Solutions 17-18

17.1. Prove that any pair of parallel lines can be transformed to any other pair of parallel lines by an isometry.

Solution. Consider a pair of parallel lines l_1 and l_2 in the upper half-plane model. Let X be the common point (lying at the absolute) of these lines and Y_1 and Y_2 be other endpoints of these lines. By triple transitivity of isometries on points of the absolute, we can see that X, Y_1Y_2 may be mapped to the endpoints of any other pair of parallel lines.

Remark: another option is just to look at these line in the upper half-plane, assuming $X = \infty$.

17.2. Let $A, B \in \gamma$ be two points on a horocycle γ. Show that the perpendicular bisector to AB is orthogonal to γ.

Solution. Consider the situation in the upper half-plane model, and let ∞ be the centre of the horocycle. Then γ is represented by a horizontal line, the perpendicular bisector to AB is represented by a vertical ray, which is obviously orthogonal to the horocycle.

17.3. Let l_1, l_2, l_3 be three lines in $\mathbb{H}^2$, let r_i be the reflection with respect to l_i and let $f = r_3 \circ r_2 \circ r_1$. Show that f is either a reflection or a glide reflection, i.e. a hyperbolic translation along some line composed with a reflection with respect to the same line.

Assuming that the lines l_1, l_2, l_3 are not passing through the same point and not having a common perpendicular, show that f is a glide reflection.

Solution.

Step 1. Consider first the restriction of f to the absolute (parameterised by the angle $\varphi \in [0, 2\pi)$). As f is orientation-reversing, the function $f(\varphi)$ (considered modulo 2π) is monotonically decreasing. Hence, there are exactly two points where $f(\varphi) = \varphi$ (the intersection points of the graph of f with the diagonal). This implies that f preserves 2 points of the absolute.

Now, in question 14.10 we have already classified all isometries preserving two points of the absolute. In particular, for the orientation-reversing case we have seen that there is a one-parametric family of such isometries, and that in the upper half-plane (with 0 and ∞ fixed) it may be written as $-a\bar{z}$, $a \in R_+$. Notice that this is a composition of a hyperbolic translation along 0∞ and a reflection with respect to the same line.

Step 2. Now, we need to show that the hyperbolic translation mentioned above is non-trivial (not id) whenever the lines l_1, l_2, l_3 are having neither common point nor common perpendicular.

Suppose the contrary, i.e. that f is a reflection r with respect to a line l, i.e. $r_3 \circ r_2 \circ r_1 = r$. This implies that $r_3 \circ r_2 = r \circ r_1$. If $l_3 \cap l_2 \neq \emptyset$ (i.e. some point X either in $\mathbb{H}^2$ or in $\partial\mathbb{H}^2$), then the point X is preserved by $r_3 \circ r_2$, and hence is preserved by $r \circ r_1$. This implies that X is a common point of all four lines l_1, l_2, l_3 and l, which contradicts to the assumption that the lines l_1, l_2, l_3 have no common point. If $l_3 \cap l_2 = \emptyset$ then l_3 and l_4 have a comon perpendicular $l^\perp$ which is preserved by both r_3 and r_2, and hence is preserved by $r_3 \circ r_2$. Therefore, $l^\perp$ is also preserved by $r \circ r_1$. This implies that $l^\perp$ is a common perpendicular for l and l_1. So, $l^\perp$ is perpendicular to each of l_1, l_2, l_3 and l, which contradicts to the assumption that the lines l_1, l_2, l_3 have no common perpendicular.

17.4. Given an isometry f of the hyperbolic plane such that the distance from A to $f(A)$ is the same for all points $A \in \mathbb{H}^2$, show that f is an identity map.

Solution. If f is not an identity, then, by classification of isometries, it is either a reflection, or a rotation, or a parabolic translation, or a hyperbolic translation, or a glide reflection. For each of these transformations we will show that there are points mapped to arbitrarily large distance.

Indeed, let l be a line with endpoints X and Y such X is not preserved by f (this is possible as a non-trivial isometry cannot preserve more than two points of the absolute by Corollary 6.16). Let $X' = f(X)$. Consider a point $T = T_t$ running along l from Y to X when t runs from $-\infty$ to ∞.

Then the distance $d(f(T_t), T_t)$ tends to $d(f(X), X)$ as a continuous function of t, but as $X \neq f(X)$ are two points of the absolute, $d(f(X), X) = \infty$. So, for every constant C there is a point T_t such that $d(f(T_t), T_t) > C$. So, a non-trivial isometry cannot move all points by the same distance.

17.5. Let a and b be two vectors in the hyperboloid model such that $\langle a, a\rangle > 0$ and $\langle b, b\rangle > 0$. Let l_a and l_b be the lines determined by equations $\langle x, a\rangle = 0$ and $\langle x, b\rangle = 0$ respectively. And let r_a and r_b be reflections with respect to l_a and l_b.

(a) For $a = (0, 1, 0)$ and $b = (1, 0, 0)$ write down r_a and r_b.
Find $r_b \circ r_a(v)$, where $v = (0, 1, 2)$.

(b) What type is the isometry $\phi = r_b \circ r_a$ for $a = (1, 1, 1)$ and $b = (1, 1, -1)$?
(Hint: you don't need to compute r_a and r_b).

(c) Find an example of a and b such that $\phi = r_b \circ r_a$ is a rotation by $\pi/2$.

Solution.

(a) $r_a(x) = x - 2\frac{\langle x,a\rangle}{\langle a,a\rangle}a, \quad r_b(x) = x - 2\frac{\langle x,b\rangle}{\langle b,b\rangle}b;$
$\langle a, a\rangle = 1$, $\langle v, a\rangle = 1$, so,
$u := r_a(v) = r_a((0, 1, 2)) = (0, 1, 2) - 2\frac{1}{1}(0, 1, 0) = (0, -1, 2).$
$\langle b, b\rangle = 1$, $\langle u, b\rangle = 0$, so,
$r_b \circ r_a(v) = r_b(u) = (0, -1, 2) - 0 = (0, -1, 2).$

(b) To find the type of isometry $\phi = r_b \circ r_a$ it is sufficient to determine weather the lines l_a and l_b are intersecting, or parallel, or ultra-parallel:

- if they do intersect $\quad\phi$ is elliptic;
- if they are parallel $\quad\phi$ is parabolic;
- if they are ultra-parallel $\quad\phi$ is hyperbolic.

The behaviour of two lines is determined by the value $Q = \frac{\langle a,b\rangle^2}{\langle a,a\rangle\langle b,b\rangle}$:

- l_a intersects l_b if $Q < 1$;
- l_a is parallel to l_b if $Q = 1$;
- l_a is ultra-parallel to l_b if $Q > 1$.

In our case, $Q = \frac{9}{1\cdot 1} > 1$, so that the lines are ultra-parallel. This implies that ϕ is hyperbolic.

(c) To get a rotation by $\pi/2$ we need to find two lines making the angle $\pi/4$. The easiest way to get such a pair of lines is to put their intersection into the centre of the model where the angles do coincide with Euclidean ones.

Take the lines defined by $a = (1, 0, 0)$ and $b = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0))$. Then $\cos^2(\angle ab) = Q = \frac{(\frac{\sqrt{2}}{2})^2}{1\cdot 1} = \frac{2}{4}$. So, $\angle ab = \arccos\frac{\sqrt{2}}{2} = \pi/4$.

18.1 Let l be a line on the hyperbolic plane and let E_l be the equidistant curve for l.

(a) Let C_1 and C_2 be two connected components of the same equidistant curve E_l. Show that that C_1 is also equidistant from C_2, i.e. given a point $A \in C_1$ the distance $d(A, C_2)$ from A to C_2 does not depend on the choice of A.

(b) Let $A \in E_l$ be a point on the equidistant curve, and let $A_l \in l$ be the point of l closest to A. Show that the line AA_l is orthogonal to the equidistant curve.

(c) Let $P, Q \in l$ be two points on l. Let $A \in E_l$ be a point of the equidistant curve such that the segments AP and AQ contain no point of E_l except A. Continue the rays AP and AQ till the next intersection points with E_l, denote the resulting intersection points by B and C. Let T be a curvilinear triangle ABC (with geodesic sides AB and AC, but BC being a segment of the equidistant curve). Assuming that all angles of ABC are acute show that the area of T does not depend on the choice of $A \in E_l$.

(d) With the assumptions of (c), show that the area of the geodesic triangle ABC does not depend on the choice of A.

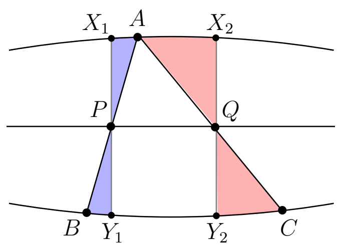
Solution.

(a) Any hyperbolic translation along the line l preserves both C_1 and C_2 (not pointwise) and moves A along C_1. Moreover, for any $B \in C_1$ there is a suitable translation T along l such that $T(A) = B$. So, the distance from B to C_2 is the same as $d(A, C_2)$.

(b) In the upper half-plane model, let l be a vertical ray on the line $x = 0$. Then the equidistant curve is the union of two rays from the origin, the line AA_l is represented by a half of a circle centred at the origin and is obviously orthogonal to the rays forming the equidistant curve. As the upper half-plane model is conformal, this implies that AA_l is orthogonal to E_l in the sense of hyperbolic geometry.

(c) Let l_P the line through P orthogonal to l and let X_1 and Y_1 be the intersections of l_P with C_1 and C_2 respectively lying on distance c_0 from P. Similarly, we construct the line l_Q through Q, $l_Q \perp l$, and its intersection points X_2 and Y_2 with C_1 and C_2.

Consider the curvilinear triangles PAX_1 and PBY_1. The rotation R by π around P swaps these triangles (indeed, R preserves all lines through P and swaps the circles C_1 and C_2). This implies that these curvilinear triangle have equal areas. Similarly, the curvilinear triangles QAX_2 and QCY_2 have equal areas. So, the area of the curvilinear triangle ABC coincides with the area of curvilinear quadrilateral $X_1X_2Y_2Y_1$ (with geodesic sides X_1X_2 and Y_1Y_2, but sides X_1X_2 and Y_1Y_2 being the segments of the equidistant curve). The later area does not depend on the choice of A. Notice, that here we use that ABC is acute-angled (if angle B or C is obtuse the diagram is more complicated).



(d) It is sufficient to prove that the distance between B and C does not depend on the choice of A (then the area of ABC differs from the area of T by the area of a lune BC formed by the geodesic segment and a segment of the equidistant curve).

To see that $d(B, C)$ is independent of the choice of A, consider the orthogonal projections A_l, B_l and C_l of the points A, B, C to the line l. Clearly, $d(B_l, P) = d(A_l, P)$ and $d(C_l, Q) = d(A_l, Q)$. This implies that $d(B_l, C_l) = 2d(P, Q)$, (here we use that ABC is acute-angled and hence, $A_l \in PQ$), which does not depend on A. Therefore, $d(B, C)$ does not depend on A.

18.2. (*)

(a) Let l and l' be ultra-parallel lines. Let γ be an equidistant curve for l intersecting l' in two points A and B. Denote by h the common perpendicular to l and l' and let $H = h \cap l'$ be the intersection point. Show that $AH = HB$.

(b) Let l be a line and γ be an equidistant curve for l. For two points A, B on γ, show that the perpendicular bisector of AB is also orthogonal to l.

(c) Let ABC be a triangle in the Poincare disc model. Let γ be a Euclidean circumscribed circle (i.e. a circumscribed circle for ABC considered as a Euclidean triangle). Suppose that γ intersects the absolute at points X and Y. Show that the (hyperbolic) perpendicular bisector to AB is orthogonal to the hyperbolic line XY

(d) Show that three perpendicular bisectors in a hyperbolic triangle are either concurrent, or parallel, of have a common perpendicular.

Solution.

(a) Let l be the imaginary axis in the upper half-plane. Then γ is represented by some other Euclidean ray emanating from 0, and h is represented by (a part of) some Euclidean circle centred at 0. Hence, h is orthogonal to γ. Now, consider the reflection r_h with respect to h. It preserves the line 0∞ (not pointwise), so, it preserves the equidistant curve γ. Also, it preserves the line l' (as l' is orthogonal to h. So, the intersection $A \in l' \cap \gamma$ should be mapped by r_h to another point in $l' \cap \gamma$, which is B. This implies that h is the perpendicular bisector AB.

(b) This is just another wording of part (a). Let l' be the line AB, then we have proved that the common perpendicular to l and l' coincides with the perpendicular bisector of AB. In particular, the latter is orthogonal to l.

(c) The curve γ is an equidistant curve to the line XY. Indeed, applying a Möbius transformation mapping the Poincare disc to the upper half-plane and the points X and Y to 0 and ∞, we take γ to some Euclidean line through 0, and the perpendicular bisector of AB is mapped to the perpendicular bisector of the image. The latter is orthogonal to 0∞ (by part (b)).

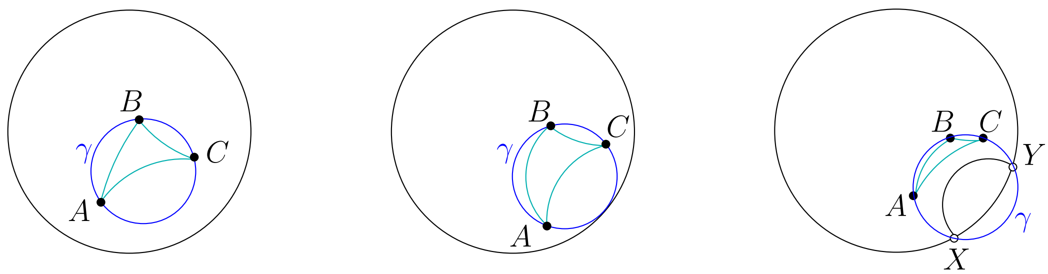
(d) Consider the triangle ABC in the Poincare disc model. Let γ be the Euclidean circle through A, B, C. Consider three cases: γ lies inside hyperbolic plane, is tangent to the absolute or intersects the absolute at two different points.

If γ intersects the absolute at two points X and Y, then as shown in part (c) all perpendicular bisectors are orthogonal to XY.

If γ is tangent to the absolute at X, then mapping this to the upper half-plane (with X mapped to ∞) we see that γ is a horocycle. It is shown in Question 17.2 that all perpendicular bisectors are orthogonal to γ, i.e. in the upper half-plane they are all represented by vertical rays - i.e. are parallel to each other.

If γ lies entirely inside the hyperbolic plane, it actually represents a hyperbolic circle. So, ABC has a circumscribed circle, whose centre is the point of concurrence of all three perpendicular bisectors.

Here are the diagrams showing what can happen in (c) and (d):



or, even more precisely: