absolute at the points $\langle x,x\rangle=0$, $x_2=0$, i.e. in $x_1^2-x_3^2=0$ which gives two solutions for $x_3>0$: $X=(-1,0,1)$ and $Y=(1,0,1)$.To find the distance $d(u,v)$ we need to find a cross-ratio of four lines spanned by u,v,X and Y.

To find the cross-ratio of four lines we need to intersect all four lines by some line l (and the result does not depend on the choice of l!). Choose l to be the horizontal line through $(0,0,1)$ (it is given by equations $x_3=1$, $x_2=0$). Renormalizing $v=(v_1,0,v_3)$ so that it belongs to the plane $x_3=1$ we get $v'=(\frac{v_1}{v_3},0,1)$. So, using the line $x_3=1$, $x_2=0$ we get

$$\big|[u,v,Y,X]\big| = \big|[0,\frac{v_1}{v_3},1,-1]\big| = \big|\frac{1-0}{1-v_1/v_3}/\frac{-1-0}{-1-\frac{v_1}{v_3}}\big| =$$

$$= \big|\frac{v_1+v_3}{v_1-v_3}\big| = \big|\frac{(v_1+v_3)^2}{v_1^2-v_3^2}\big| = (v_1+v_3)^2,$$

so that

$$d(u,v)=\frac{1}{2}\big|ln|[u,v,X,Y]|\big| = \frac{1}{2}ln(v_1+v_3)^2 = ln(v_1+v_3),$$

which implies $e^d=v_1+v_3$, and

$$\cosh d = \frac{v_1+v_3+\frac{1}{v_1+v_3}}{2} = \frac{v_3+\frac{1+v_1(v_1+v_3)}{v_1+v_3}}{2} = \frac{v_3+\frac{1+v_1^2+v_1v_3)}{v_1+v_3}}{2} = \frac{v_3+\frac{v_3^2+v_1v_3)}{v_1+v_3}}{2} = v_3$$

On the other hand,

$$\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle} = \frac{v_3^2}{(-1)(-1)} = v_3^2.$$

Thus,

$$\cosh^2 d(u,v) = \big|\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}\big|.$$

(b) Let $t\in l_v$ be an orthogonal projection of u to l_v, i.e. the line tu is perpendicular to l_v. Clearly, $d(u,l_v)=d(u,t)$.

Without loss of generality we may assume that $u=(0,0,1)$ and $t=(t_1,0,t_3)$, $t_1^2-t_3^2=-1$ By part (a),

$$\cosh^2 d(u,l_v) = \cosh^2 d(u,t) = \big|\frac{t_3^2}{(-1)(-1)}\big| = t_3^2.$$

Therefore,

$$\sinh^2 d(u,l_v) = \cosh^2 d(u,l_v) - 1 = t_3^2-1 = t_1^2.$$

Now, let us find the equation for the line l_v. The line tu corresponds to the plane given by the equation $x_2=0$. The whole pattern (i.e. hyperboloid, the point u, the line l_v the line tu) is symmetric with respect to this plane. Hence, the vector v defining the line l_v has zero second coordinate $v_2=0$, which implies $v=(v_1,0,v_3)$. Since the line l_v contains the point $t=(t_1,0,t_3)$, we have $\langle v,t\rangle=0$, i.e. $v_1t_1-v_3t_3=0$. This implies $v=\lambda(t_3,0,t_1)$, or simply $v=(t_3,0,t_1)$ after rescaling $\langle v,v\rangle=1$. Hence,

$$\big|\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}\big| = \big|\frac{t_1^2}{(-1)\cdot 1}\big| = t_1^2,$$

which coincides with the value of $\sinh^2 d(u,l_v)$.

(c)
- $\underline{Q < 1}$. Applying an isometry, we may assume that the point of intersection of l_u and l_v is $(0, 0, 1)$. Then the planes through the origin representing the lines l_u and l_v are vertical planes (passing through the third coordinate axis), these planes are represented by vectors $(u_1, u_2, 0)$, $(v_1, v_2, 0)$ (to see that notice, that the vertical planes are symmetric with respect to the plane $x_3 = 0$). Furthermore, due to the rotational symmetry, the angles at the point $(0, 0, 1)$ are Euclidean angles, i.e. ϕ (or $\pi - \phi$) coincides with the angle between $(u_1, u_2, 0)$ and $(v_1, v_2, 0)$. By Euclidean formula for computation of angles we get

  $$\cos\phi = \pm\frac{\langle u, v\rangle}{\sqrt{\langle u, u\rangle\langle v, v\rangle}}$$

  (we may use pseudo-scalar product $\langle\cdot,\cdot\rangle$ in a Euclidean formula since the third coordinate is zero).
- $\underline{Q > 1}$. Let h be a line orthogonal to both l_u and l_v. Let $h_u = h \cap l_u$ and $h_v = h \cap l_v$ be the intersection points. Then $d(l_u, l_v) = d(h_u, h_v)$.
  Without loss of generality we may assume $h_u = (0, 0, 1)$ and $h_v = (t_1, 0, t_3)$, $t_1^2 - t_3^2 = 1$ (so that h corresponds to the plane $x_2 = 0$), see Fig. 3 for the projection to the plain $x_3 = 1$. Then l_u and l_v are represented by the vectors $u = (1, 0, 0)$ and $v = (t_3, 0, t_1)$ (since $\langle h_v, v\rangle = 0$ and $v_2 = 0$). This implies that

  $$\cosh^2 d(h_u, h_v) = \left|\frac{\langle h_u, h_v\rangle^2}{\langle h_u, h_u\rangle\langle h_v, h_v\rangle}\right| = \frac{t_3}{|t_1^2 - t_3^2|} = \left|\frac{\langle u, v\rangle^2}{\langle u, u\rangle\langle v, v\rangle}\right|,$$

  This proves the theorem since $d(l_u, l_v) = d(h_u, h_v)$.
- $\underline{Q = 1}$. The result for this case follows from two previous ones by continuity.

16.4. (*) Consider the two-sheet hyperboloid model $\{u = (u_1, u_2, u_3) \in R^{2,1} \mid \langle u, u\rangle = -1,\ u_3 > 0\}$, where $\langle u, u\rangle = u_1^2 + u_2^2 - u_3^2$.

(a) For the vectors

$$\begin{array}{lll} v_1 = (2,1,2) & v_2 = (0,1,2) & v_3 = (3,4,5) \\ v_4 = (1,0,0) & v_5 = (0,1,0) & v_6 = (1,1,2) \end{array}$$

decide if v_i corresponds to a point in $\mathbb{H}^2$, or a point in the absolute, or a line in $\mathbb{H}^2$.

(b) Find the distance between the two points of $\mathbb{H}^2$ described in (a).

(c) Which pair the lines in (a) is intersecting? Which lines are parallel? Which are ultra-parallel?

(d) Find the distance between the pair of ultra-parallel lines in (a).

(e) Does any of the points in (a) lie on any of the three lines?

(f) Find the angle between the pair of intersecting lines.

Solution.

(a) We need to check $\langle v_i, v_i\rangle$: if it is negative, v_i corresponds to a point of hyperbolic plane, if it is equal to zero, v_i is a point of the absolute, if it is positive, then v_i corresponds to a line (more precisely, it is a normal vector to plane through $(0, 0, 0)$ which determines a line in the model).

$$\begin{array}{ll} \langle v_1, v_1\rangle = 4 + 1 - 4 = 1 > 0, & \text{line;} \\ \langle v_2, v_2\rangle = 0 + 1 - 4 = -3 < 0, & \text{point} \\ \langle v_3, v_3\rangle = 9 + 16 - 25 = 0, & \text{point of the absolute;} \\ \langle v_4, v_4\rangle = 1 + 0 - 0 = 1 > 0, & \text{line;} \\ \langle v_5, v_5\rangle = 0 + 1 - 0 = 1 > 0, & \text{line;} \\ \langle v_6, v_6\rangle = 1 + 1 - 4 = -2 < 0, & \text{point.} \end{array}$$

(b)

$$\cosh^2(d(v_2, v_6)) = \frac{\langle v_2, v_6\rangle^2}{\langle v_2, v_2\rangle\langle v_6, v_6\rangle} = \frac{(0 + 1 - 4)^2}{(-3)(-2)} = \frac{9}{6} = \frac{3}{2}.$$

So, $d(v_2, v_6) = \text{arcCosh}\sqrt{\frac{3}{2}}$.

(c) $|\frac{\langle v_1,v_4\rangle^2}{\langle v_1,v_1\rangle\langle v_4,v_4\rangle}| = \frac{4}{1\cdot 1} = 4 > 1$, so, v_1 and v_4 are ultra-parallel lines.
$|\frac{\langle v_1,v_5\rangle^2}{\langle v_1,v_1\rangle\langle v_5,v_5\rangle}| = \frac{1}{1\cdot 1} = 1$, so, v_1 is parallel to v_5.
$|\frac{\langle v_4,v_5\rangle^2}{\langle v_4,v_4\rangle\langle v_5,v_5\rangle}| = \frac{0}{1\cdot 1} = 0 < 1$, so, v_4 intersects v_5.

(d) $\cosh^2(d) = ||\frac{\langle v_1,v_4\rangle^2}{\langle v_1,v_1\rangle\langle v_4,v_4\rangle}| = 4$, so, $d = \mathrm{arcCosh}\,2$.

(e) A point v_i lies on a line v_j if and only if $\langle v_i, v_j\rangle = 0$.
This holds for the point v_2 and the line v_4.
This also holds for the point of the absolute v_3 and the line v_1.

(f) $\cos^2\alpha = |\frac{\langle v_4,v_5\rangle^2}{\langle v_4,v_4\rangle\langle v_5,v_5\rangle}| = 0$, so, the lines are orthogonal.