Anna Felikson, Durham University **Geometry**, 1.12.2022

Homework 9-10

9.1 Show that removing a small disc from a projective plane we get a Möbius band.

9.2 Removing a line from $\mathbb{R}^2$ or from S^2 one gets a space with two connected components.
Show that $\mathbb{R}P^2 \setminus \mathbb{R}P^1$ is a connected space.

9.3. Let the Möbius band $\mathbf{M}$ is obtained by gluing along the vertical sides of the square with vertices $(\pm 1, \pm 1)$. Let m be a midline of the Möbius band $\mathbf{M}$ (obtained from the segment of the line line $y = 0$).

(a) what is $\mathbf{M} \setminus m$?

(b) Let l be the closed line obtained from $y = 1/2$ and $y = -1/2$. What is $\mathbf{M} \setminus l$?

9.4. Let $\mathbf{C}$ be a conic $x^2 + y^2 = z^2$. What kind of space is $\mathbb{R}P^2 \setminus \mathbf{C}$?

9.5. Given a point P inside a circle and a chord AB through the point P, let M_{AB} denote the intersection point of the two lines tangent to the circle at A and B. Show that M_{AB} runs over some line as A runs over the circle.

9.6. Let $\mathbf{C}$ be a conic $x^2 + y^2 = z^2$. A triangle in $\mathbb{R}P^2$ is self-polar (with respect to $\mathbf{C}$) if its sides are polar to its vertices (not necessarily the opposite ones).

(a) Construct a self-polar triangle with two vertices on $\mathbf{C}$.

(b) Does there exist a self-polar triangle with exactly one vertex on $\mathbf{C}$?

(c) Show that there exists a self-polar triangle having no vertex on $\mathbf{C}$.
Hint: it may have some vertices at infinity.

9.7. (a) Formulate the theorem dual to Desargues' theorem.

(b) Draw an example. (Hint: send the line s to the line at infinity).

(c) Can you prove this theorem?

10.1. (a) How many non-intersecting lines can you draw in the Klein model of hyperbolic plane?
(b) The same question, but no other line should intersect more than two lines of your family.

10.2. Show that any two lines on hyperbolic plane either intersect inside the hyperbolic plane, or intersect on its boundary, or have a unique common perpendicular.

10.3. (Right-angled polygons on hyperbolic plane)

(a) Show that a hyperbolic triangle can not have more than 1 right angle.

(b) Show that there are no hyperbolic rectangles (i.e. quadrilaterals with 4 right angles).

(c) In the Klein model, construct a hyperbolic pentagon with 5 right angles.

References:

1. Lecture 17 follows the section on Pappus' and Desargues' theorems in Chapter 3 of

- V. V. Prasolov, V. M. Tikhomirov, *Geometry*, American Maths Soc., 2001.

2. Material of Lecture 18 (topology of projective plane and polarity on projective plane) may be found in Part II of

- E. Rees, Notes on Geometry, Universitext, Springer, 2004.
(the book is available on DUO in Other Resources).

3. Lectures 19 and 20 follow Lecture IV of Prasolov's book (or see pp.89-93 in Prasolov, Tikhomirov).