Solutions 5-6

5.1 A *circle* $C_{A,r}$ of radius r centred at A is the set of points on distance r from A.
Show that any spherical circle on a sphere $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is represented by a Euclidean circle.

Solution. Let B be any point on $C_{A,r}$. Then $\angle AOB = r$. Let H be the orthogonal projection of B to the line AO. Then $|OH| = \cos r$ does not depend on B (depending only on r), which implies that the spherical circle $C_{A,r}$ lies in the Euclidean plane Π passing through H and orthogonal to OA. Moreover, $|HB| = \sin r$, so, the points of $C_{A,r}$ form a Euclidean circle in the Euclidean plane Π.

5.2 (*) Prove that in a spherical triangle (a) the perpendicular bisectors are concurrent;
(b) the angle bisectors are concurrent.

Solution. (a) A perpendicular bisector of AB is a locus of points on the same distance from A and B. Let Q be an intersection point of the perpendicular bisectors of AB and BC (exists as every pair of lines on the sphere intersects). Then Q lies on the same distance from A and B, but also it lies on the same distance from B and C. So, we conclude that it lies on the same distance from A and C, which implies that the perpendicular bisector for AC also passes through Q.

(b) An angle bisector is a locus of points on the same distance from the rays forming the angle. So, we can repeat the reasoning of the case (a): an intersection point of two angle bisectors lies on the same distance from all the sides of the triangle, so lies on all three angle bisectors.

5.3 Given SAS congruence law for spherical triangles, derive the ASA law.

Solution. Let $A_1B_1C_1$ and $A_2B_2C_2$ be two spherical triangles and $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ be their polar triangles. Suppose that $A_1B_1 = A_2B_2$, $\angle A_1B_1C_1 = \angle A_2B_2C_2$ and $B_1C_1 = B_2C_2$.

Recall that, given a triangle with angles (α, β, γ) and sidelengths (a, b, c), the polar triangle has angles $(\pi - a, \pi - b, \pi - c)$ and sidelengths $(\pi - \alpha, \pi - beta, \pi - \gamma)$. This implies that $\angle A_1'C_1'B_1' = \angle A_2'C_2'B_2'$, $A_1'C_1' = A_2'C_2'$ and $\angle B_1'A_1'C_1' = \angle B_2'A_2'C_2'$. By ASA law of confgruence of triangles this implies that the triangle $A_1'B_1'C_1'$ is congruent to the triangle $A_2'B_2'C_2'$, i.e. that all the angles and sidelengths of $A_1'B_1'C_1'$ are equal to the respective angles and sidelengths of $A_2'B_2'C_2'$.

Now, apply the polar correspondence again. By the bipolar theorem the triangles $A_1B_1C_1$ and $A_2B_2C_2$ are polar to the triangles $A_1B_1C_1$ and $A_2B_2C_2$. This implies that all angles and sidelengths of $A_1B_1C_1$ are equal to the respective elements of $A_2B_2C_2$. Hence, $A_1B_1C_1$ is congruent to $A_2B_2C_2$.

5.4 (*) A *self-polar* triangle is a triangle polar to itself.

(a) Show that a self-polar triangle does exist.

(b) Show that all self-polar triangles are congruent.

Solution. (a) Consider a triangle ABC with angles $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$ and sidelengths $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. Clearly, A is a polar point to the spherical line BC (contained in the same hemisphere as A). Similar properties hold for B and C. So ABC is self-polar.

To see that such a triangle exists, we can take two points B and C on distance $\pi/2$ and the point $A = Pol(BC)$ polar to the line BC. Another way to construct this triangle will be to use coordinates: points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ work nicely for A, B, C.

(b) Let ABC be a self-polar triangle. Then by definition of the polar triangle A is a polar point to the spherical line BC (contained in the same hemisphere as A). This implies that $|AB| = |AC| = \frac{\pi}{2}$. Similar consideration for the vertex B shows $BC = BA = \frac{\pi}{2}$.

So, all sidelengths of a self-polar triangle are $\frac{\pi}{2}$. By SSS theorem of congruence of triangles this implies that all self-polar triangles are congruent.

5.5 On the planet Polaris the whole polar to each point of the dry land lies in the ocean.

(a) How many continents may be on the Polaris if every continent is a disc?
Here by a disc centred at p_0 of radius r we mean the set $\{p \in S^2 \mid d(p, p_0) < r\}$.
Is the number of continent bounded? Can it be odd?

(b) Is it possible that the whole polar to each point of the ocean belongs to the dry land?

Solution. (a) Let $C_N(r)$ be a continent of radius r centred at the north pole. Then the poles of the points in this continent sweep the belt around the equator (more precisely, it is the set of all points lying on distance r or close to the equator). We need to place several continents so that the corresponding belts never intersect other continents. To do that we can take $2(2n+1)$ points on the equator, lying on equal distances from each other. Notice that the polars to that points never pass through other points in the family (as there are even number of them at each half-circle). Now, if we will take r small enough and will replace each points by a disc of radius r, then the polar circles (where we are forbidden to place other continents) will be replaced by the belts (thin enough not to intersect any continent).

So, we can have as many continents as we want. We can also remove one continent from the picture without breaking any rules of the game - so, we will be able to obtain any number of continents (including the odd numbers).

(b) Let the North pole N be covered by some continent. Them the whole equator should lie in the ocean. Take a point A in the equator. Then the polar to A contains intersects the equator (as any two lines in the sphere do intersect), and so, can not consist entirely of the points in dry land.

5.6 Prove the formulas for a spherical triangle with right angle γ:

(a) $\tan a = \tan\alpha \sin b$ (b) $\tan a = \tan c \cos\beta$.

Solution.

(a) By the sine law we have $\sin a = \sin\alpha \sin b / \sin\beta$, which implies

$$\tan a = \frac{\sin a}{\cos a} = \frac{1}{\cos a}\frac{\sin\alpha \sin b}{\sin\beta} = \frac{\sin\alpha}{\cos\alpha}\frac{\cos\alpha}{\cos a}\frac{\sin b}{\sin\beta} = \tan\alpha \sin b \frac{\cos\alpha}{\cos a \sin\beta}.$$

Now, we are left to check that $\cos\alpha = \cos a \sin\beta$, but this follows immediately from the second cosine law

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

and the assumption that $\gamma = \pi/2$.

(b) By the sine law we have $\sin a = \sin\alpha \sin c / \sin\gamma$, which implies

$$tana = \frac{\sin a}{\cos a} = \frac{1}{\cos a}\frac{\sin c \sin\alpha}{\sin\gamma} = \frac{\sin c}{\cos c}\frac{\cos c \sin\alpha}{\cos a \cdot 1} = \tan c \frac{\cos a \cos b \sin\alpha}{\cos a} =$$

$$= \tan c \cos b \sin\alpha = \tan c \cos\beta$$

In the fourth equality we use the cosine law for a right-angled triangle $\cos c = \cos a \cos b$. The last equality uses $\cos b \sin\alpha = \cos\beta$ which immediately follows from the second cosine law

$$\cos\beta = -\cos\alpha\cos\gamma + \sin\alpha\sin\gamma\cos b.$$

5.7 Let T be a spherical triangle with three right angles. Let r and R be the radii of the inscribed and superscribed circles for T. Find the ratio $\sin R/\sin r$.

Solution. Consider three angle bisectors of T: they meet at a point (denote it O), and since T is regular, they are also orthogonal to the sides and pass through the midpoints of the corresponding sides. Moreover, the bisectors decompose T into 6 congruent triangles with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}$ (all six triangles meet in the centre O, so that the angle of each of them at O is $\frac{\pi}{3}$).

So, R is the largest side of this triangle (the one opposite to the right angle), and r is the smallest side (opposite to the angle $\frac{\pi}{4}$). By the sine law we have

$$\frac{\sin R}{\sin\frac{\pi}{2}} = \frac{\sin r}{\sin\frac{\pi}{4}},$$

so, $\frac{\sin R}{\sin r} = 1/\frac{\sqrt{2}}{2} = \sqrt{2}$.

5.8 (*) For a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{2\pi}{3}$ on the unit sphere find the length of the side opposite to the angle $\frac{2\pi}{3}$.

Solution. Applying the second law of cosines

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a$$

to the triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{2\pi}{3}$ we get

$$\cos\frac{2\pi}{3} = -\cos\frac{\pi}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{2}\sin\frac{\pi}{4}\cos a.$$

Since $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ we have

$$-\frac{1}{2} = \frac{\sqrt{2}}{2}\cos a.$$

Thus, $\cos a = (-\frac{1}{2})/(\frac{\sqrt{2}}{2}) = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$.

Hence, $a = \frac{3\pi}{4}$.

5.9 (a) Given a spherical line segment of length α, prove that the polars of all spherical lines intersecting this segment sweep out a set of area 4α.

(b) Given several spherical line segments whose sum of lengths is less than π, prove that there exists a spherical line disjoint from each.

Solution. (a) The polars to the points in α sweep two digons, each of angle α. As the area of the digon of angle α is 2α, the statement follows.

(b) Consider the union U of the polars to all points in all the line segments we are given. From (a) it follows that the area of U is less than 4π, which implies that there is a point A in the sphere not lying in the set U. Let l be a line polar to A. We want to prove that l is disjoint from all the line segments we started from.

Let $B \in l$ be a point. If B belongs to some of the initial segments, then A belongs to its polar, and hence, $A \in U$. This contradicts to the assumption. So B does not belong to any of the initial segments. As B was any point in l, l is disjoint from the initial segments.

6.1 (a) Find the area of a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{3}$. Which part of the area of the whole sphere does it make?

(b) The same question for the triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{4}$.

Solution. (a) Without loss of generality we may assume that the radius of the sphere is 1. Then the area of a spherical triangle with angles α, β, γ is $S_{(\alpha,\beta,\gamma)} = \alpha + \beta + \gamma - \pi$.

So, $S_{(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{3})} = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{3} - \pi = \frac{\pi}{6}$.

The area S of the whole unit sphere is 4π, hence, $S_{(\frac{\pi}{2},\frac{\pi}{3},\frac{\pi}{3})}$ is $\frac{1}{24}$ part of the area of the sphere.

(b) One can compute in the same way as in (a). Another possibility is to notice that the triangle with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})$ may be decomposed into two triangles with angles $(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})$. So, $S_{(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4})} = \frac{1}{2} S_{(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3})} = \frac{\pi}{12}$, which is $\frac{1}{48}$ part of the area of the sphere.

6.2 (a) Find the area of a spherical quadrilateral with angles $\alpha, \beta, \gamma, \delta$.

(b) Given angles of a spherical n-gone, find its area.

Solution. (a) A diagonal decomposes a quadrilateral into two triangles. The area of each triangle is its sum of angles minus π. So, the area of the quadrilateral equals to the sum of angles of two triangles minus 2π. Since the sum of angles of two triangles is equal to the sum of angles of the quadrilateral, we get

$$S_{quadrilateral} = \alpha + \beta + \gamma + \delta - 2\pi.$$

(b) Choose a vertex A of the n-gon. The diagonals incident to A decompose the n-gon into $n-2$ triangles, whose total sum of angles equals to the sum of angles of the n-gon. Hence, the area of the n-gon with angles $\alpha_1, \ldots, \alpha_n$ is

$$S_{n-gon} = \sum_{i=1}^{n} \alpha_i - (n-2)\pi.$$

6.3 Let $Ant : S^2 \to S^2$ be the antipodal map (which takes every point of the sphere to its antipodal). Write Ant as a composition of reflections.

Solution. One way to do that is to take the regular triangle T with three right angles and consider a composition φ of three reflections with respect to its sides. It is straightforward to check that this composition takes the vertices of T to their antipodes. As the images of three points completely define the isometry of the sphere, φ takes every point of the sphere to its antipode.

6.4 Show that the group $Isom^+(S^2)$ of orientation-preserving isometries of the sphere is generated by rotations by angle π.

Solution. As all orientation-preserving isometries of S^2 are rotations, we only need to show that every rotation is a composition of rotations by π. Let $R_{N,\alpha}$ be a rotation about the North pole N by the angle α. It is a composition of two reflections $r_2 \circ r_1$ with respect to two lines l_1 and l_2 passing through N and forming an angle $\alpha/2$. Let r_3 be the reflection with respect the equator. Then

$$R_{N,\alpha} = r_2 \circ r_1 = r_2 \circ (r_3 \circ r_3) \circ r_1 = (r_2 \circ r_3) \circ (r_3 \circ r_1).$$

As the equator is orthogonal to both l_1 and l_2, the isometries $r_2 \circ r_3$ and $r_3 \circ r_1$ are actually rotations by π.

6.5 (*) Prove that (a) the medians and (b) the altitudes of a spherical triangle are concurrent.

Solution: The <u>idea</u> of the solution is the same for the case of medians and altitudes. We will use a projection p from the centre O of the sphere to some plane, such that a spherical triangle will be mapped to a Euclidean triangle and the medians (altitudes) of the sperical triangle will be mapped to the medians (respectively, altitudes) of the Euclidean triangle. As the medians (respectively altitudes) of a Euclidean triangle are concurrent (see E17, E21) at some point X, we conclude that the medians (altitudes) of a spherical triangle are concurrent at the point which projects to X.

(a) Let ABC be a spherical triangle and pet p be a projection of the sphere from the center of the sphere to the plane ABC. It is sufficient to show that this projection takes a median of a spherical triangle to the median of the corresponding Euclidean triangle.

To prove this, let M be a midpoint of AB on the sphere and $M' = p(M) = MO \cap AB$ be its projection to ABC. Then $\angle MOA = \angle MOB$ and hence $\angle M'OA = \angle M'OB$, the later implies that the (Euclidean) triangles $M'OA$ and $M'OB$ are congruent by SAS ($OB = OA$ is the radius of the sphere and AM' is the common side). Hence $M'A = M'B$, so that M' is the (Euclidean) midpoint of AB. So, the projection of the spherical midpoint of AB is the Euclidean midpoint, and hence, the projection takes a spherical median to a Euclidean one.

(b) First, suppose that the spherical triangle ABC has two (or more) right angles, $\angle A = \angle B = \frac{\pi}{2}$. Then the altitude from C is not uniquely determined (more precisely, every line through C is orthogonal to AB). It is clear that in this case the altitudes of ABC are not necessirily concurrent.

Suppose now that the spherical triangle ABC has at most one right angle. Then each of the three altitudes of ABC is uniquely determined (by the conditions that it passes through a vertex and is orthogonal to a side). Below we prove that in this case the altitudes of ABC are concurrent.

The idea is the same as for the case of medians, however, we will use another projection.

Let Π_C be a plane in $\mathbb{E}^3$ tangent to the sphere at the point C. Let p_C be a projection of the sphere to the plane Π_C from the centre O of the sphere. It is clear that p_C takes segments of spherical lines to segments of Euclidean lines (obtained as intersection of the plane Π_C with the corresponding plane through O). Let ABC be a spherical triangle (with at most one right angle) and let $A'B'C = p_C(ABC)$. We need to prove that p_C takes all altitudes of the spherical triangle ABC to the altitudes of the Euclidean triangle $A'B'C$. Notice that each of the three pairs (side of ABC, corresponding altitude) contains a line through C (it is a side for sides AC and BC and an altitude for the side AB). So, the required statement follows immediately from the next claim:

> **Claim:** *Let Π_C be a plane in $\mathbb{E}^3$ tangent to the sphere at the point C.*
> *Let α be a plane, $O, C \in \alpha$ and let β be any plane through O orthogonal to α.*
> *Let $l_\alpha = \Pi_C \cap \alpha$ and $l_\beta = \Pi_C \cap \beta$ be non-empty. Then l_α is orthogonal to l_β.*
>
> Proof of the claim. Let $\mathbf{v}_\alpha$ be a normal vector to the plane α. Then each plane Π orthogonal to α contains a line parallel to $\mathbf{v}_\alpha$. Moreover, there exists such a line (parallel to $\mathbf{v}_\alpha$ and lying in Π) through every point in Π. In particular, this holds for each of the planes Π_c and β (as both of them are orthogonal to α). Let Q be a point in $l_\beta = \Pi_C \cap \beta$. Then the line through Q parallel to $\mathbf{v}_\alpha$ lies both in Π_C and β, so this line coincides with the intersection line l_β. Since $\mathbf{v}_\alpha$ is orthogonal to α, it is orthogonal to every line in α, in particular, to l_α. This implies that l_β is orthogonal to l_α.

Remark 1. Were does this proof fails for the triangles with 2 right angles? Is the projection p_C of ABC always well defined?

Remark 2. As the solution both for medians and altitudes refers to the corresponding Euclidean theorems, it looks reasonable to refresh/to learn the proofs of the Euclidean theorems. For example, one can find the proofs on the cut-the-knot.

6.6 Given a spherical triangle ABC and the midpoints M and N on the sides AB and AC respectively, show that $MN > BC/2$.

Solution: Let $AN = b$, $AM = c$, $MN = a$ and $CB = d$ (note that we choose a bit non-standard notation here to avoid working with half-distances).

We need to show that $MN > BC/2$, or equivalently that $d < 2a$, which is equivalent to that $\cos d > \cos(2a)$. It is sufficient to check that $\cos d - 2\cos^2 a + 1 > 0$ (here we use that $\cos(2a) = 2\cos^2 a - 1$).

We will use the first cosine rule applied to triangles AMN and ABC to compute a and d respectively and to show that $\cos d - 2\cos^2 a + 1 > 0$:

$$
\begin{aligned}
&\cos d - 2\cos^2 a + 1 = \\
&\quad (\cos(2b)\cos(2c) + \sin(2b)\sin(2c)\cos\alpha) - 2(\cos b\cos c + \sin b \sin c\cos\alpha)^2 + 1 = \\
&\quad (2\cos^2 b - 1)(2\cos^2 c - 1) + 4\cos b\cos c\sin b\sin c\cos\alpha - \\
&\qquad 2(\cos^2 b\cos^2 c + 2\cos b\cos c\sin b\sin c\cos\alpha + \sin^2 b\sin^2 c\cos^2\alpha) + 1 = \\
&\qquad 2(\cos^2 b\cos^2 c - \cos^2 b - \cos^2 c + 1 - \sin^2 b\sin^2 c\cos^2\alpha) > \\
&\qquad 2(\cos^2 b\cos^2 c - \cos^2 b - \cos^2 c + 1 - \sin^2 b\sin^2 c) = \\
&\qquad\qquad 2((\cos^2 b - 1)(\cos^2 c - 1) - \sin^2 b\sin^2 c) = 0.
\end{aligned}
$$