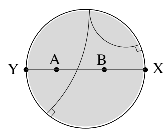
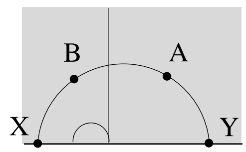
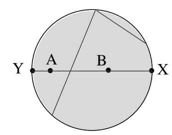
Models of hyperbolic geometry

model	Poincaré disk	Upper half-plane	Klein disk	two-sheet hyperboloid
$\mathbb{H}^2$	$\{z \in \mathbb{C} \mid \|z\| < 1\}$	$\{z \in \mathbb{C} \mid Imz > 0\}$	$\{z \in \mathbb{C} \mid \|z\| < 1\}$	$\{v \in \mathbb{R}^{2,1} \mid \langle v, v\rangle = -1,\ v_3 > 0\}$ where $\langle v, u\rangle = v_1u_1 + v_2u_2 - v_3u_3$
$\partial\mathbb{H}^2$ (absolute)	$\{z \in \mathbb{C} \mid \|z\| = 1\}$	$\{z \in \mathbb{C} \mid Imz = 0\}$	$\{z \in \mathbb{C} \mid \|z\| = 1\}$	$\{v \in \mathbb{R}^{2,1} \mid \langle v, v\rangle = 0,\ v_3 > 0\}$ $v \sim \lambda v$
lines				$\{v \| \langle v, a\rangle = 0\}$ where $\langle a, a\rangle > 0$
distance	$d(A, B) = \|ln[A, B, X, Y]\|$ $X, Y =$ the "endpoints" of the line AB		$d(A, B) = \frac{1}{2}\|ln[A, B, X, Y]\|$	$d(A, B) = \frac{1}{2}\|ln[A, B, X, Y]\|$ cross-ratio of four lines*
formula		$\cosh d(u, v) = 1 + \frac{\|u-v\|^2}{2Im(u)Im(v)}$		$Q = \left\|\frac{\langle u,v\rangle^2}{\langle u,u\rangle\langle v,v\rangle}\right\|$ if $\langle u, u\rangle < 0$, $\langle v, v\rangle < 0$ $Q = \cosh^2 d(pt, pt)$ if $\langle u, u\rangle < 0$, $\langle v, v\rangle > 0$ $Q = \sinh^2 d(pt, line)$ if $\langle u, u\rangle > 0$, $\langle v, v\rangle > 0$ $Q < 1$, intersecting lines $Q = \cos^2 \alpha$ $Q = 1$, parallel lines $Q > 1$, ultraparallel lines $Q = \cosh^2 d(line, line)$
isometries**	Möbius transformations		Projective tr	Linear transformations of $\mathbb{R}^{2,1}$
orientation-preserving isometries		$\frac{az+b}{cz+d}$ $a, b, c, d \in \mathbb{R},\ \ ad - bc = 1$		
orientation-reversing isometries		$\frac{a\bar{z}+b}{c\bar{z}+d}$ $a, b, c, d \in \mathbb{R},\ \ ad - bc = -1$		
reflections	Euclidean inversions or reflections			$r_a(v) = v - 2\frac{\langle v,a\rangle}{\langle a,a\rangle}a$
circles	Euclidean circles		ellipses	plane sections of the hyperboloid
angles	angles=Euclidean angles		distorted angles good for right angles***	

***Right angles in the Klein model**.
Let l be a hyperbolic line.
Let $\bar{l}$ be a Euclidean line containing the segment which represents l in the Klein model.
Let $X_1(l)$ and $X_2(l)$ be the endpoints of l (intersections of $\bar{l}$ with the unit circle).
Let $t_1(l)$ and $t_2(l)$ be tangent lines to the unit circle at the points $X_1(l)$ and $X_2(l)$.
Let $T(l) = t_1(l) \cap t_2(l)$ (if $t_1 || t_2$, i.e. l is represented by a diameter, then $T(l)$ is a point at infinity).

Thm. l' is orthogonal to l if and only if $T(l) \in l'$.

In particular, if l is represented by a **diameter**, then $l' \perp l$ if and only if $\bar{l}' \perp \bar{l}$ (in Euclidean sinse).