Anna Felikson, Durham University **Geometry**, 1.11.2022

Assignment 5-6
Starred problems due on Friday, 18 November

5.1 A *circle* $C_{A,r}$ of radius r centred at A is the set of points on distance r from A.
Show that any spherical circle on a sphere $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ is represented by a Euclidean circle.

5.2 (*) Prove that in a spherical triangle (a) the perpendicular bisectors are concurrent;
(b) the angle bisectors are concurrent.

5.3 Given SAS congruence law for spherical triangles, derive the ASA law.

5.4 (*) A *self-polar* triangle is a triangle polar to itself.

(a) Show that a self-polar triangle does exist.

(b) Show that all self-polar triangles are congruent.

5.5 On the planet Polaris the whole polar to each point of the dry land lies in the ocean.

(a) How many continents may be on the Polaris if every continent is a disc?
Here by a disc centred at p_0 of radius r we mean the set $\{p \in S^2 \mid d(p, p_0) < r\}$.
Is the number of continent bounded? Can it be odd?

(b) Is it possible that the whole polar to each point of the ocean belongs to the dry land?

5.6 Prove the formulae for a spherical triangle with right angle γ:

(a) $\tan a = \tan\alpha \sin b$ (b) $\tan a = \tan c \cos\beta$.

5.7 Let T be a spherical triangle with three right angles. Let r and R be the radii of the inscribed and superscribed circles for T. Find the ratio $\sin R/\sin r$.

5.8 (*) For a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{2\pi}{3}$ on the unit sphere find the length of the side opposite to the angle $\frac{2\pi}{3}$.

5.9 (a) Given a spherical line segment of length α, prove that the polars of all spherical lines intersecting this segment sweep out a set of area 4α.

(b) Given several spherical line segments whose sum of lengths is less than π, prove that there exists a spherical line disjoint from each.

6.1 (a) Find the area of a spherical triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{3}$. Which part of the area of the whole sphere does it make?

(b) The same question for the triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{\pi}{4}$.

6.2 (a) Find the area of a spherical quadrilateral with angles $\alpha, \beta, \gamma, \delta$.

(b) Given the angles of a spherical n-gone, find its area.

6.3 Let $Ant : S^2 \to S^2$ be the antipodal map (which takes every point of the sphere to its antipodal).
Write Ant as a composition of reflections.

6.4 Show that the group $Isom^+(S^2)$ of orientation preserving isometries of the sphere is generated by rotations by angle π.

6.5 (*) Prove that (a) the medians and (b) the altitudes of a spherical triangle are concurrent.
Remark: for part (b) assume that the triangle has at most one right angle.
Hint: use some projection to reduce the question to the similar questions on $\mathbb{E}^2$.

6.6 Given a spherical triangle ABC and the midpoints M and N on the sides AB and AC respectively, show that $MN > BC/2$.

References:

1. Lectures 9-11 follow Prasolov's book (see Lecture I and pp. 48-49).
   You can find the same material in pp. 83-87 of Prasolov, Tikhomirov.
2. Lecture 12 does not follow any particular text. The spirit follows the paper by Oleg Viro:
   - O. Viro, *Defining relations for reflections. I*, arXiv:1405.1460v1.
3. Another discussion of the isometry group of the sphere may be found in
   - G. Jones, *Algebra and Geometry*, Lecture notes (Section 2.2).