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Christmas Problems

Snowed problems to be sent to Santa Claus
by Saturday, 24 December

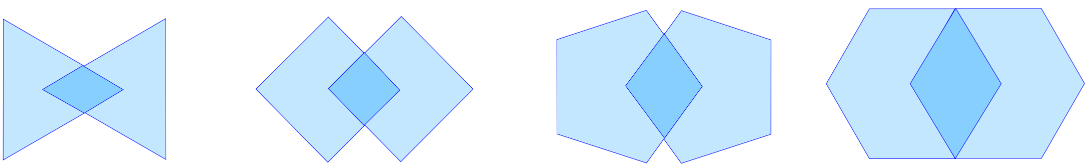
1. (*) The White Witch.

The White Witch plays the following game. She picks polygons in the plane, taking one at each move, and reduces the temperature at each point inside the choosen polygon P_0 by $10°C$. For the initial move she chooses a regular polygon P_0. For every further move i, she is picking a regular polygon P_i equal to P_0 so that the centre of P_i is at one of the vertices of some of the polygons P_j, $j < i$, obtained at one of the previous moves, and a vertex of P_i is at the centre of P_j.

She is playing a fair game and never uses the same polygon twice for different moves. When she started the game, the temperature was $+20°$ everywhere.

What will happen after the White Witch will perform all possible moves? Will it be equally cold everywhere? Will the temperature drop below freezing?

- Solve the question when P is a regular triangle. (A pattern shooud apper...)
- The same question for a quadrilateral.
- And the same for a hexagon... (which of the three games is cooler?)
- Now, consider the same game for a pentagon.
  Prove that this is a very sad modification of the game...

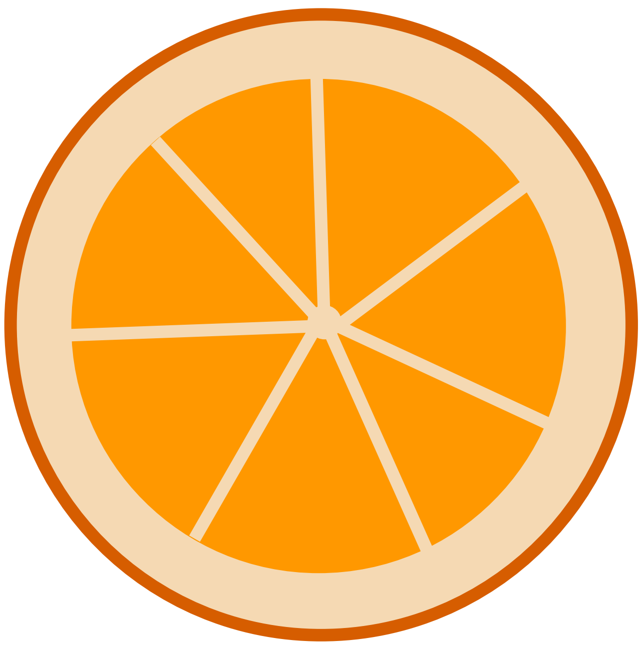


2. Buying spherical oranges.

A spherical orange is a disc on a sphere. These oranges have quite thick peel: an orange of radius r has only a $\frac{4}{5}r$-disc of pulp and the rest is the peel. Oranges are good and tasty, the peel is rubbish.

a) A shop sells big and small oranges (both are tasty) for the same price per unit of total area. What costs less per unit of area of pulp, the big ones or the small ones?

b) The shop also sells Euclidean oranges for the same price.
Which would you prefer to buy, spherical or Euclidean ones?



3. Cutting a triangular cake.

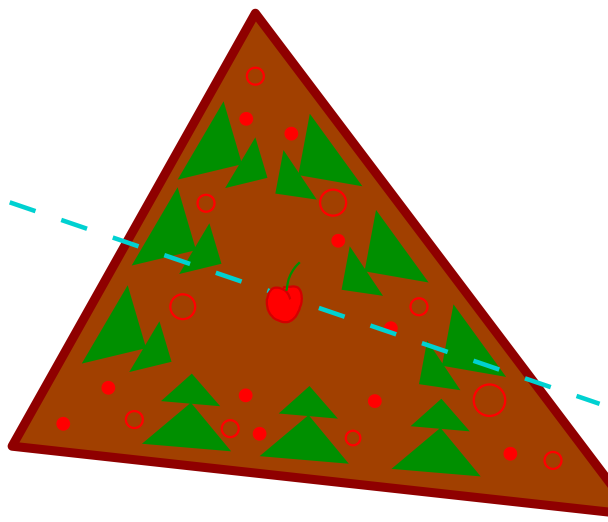
Alice and Bob have a triangular cake to share. Alice chooses a point A inside the cake, then Bob chooses a straight line through A to cut the cake in two parts. Alice gets the **smaller** part and Bob the bigger.

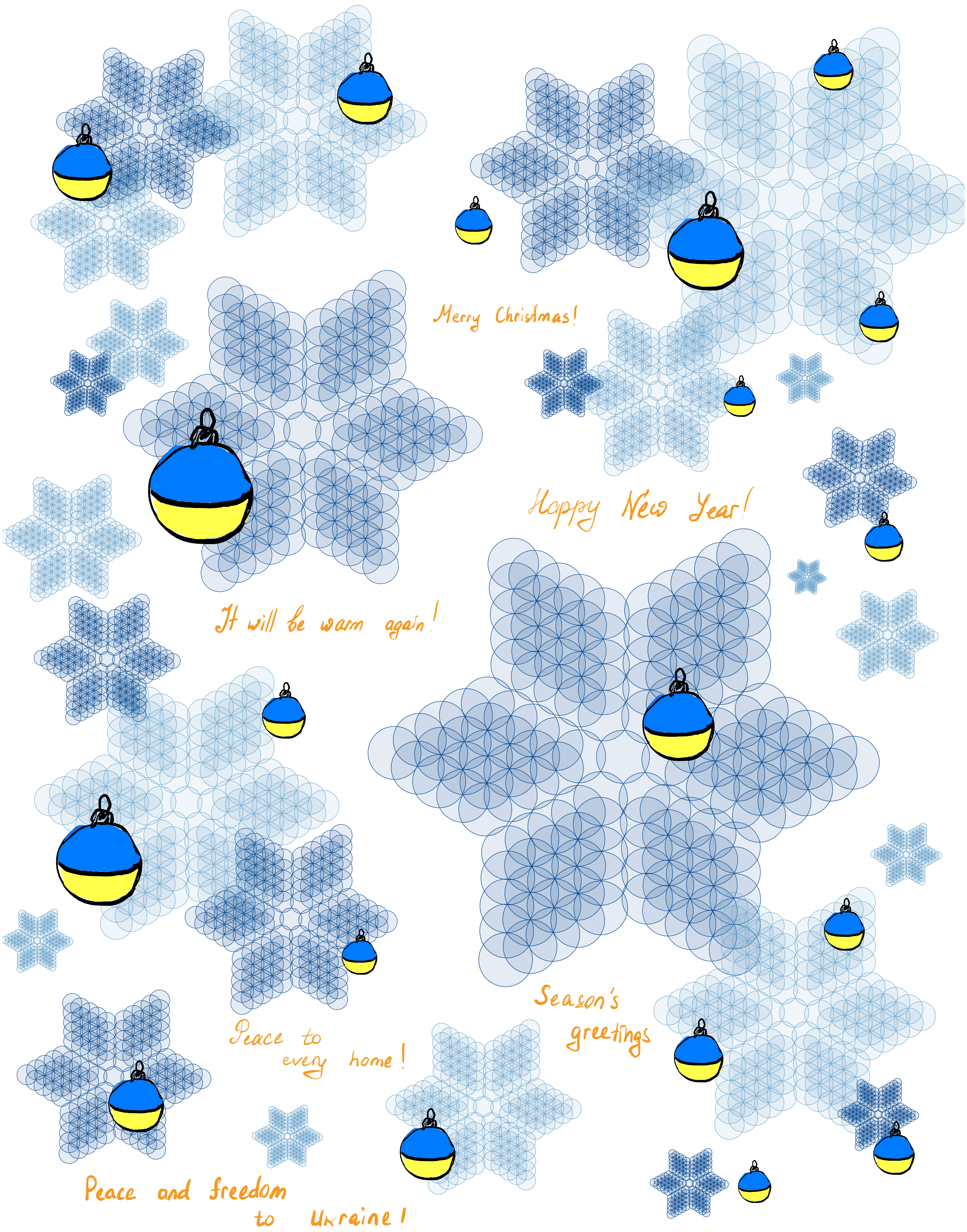
(a) Where should Alice place the point to get as much of the cake as possible?

(b) Now, modify the rules so that Alice gets the **bigger** part. Show that then Bob can always get exactly a half (if he could guess the right line l).

(c) In the settings of (b), can you propose an explicit algorithm allowing Bob to construct a line l which cuts the cake almost into two halves (i.e. as close to the two halves as Bob wants)?

(d) The same questions if the cake is a spherical triangle. (**Warning:** this question is difficult!)




Merry Christmas!
Happy New Year!
It will be warm again!
Season's greetings
Peace to every home!
Peace and freedom to Ukraine!