Assignment 7-8
Starred problems due on Friday, 2 December

7.1. (a) For an affine transformation $f = A\mathbf{v} + \mathbf{b}$ in affine 2-dimensional space, find f^{-1}.
(b) Find an affine transformation $f(\mathbf{v}) = A\mathbf{v} + \mathbf{b}$ which maps the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(0, 1)$, $(2, 4)$, $(4, 4)$ respectively.
(c) Find an affine transformation $g(\mathbf{v}) = C\mathbf{v} + \mathbf{d}$ which maps the points $(4, 7)$, $(9, 6)$, $(-2, 8)$ to the points $(0, 0)$, $(1, 0)$, $(0, 1)$ respectively.
(d) Use (b) and (c) to find an affine transformation $h(\mathbf{v}) = E\mathbf{v} + \mathbf{q}$ which maps the points $(4, 7)$, $(9, 6)$, $(-2, 8)$ to the points $(0, 1)$, $(2, 4)$, $(4, 4)$ respectively.

7.2. (*) Through each vertex of a triangle on Euclidean plane, two lines deviding the opposite side into three equal parts are drawn. Let P be the hexagon bounded by these six lines. Prove that the diagonals joining the opposite vertices of P are concurrent.

7.3. Prove that an arbitrary convex pentagon $ABCDE$ with sides parallel to its diagonals (i.e. such that $AB||CE, BC||DA$, etc) can be affinely transformed into a regular pentagon.

7.4. (*) (a) Use similarity of triangles (or any other arguments of affine geometry) to prove Theorem of Menelaus:

Given a triangle ABC, and a transversal line that crosses BC, AC and AB at points D, E and F respectively, with D, E, and F distinct from A, B and C, then

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

(Note that at least one of the sides will have to be extended to get the intersection point).

(b) The theorem was known before Manelaus. Menelaus proved the spherical version of the theorem (for the sphere of radius 1):

$$\frac{\sin AF}{\sin FB} \cdot \frac{\sin BD}{\sin DC} \cdot \frac{\sin CE}{\sin EA} = 1.$$

Use the sine law to prove the spherical version of Theorem of Menelaus.

7.5. Three pegs on a plane form an isosceles right triangle with a leg of length 3. The pegs may move to an arbitrary distance but on a line parallel to the line formed by the other two. Is it possible to eventually get the three pegs at the vertices of a right triangle with legs 2 and 4?

7.6. (a) Find a projective transformation f which takes the points $0, 1, \infty$ of the projective line to the points $3, 4, 0$ respectively.
(b) Find the image of the point 2 under the transformation f in part (a) and use it to check that f preserves the cross-ratio of the points $0, 1, 2, \infty$.

7.7. Let $A, B, C, D \in \mathbb{R}^2$ be four collinear points and O be a point not on the same line. Suppose that OB is a median of AOC and OC is a median of BOD. Find the cross-ratio of the lines OA, OB, OC and OD.

7.8. (*) (a) Show that $[A, B, C, D] = [C, D, A, B] = [B, A, D, C] = [D, C, B, A]$.
(b) Given $[A, B, C, D] = \lambda$ find $[A, B, D, C]$ and $[A, C, B, D]$.
(c) Assuming that $[A, B, C, D] = \lambda$, find all other possible values for $[X_1, X_2, X_3, X_4]$, where (X_1, X_2, X_3, X_4) is a permutation of (A, B, C, D).

8.1. Find the projective transformation of $\mathbb{R}P^2$ sending the points $(1 : 0 : 0), (0 : 1 : 0), (1 : 0 : 1)$ and $(3 : -1 : 2)$ to the points $(2 : -1 : 1)$, $(0 : 1 : 1)$, $(1 : 0 : 2)$, $(0 : 0 : 1)$.

8.2. How many projective transformations send a quadrilateral to itself?

8.3. (*) Calculate the cross ratio of the following four points lying on the infinite line: $(1:2:0)$, $(2:3:0)$, $(3:4:0)$, $(4:1:0)$.

References:

1. Most ideas of Lecture 13 and more information on affine geometry may be found in
   - G. Jones, *Algebra and Geometry*, Lecture notes (Section 3).
2. Lectures 14 and 15 a follow Prasolov's book (see Lecture II and III).
3. Lecture 16 closely follows a section on Pappus' and Desargues's theorems in Chapter 3 of
   - V. V. Prasolov, V. M. Tikhomirov, *Geometry*, American Maths Soc., 2001.
4. One can also find a great exposition of affine and projective geometry in
   - N. Peyerimhoff, *Geometry, Lecture notes*, (sections 2 and 3, in particular, Section 3.3).